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The generalized Balescu–Lenard collision operator

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The generalization of the Balescu–Lenard collision operator to its fully electromagnetic counterpart in Kaufman's action-angle formalism is derived and its properties investigated. The general form may be specialized to any particular geometry where the unperturbed particle motion is integrable, and thus includes cylindrical plasmas, inhomogeneous slabs with non-uniform magnetic fields, tokamaks and the particularly simple geometry of the standard operator as special cases. The general form points to the commonality between axisymmetric, turbulent and ripple transport, and implies properties (e.g. intrinsic ambipolarity) that should be shared by them, under appropriate conditions. Along with a turbulent 'anomalous diffusion coefficient' calculated for tokamaks in previous work, an 'anomalous pinch' term of closely related structure and scaling is also implied by the generalized operator.

1. Introduction

The principal objective of this paper is the generalization of the Balescu–Lenard (BL) collision operator (Balescu 1960; Lenard 1960) from the uniform unmagnetized electrostatic context, in which the standard BL operator is derived, to its electromagnetic counterpart in the action-angle formalism initially developed by Kaufman (1971, 1972). By specializing this one general (though explicit) form, one can obtain the appropriate BL operator in the uniform unmagnetized case, or the uniform but magnetized case for which Montgomery & Turner (1974) obtained the Landau operator, or the cases of a nonuniform magnetized slab, a cylindrical plasma or a tokamak, basically by choosing the appropriate triplet of canonical invariants $\mathbf{J} \equiv (J_1, J_2, J_3)$.

The resultant collision operator in $\mathbf{J}$ space permits one to view the effects on transport due to binary Coulomb collisions on the same formal footing as the effects due to longer-wavelength electromagnetic perturbations, either from internally generated fluctuations (producing 'turbulent' transport), or from externally applied and/or coherent perturbations (for 'ripple' transport). Thus these three types of tokamak transport, normally considered separately, emerge as arising from the same process of diffusion (and drag) in action space. The manifest commonality encourages one to look for the counterparts of significant properties of one transport mechanism in the other two. For example, it is well known that axisymmetric ('neoclassical') tokamak transport (Hinton & Hazeltine 1976) is 'intrinsically ambipolar', i.e. that the electron and ion particle fluxes are equal, independent of the strength of the radial electric field, and that like-particle collisions produce no net particle transport, a consequence of the fact that the binary Coulomb collision operator $C_b(\mathbf{p})$ conserves angular momentum p_ζ. Here it will be shown that the generalized BL operator $C(\mathbf{J})$ also conserves p_ζ, with the same implications for the longer-wavelength fluctuations whose effect it includes; turbulent or ripple tokamak transport due to a steady-state self-consistent spectrum should be intrinsically ambipolar. Further discussion of this and other implications for transport which may be drawn from general considerations of the structure of $C(\mathbf{J})$ will be given in §5.

This work may be regarded as an additional step in the development of a more unified transport theory from the action-angle viewpoint, in which a number of steps have already been taken. Kaufman's original quasi-linear diffusion tensor $\mathsf{D}(\mathbf{J})$ is adequate to study any collisionless diffusive tokamak transport problem where the perturbing fields are not self-consistent. Thus it has been fruitfully applied to problems in collisionless ripple transport (Mynick & Krommes 1980), and to transport in the presence of a background of non-self-consistent microturbulence (Mynick & Krommes 1980; Hazeltine, Mahajan & Hitchcock 1981).

More recently, the action-angle framework has been extended (Bernstein & Molvig 1983; Cohen *et al.* 1984), to incorporate collisional effects. There the binary collision operator $C_b(\mathbf{J})$ in action space is obtained simply by transforming the usual Landau form $C_b(\mathbf{p})$ to $\mathbf{J}$ space. This formulation permitted the treatment of neoclassical transport in the action-angle formalism.

In Mynick (1986) the basic commonality between the transport induced by external ripple and internally generated modes was stressed. This work shares with Bernstein & Molvig (1983) and Cohen *et al.* (1984) a somewhat hybrid character: the action-angle representation is used to deal with the unperturbed motion efficiently, while collisions are included by transforming the binary operator $C_b(\mathbf{p})$ into $\mathbf{J}$ space. The operator $C(\mathbf{J})$ derived in this work provides a basis for eliminating this hybrid feature, and for extending the range of transport mechanisms that may be viewed in a unified manner.

The remainder of the paper is organized as follows. In §2 we introduce some notation, and review the features of the action-angle formalism that are necessary for the work of the remaining sections. The interested reader is referred to Kaufman (1971, 1972) for further development of this formalism. The derivation of $C(\mathbf{J})$ is given in §3, with the final result given in (43), supplemented by the definitions in (35), (36) and (42). The derivation is quite analogous to the standard one, and the result (43) only slightly more complicated than the standard result, though it is a great deal more general. As explained in §3, the key to making this generalization is using 'natural' basis sets to represent the particle distribution functions and the fields that mediate the particle interactions. In §3.3 the general form is specialized to the uniform unmagnetized electrostatic case, and the standard BL operator is recovered. Section 4 explores the properties one expects of a collision operator, including conservation laws and an H theorem. In §5 we indicate some of the ways in which $C(\mathbf{J})$ permits a unification of elements in transport theory normally regarded separately, and a number of additional implications for transport.

2. Review of notation and the action-angle formalism

We denote the set of variables specifying the position of a particle in its 6-dimensional phase space by $z \equiv \{z^i\}$ $(i = 1, \dots, 6)$. Following Kaufman (1971, 1972), we make the particular choice $z = (\boldsymbol{\theta}, \mathbf{J})$, where $\mathbf{J} \equiv (J_1, J_2, J_3)$ are the action invariants of the unperturbed motion and $\boldsymbol{\theta} \equiv (\theta_1, \theta_2, \theta_3)$ are their conjugate angles. Because the Js are constants of the unperturbed motion, the unperturbed Hamiltonian H_0 is independent of $\boldsymbol{\theta}$:

$$H(z) = H_0(\mathbf{J}) + h(\boldsymbol{\theta}, \mathbf{J}, t), \tag{1}$$

where h is the perturbing Hamiltonian. Thus, from Hamilton's equations,

$$\dot{\boldsymbol{\theta}} = \partial_{\mathbf{J}} H = \boldsymbol{\Omega}(\mathbf{J}) + \partial_{\mathbf{J}} h \tag{2}$$

and

$$\dot{\mathbf{J}} = -\partial_{\boldsymbol{\theta}} H = -\partial_{\boldsymbol{\theta}} h(z, t)$$

$$= -\sum_{\mathbf{l}} i\mathbf{l} h(\mathbf{l}, \mathbf{J}, t)\, e^{i\mathbf{l}.\boldsymbol{\theta}}. \tag{3}$$

Here $\partial_x \equiv \partial/\partial x$ denotes a partial derivative with respect to any variable x, and if x is a vector (e.g. $x = \mathbf{J}$), a gradient in the space of that vector is denoted. $\boldsymbol{\Omega} \equiv \partial_{\mathbf{J}} H_0$ is the unperturbed time-rate of change of $\boldsymbol{\theta}$, usually large compared with $\partial_{\mathbf{J}} h$ in (2). In the final form in (3), $h(z, t)$ has been written as a Fourier series in $\boldsymbol{\theta}$, with Fourier coefficients

$$h(\mathbf{l}, \mathbf{J}, t) \equiv (2\pi)^{-3} \oint d\boldsymbol{\theta}\, e^{-i\mathbf{l}.\boldsymbol{\theta}} h(z, t), \tag{4}$$

and with $\mathbf{l}$ a three-component vector index. This is an advantageous representation, because for any function $g(\mathbf{J})$, $g(\mathbf{J})\, e^{i\mathbf{l}.\boldsymbol{\theta}}$ is an eigenfunction of the unperturbed Liouville operator (Lewis & Symon 1979) $L_0 \equiv \{\ , H_0\} \equiv \partial_{\mathbf{J}} H_0 . \partial_{\boldsymbol{\theta}} - \partial_{\boldsymbol{\theta}} H_0 . \partial_{\mathbf{J}}$:

$$L_0 g(\mathbf{J})\, e^{i\mathbf{l}.\boldsymbol{\theta}} = (i\mathbf{l}.\boldsymbol{\Omega})\, g(\mathbf{J})\, e^{i\mathbf{l}.\boldsymbol{\theta}}. \tag{5}$$

This property permits one to deal with unperturbed particle motion for complex geometries essentially as easily as for the unmagnetized case.

For non-relativistic particle motion the Hamiltonian is

$$H(\mathbf{r}, \mathbf{p}) = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi \tag{6}$$

$$= H_0 + h + h_2,$$

with $\mathbf{r}$ the particle position, $\mathbf{p} \equiv M\mathbf{v} + (e/c)\mathbf{A}$ the canonical momentum, $\mathbf{v} \equiv \dot{\mathbf{r}}$ the particle velocity, $\mathbf{A}(\mathbf{r}, t)$ the vector potential and $\Phi(\mathbf{r}, t)$ the electrostatic potential. Writing $\mathbf{A} = \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}, t), \Phi = \Phi_0(\mathbf{r}) + \Phi_1(\mathbf{r}, t)$, with $(\mathbf{A}_0, \Phi_0)$ the unperturbed and $(\mathbf{A}_1, \Phi_1)$ the perturbing potentials in (6), one arrives at the expansion in $(\mathbf{A}_1, \Phi_1)$ given on the second line there, with

$$H_0(z) = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}_0\right)^2 + e\Phi_0, \tag{7}$$

$$h(z, t) = -\frac{e}{c}\mathbf{v}.\mathbf{A}_1 + e\Phi_1 \equiv -\frac{e}{c} v . A_1, \tag{8}$$

and $h_2 = (1/2M)(e/c)^2|\mathbf{A}_1|^2$, which is neglected as being of higher order in the perturbing potentials. In the final form for h in (8), we adopt 4-vector notation for compactness and for ease of making gauge transformations: we write $r^\mu = (\mathbf{r}, r^4 \equiv ct)$, $v^\mu \equiv dr^\mu/dt = (\mathbf{v}, c)$, $A^\mu = (\mathbf{A}, \Phi)$, with metric $(1,1,1,-1)$, so that $A_\mu = (\mathbf{A}_1, -\Phi)$ and $v.A \equiv v^\mu A_\mu \equiv \mathbf{v}.\mathbf{A} - c\Phi$.

The expressions given are all that are needed for the derivation of $C(\mathbf{J})$. However, it is useful to be able to attach some physical significance to the variables $(\boldsymbol{\theta}, \mathbf{J})$. For a uniform unmagnetized plasma, $\mathbf{J} = \mathbf{p} \equiv m\mathbf{v}$ and $\boldsymbol{\theta} = \mathbf{r}$. For a plasma slab, with magnetic field $\mathbf{B}(x) = \hat{\mathbf{z}}B(x) = \hat{\mathbf{z}}\,\partial_x A_y(x)$ and plasma variation in the $\hat{\mathbf{x}}$ direction alone (this includes a uniform magnetized plasma as a particularly simple special case), the invariants are $\mathbf{J} = (J_g, p_y, p_z)$ and their conjugate co-ordinates $\boldsymbol{\theta} = (\theta_g, Y, z)$. Here $J_g \equiv Mv_\perp^2/2\Omega$ is Mc/e times the usual magnetic moment μ, with $\Omega \equiv eB/Mc$ the particle gyrofrequency (species label suppressed), and e and M the particle charge and mass respectively. θ_g is the gyrophase. $p_y \equiv Mv_y + (e/c)A_y(x) \equiv (e/c)A_y(X)$ is the canonical y momentum, conjugate to the guiding-centre y co-ordinate Y, and defining the guiding-centre x co-ordinate X. The pair $(z, p_z \equiv Mv_z)$ have their usual meaning.

As a last example, for a tokamak one may choose (Kaufman 1972) $\mathbf{J} = (J_g, J_b, p_\zeta)$ and $\boldsymbol{\theta} = (\theta_g, \theta_b, \zeta_0)$, where J_b is the longitudinal invariant (the 'bounce action'), θ_b the bounce phase and ζ_0 the bounce-averaged value of the toroidal azimuth ζ. The other variables have meanings already introduced. Forming the scalar product of Hamilton's equation $\mathbf{p} = M\mathbf{v} + (e/c)\mathbf{A}$ with the covariant basis vector $\mathbf{e}_\zeta \equiv R\hat{\zeta}$, one has

$$p_\zeta = Mv_\zeta + \frac{e}{c}A_\zeta(r) \equiv \frac{e}{c}A_\zeta(r_b), \tag{9}$$

where $v_\zeta \equiv \mathbf{e}_\zeta.\mathbf{v} = R^2\dot{\zeta}$. In the last form in (9) we define a minor radial variable, the 'banana centre' $r_b(p_\zeta)$ of a particle, in analogy to the definition for the guiding-centre position $X(p_y)$ made above for the slab example. As there, the definition serves to emphasize the predominantly spatial character of diffusion in p_ζ (or p_y); change in p_ζ corresponds to change in the bounce-averaged minor radius of a particle, related by the conversion factor

$$\frac{\partial p_\zeta}{\partial r_b} = \frac{e}{c}\frac{\partial A_\zeta}{\partial r}\bigg|_{r_b} \approx -\frac{e}{c}RB_p\bigg|_{r_b}, \tag{10}$$

with B_p the poloidal field.

3. Derivation of $C(\mathbf{J})$

We follow the Fokker–Planck approach (see e.g. Ichimaru 1973) in deriving C. As usual, the first two Fokker–Planck coefficients may be manipulated so that the Fokker–Planck equation (truncated after the second coefficient) reads

$$\partial_t f_0(\mathbf{J}) = Cf_0(\mathbf{J}), \tag{11}$$

where f_0 is the $\boldsymbol{\theta}$ average ($\mathbf{l} = \mathbf{0}$ component) of f, and

$$Cf_0(\mathbf{J}) \equiv \partial_{\mathbf{J}}.[\mathbf{D}(\mathbf{J}).\partial_{\mathbf{J}} f_0 - \mathbf{F}(\mathbf{J})f_0]. \tag{12}$$

The diffusion tensor $\mathbf{D}$ is defined as

$$\mathbf{D}(\mathbf{J}) \,.\int_0^\infty d\tau \langle \dot{\mathbf{J}}(t)\,\dot{\mathbf{J}}(t-\tau)\rangle, \tag{13}$$

where $\langle\ \rangle$ denotes an ensemble average and $\dot{\mathbf{J}}(t)$ means $\dot{\mathbf{J}}(z(t),t)$. The friction term $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{J}) \equiv \langle \dot{\mathbf{J}}^p(t)\rangle, \tag{14}$$

where $\dot{\mathbf{J}}^p(t) \equiv \dot{\mathbf{J}}^p(z(t),t)$ means that portion of $\dot{\mathbf{J}}$ due to the 'polarization fields' induced by the test particle, moving along its unperturbed trajectory at $z(t) \equiv (\boldsymbol{\theta}(t), \mathbf{J})$. $\dot{\mathbf{J}}^p$ is negligible in $\mathbf{D}$.

3.1. *Calculation of* $\mathbf{D}$

We first evaluate $\mathbf{D}$. We use (3) in (13), writing $h(\mathbf{l},\mathbf{J},t)$ as the Fourier integral $(2\pi)^{-1}\int d\omega\, h(\mathbf{l},\mathbf{J},\omega)\, e^{-i\omega t}$, and using $\boldsymbol{\theta}(t-\tau) \approx \boldsymbol{\theta}(t) - \boldsymbol{\Omega}\tau$. The τ integration can then be done and, using Plemelj's formula, we obtain

$$\mathbf{D}(\mathbf{J}_1) = \sum_{\mathbf{l}_1,\mathbf{l}_3} \mathbf{l}_1\mathbf{l}_3 \int \frac{d\omega_1}{2\pi}\frac{d\omega_3}{2\pi}\pi\delta(\omega_3 - \mathbf{l}_3 \,.\, \boldsymbol{\Omega}_1)$$
$$\times \langle h^*(\mathbf{l}_1, \mathbf{J}_1, \omega_1)\, h(\mathbf{l}_3, \mathbf{J}_1, \omega_3)\, e^{i(\omega_1-\omega_3)\tau}\, e^{-i(\mathbf{l}_1-\mathbf{l}_3)\,.\,\boldsymbol{\theta}_1(t)}\rangle. \tag{15}$$

Here $\boldsymbol{\Omega}_1 \equiv \boldsymbol{\Omega}(\mathbf{J}_1)$, and in general the subscript 1 refers to the particle or phase point being scattered.

The ensemble average includes an average over $\boldsymbol{\theta}_1(t)$, yielding a $\delta(\mathbf{l}_1 - \mathbf{l}_3)$ in (15). (Here $\delta(\mathbf{l})$ denotes a Dirac δ for those components l_i of $\mathbf{l}$ whose θ_i has an infinite domain (e.g. $\theta_i \to z$ for a slab), and a Kronecker δ when θ_i has domain $[0, 2\pi]$ (e.g. $\theta_i \to \theta_g$).) Thus

$$\mathbf{D}(\mathbf{J}_1) = \sum_{\mathbf{l}_1} \mathbf{l}_1\mathbf{l}_1 \int \frac{d\omega_1}{2\pi}\frac{d\omega_3}{2\pi}\pi\delta(\omega_3 - \mathbf{l}_1 \,.\, \boldsymbol{\Omega}_1)$$
$$\times \langle h^*(\mathbf{l}_1, \mathbf{J}_1, \omega_1)\, h(\mathbf{l}_1, \mathbf{J}_1, \omega_3)\, e^{i(\omega_1-\omega_3)t}\rangle. \tag{16}$$

Assuming in addition that the fluctuation levels $h(\omega_1)$ and $h(\omega_3)$ in (15) have randomly correlated phases unless $\omega_1 = \omega_3$, the ensemble average includes an additional $2\pi\delta(\omega_1 - \omega_3)$, resulting in

$$\mathbf{D}(\mathbf{J}_1) = \sum_{\mathbf{l}_1} \mathbf{l}_1\mathbf{l}_1 \int \frac{d\omega_1}{2\pi}\pi\delta(\omega_1 - \mathbf{l}_1 \,.\, \boldsymbol{\Omega}_1)\langle |h(\mathbf{l}_1, \mathbf{J}_1, \omega_1)|^2\rangle. \tag{17}$$

In the quasilinear calculation of Kaufman (1972), where this standard random-phase assumption is made for a discrete set of normal modes a, the integral $(2\pi)^{-1}\int d\omega$ in (17) is replaced by a discrete sum Σ_a. For the present BL calculation, this random-phase assumption must be justified from computing the fluctuation spectrum, which we now undertake.

3.1.1. *Fluctuations*

The fluctuation level $h(1,t) \equiv h(z_1,t)$ at phase point $z_1 \equiv (\boldsymbol{\theta}_1, \mathbf{J}_1)$ is the sum of the contributions $h(1,t\,|\,i)$ from each particle i in the plasma:

$$h(1,t) = \sum_i h(1,t\,|\,i). \tag{18}$$

By the linearity of Fourier transforms, an analogous relation holds for $h(1,\omega)$ and for $h(\mathbf{l}_1,\mathbf{J}_1,\omega)$.

From the definition (8) of h, it can be seen that we need the contribution $A_{1\mu}(\mathbf{r}_1,\omega\,|\,i)$ to the 4-potential $A_{1\mu}(\mathbf{r}_1,\omega)$ arising from particle i. This requires inversion of the Maxwell equations to obtain the fields $A_{1\mu}$ from the sources $j^\nu \equiv (\mathbf{j}, c\rho)$. To make maximum use of existing results, we choose a gauge in which $\Phi_1 = 0$, so $A_{1\mu} = (\mathbf{A}_1, 0)$. After performing the inversion, we shall then transform back to arbitrary gauge. Alternatively, it is not difficult to generalize the existing 3-vector expressions needed for the derivation given here to be valid for arbitrary gauge, arriving at the same result (29) or (32) for $h(1,\omega\,|\,i)$. However, doing this brings in a significant amount of extra notation associated with a relativistic formulation, extraneous to the main issues considered here, and so we shall follow the present 3-vector, special-gauge route.

With this choice of gauge, the Maxwell equations may be written (Kaufman 1972) as

$$\mathbf{\Delta}(\mathbf{x},\omega)\,.\,\mathbf{A}_1(\mathbf{x},\omega) = -\left(\frac{c}{\omega}\right)^2\frac{4\pi}{c}\mathbf{j}_{\text{ext}}(\mathbf{x},\omega), \tag{19}$$

where $\mathbf{\Delta}$ is defined by

$$\mathbf{\Delta}(\mathbf{x},\omega)\,.\,\mathbf{A}(\mathbf{x}) \equiv \mathbf{A}(\mathbf{x}) + \int d\mathbf{x}'\,\boldsymbol{\chi}(\mathbf{x},\mathbf{x}',\omega)\,.\,\mathbf{A}(\mathbf{x}') - \frac{c^2}{\omega^2}\nabla\times\nabla\times\mathbf{A}. \tag{20}$$

The susceptibility $\boldsymbol{\chi} = \Sigma_s \boldsymbol{\chi}_s$ in (20) (s is the species label) is derived within the action-angle framework in Kaufman (1972). However, the result given there is missing a term (Kaufman, private communication) arising from calculating the contribution $\mathbf{v}_0 f_1$ to the current density, but overlooking the ('adiabatic') contribution $\mathbf{v}_1 f_0 \equiv -(e/Mc)\,\mathbf{A}_1 f_0$. With this missing term restored, the result is

$$\boldsymbol{\chi}_s(\mathbf{x},\mathbf{x}',\omega) \equiv -\frac{\omega_s^2(\mathbf{x})}{\omega^2}\mathbf{I}\delta(\mathbf{x}-\mathbf{x}') - \frac{4\pi}{\omega^2}(2\pi)^3\sum_{\mathbf{l}_2}\int d\mathbf{J}_2\,\mathbf{j}^*(\mathbf{x}\,|\,\mathbf{l}_2,\mathbf{J}_2)\frac{\mathbf{l}_2\,.\,\partial_{\mathbf{J}_2} f_{s0}}{\mathbf{l}_2\,.\,\mathbf{\Omega}_2 - \omega}\mathbf{j}(\mathbf{x}'\,|\,\mathbf{l}_2,\mathbf{J}_2), \tag{21}$$

where $\omega_s(\mathbf{x}) \equiv (4\pi n_{0s}(\mathbf{x})\,e_s^2/M_s)^{\frac{1}{2}}$ is the local plasma frequency of species s, $\mathbf{j}(\mathbf{x}\,|\,2) \equiv \mathbf{j}(\mathbf{x}\,|\,z_2) \equiv e_2\,\mathbf{v}(2)\,\delta(\mathbf{x}-\mathbf{r}_2)$ is the current density at observation point $\mathbf{x}$ due to a particle at phase point $z_2 \equiv (\mathbf{r}_2,\mathbf{p}_2) = (\boldsymbol{\theta}_2,\mathbf{J}_2)$, and $\mathbf{j}(\mathbf{x}\,|\,\mathbf{l}_2,\mathbf{J}_2)$ is its Fourier transform with respect to $\boldsymbol{\theta}_2$. The adiabatic contribution is the first term on the right-hand side of (21).

The inversion of (19) to obtain $\mathbf{A}_1(\mathbf{x},\omega\,|\,i)$ in terms of the external current $\mathbf{j}_{\text{ext}}(\mathbf{x},\omega) = \mathbf{j}(\mathbf{x},\omega\,|\,i)$ due to particle i may be achieved by Green's function methods. We seek the tensor Green's function $\mathbf{G}$ satisfying

$$\mathbf{\Delta}(\mathbf{x},\omega)\,.\,\mathbf{G}(\mathbf{x},\mathbf{x}',\omega) = \mathbf{I}\delta(\mathbf{x}-\mathbf{x}'), \tag{22}$$

i.e. $\mathbf{G}$ is the operator inverse of $\mathbf{\Delta}$. Then

$$\mathbf{A}_1(\mathbf{x},\omega\,|\,i) = -\frac{c^2}{\omega^2}\frac{4\pi}{c}\int d\mathbf{x}'\,\mathbf{G}(\mathbf{x},\mathbf{x}'\omega)\,.\,\mathbf{j}(\mathbf{x}',\omega\,|\,i). \tag{23}$$

We write

$$\mathbf{j}(\mathbf{x}',\omega\,|\,i) = \int d2\,\mathbf{j}(\mathbf{x}'\,|\,2) f(2,\omega\,|\,i), \tag{24}$$

where again 2 denotes phase point z_2 and $f(2,\omega\,|\,i)$ is the contribution to $f(2,\omega)$ from particle i (cf. (31)). Putting (24) into (23), we find

$$h(1,\omega\,|\,i) \equiv -\frac{e_1}{c}\mathbf{v}(1)\,.\,\mathbf{A}_1(\mathbf{r}_1,\omega\,|\,i)$$

$$= \frac{4\pi c^2}{\omega^2}\int d2\,\frac{e_1}{c}\mathbf{v}(1)\,.\,\mathbf{G}(\mathbf{r}_1,\mathbf{r}_2,\omega)\,.\,\frac{e_2}{c}\mathbf{v}(2) f(2,\omega\,|\,i). \tag{25}$$

Kaufman & Nakayama (1970) have given an explicit solution of (22) for the case of a one-dimensional, purely electrostatic, inhomogeneous plasma. Assuming, as there, that $\mathbf{G}(\omega)$ is analytic except at simple poles at the eigenfrequencies ω_a of the plasma, the generalization of their result to the present three-dimensional case may be written

$$\mathbf{G}(\mathbf{x},\mathbf{x}',\omega) = \sum_a \mathbf{E}_a(\mathbf{x})\,[G_a(\omega)/N_a]\,\mathbf{E}_a^*(\mathbf{x}'). \tag{26}$$

Here $\mathbf{E}_a(\mathbf{x})$ is the electric field of the plasma normal modes, which have eigenfrequencies ω_a, and are mutually orthogonal (Kaufman & Nakayama 1970). $N_a \equiv \int d\mathbf{x}\,|\mathbf{E}_a(\mathbf{x})|^2$ is a normalizing factor, defined so that the $G_a(\omega)$, proportional to $(\omega-\omega_a)^{-1}$, are the eigenvalues of $\mathbf{G}$:

$$\mathbf{G}\,.\,\mathbf{E}_a(\mathbf{x}) \equiv \int d\mathbf{x}'\,\mathbf{G}(\mathbf{x},\mathbf{x}',\omega)\,.\,\mathbf{E}_a(\mathbf{x}') = G_a(\omega)\,\mathbf{E}_a(\mathbf{x}). \tag{27}$$

The crucial structural feature of expression (26) is that, by representing any $\mathbf{E}(\mathbf{x},\omega) = (i\omega/c)\,\mathbf{A}(\mathbf{x},\omega)$ in terms of the basis set $\mathbf{E}_a(\mathbf{x})$ of normal modes, $\mathbf{G}$ and its inverse $\mathbf{\Delta}$ are brought into 'diagonal' form, making the inversion of $\mathbf{\Delta}$ simple. Thus, in this representation, $\mathbf{\Delta}$ may be written in a form analogous to (26), with eigenvalues $G_a(\omega)$ in (26) replaced by

$$\Delta_a(\omega) \equiv G_a^{-1}(\omega). \tag{28}$$

This choice of a natural basis set $\mathbf{E}_a(\mathbf{x})$ for the fields, taken in order to permit inversion of the Maxwell operator $\mathbf{\Delta}$, is conceptually analogous to the use of the phase-space eigenfunctions $g(\mathbf{J})\,e^{i\mathbf{l}.\boldsymbol{\theta}}$, already discussed, which permit the inversion of the operator $\partial_t + L_0$ arising in the Vlasov equation. In the uniform unmagnetized context in which the usual BL operator is derived, translational invariance implies that the spatial dependence of the basis functions for the fields and the distribution function are *the same*, $\mathbf{A}(\mathbf{x}) \propto f(\mathbf{x}) \propto \exp i\mathbf{k}\,.\,\mathbf{x}$. This 'degenerate' situation is no longer true for the much more general case considered here, and the resultant expression for $C(\mathbf{J})$ is accordingly a bit (though not much) more complicated, as will be seen.

Using (26) and (28) in (25), we find

$$h(1,\omega\,|\,i) = 4\pi\sum_a \int d2\, h(1,\omega\,|\,a)\,[N_a\,\Delta_a(\omega)]^{-1} h^*(2,\omega\,|\,a) f(2,\omega\,|\,i), \tag{29}$$

where $h(1,\omega\,|\,a) \equiv -(e_1/c)\,\mathbf{v}(1)\,.\,\mathbf{A}_a(\mathbf{r}_1,\omega)$ is the perturbing Hamiltonian due to vector potential $\mathbf{A}_a \equiv (c/i\omega)\,\mathbf{E}_a$. At this point we may transform back to arbitrary gauge, leaving (29) unchanged.

Fourier transforming

$$f(2,t\,|\,i) \equiv \delta[z_2 - z_i(t)] \equiv \delta(\mathbf{J}_2 - \mathbf{J}_i)\,\delta[\boldsymbol{\theta}_2 - \boldsymbol{\theta}_i(0) - \boldsymbol{\Omega}_i t] \tag{30}$$

in time, and Fourier decomposing in $\boldsymbol{\theta}_2$, we have

$$f(2,\omega\,|\,i) = \delta(\mathbf{J}_2-\mathbf{J}_i)\,(2\pi)^{-3}\sum_{\mathbf{l}_2} e^{i\mathbf{l}_2\cdot(\boldsymbol{\theta}_2-\boldsymbol{\theta}_i(0))}\,2\pi\delta(\omega-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_2). \tag{31}$$

Inserting this into (29), we find

$$h(1,\omega\,|\,i) = 4\pi\sum_a\sum_{\mathbf{l}_1,\mathbf{l}_2}\alpha(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_i,\omega,a)\,e^{i\mathbf{l}_1\cdot\boldsymbol{\theta}_1}\,e^{-i\mathbf{l}_2\cdot\boldsymbol{\theta}_i(0)}\,2\pi\delta(\omega-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_i), \tag{32}$$

where

$$\alpha(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_2,\omega,a) \equiv \frac{h(\mathbf{l}_1,\mathbf{J}_1,\omega\,|\,a)\,h^*(\mathbf{l}_2,\mathbf{J}_2,\omega\,|\,a)}{N_a\,\Delta_a(\omega)}$$

measures the effectiveness of mode a in coupling particles 1 and 2. From (32), the Fourier coefficients $h(\mathbf{l}_1,\mathbf{J}_1,\omega\,|\,i)$ of $e^{i\mathbf{l}_1\cdot\boldsymbol{\theta}_1}$ required by (16) may be read off.

Paralleling the usual argument, summing $h(\mathbf{l},\mathbf{J},\omega\,|\,i)$ from (32) over particle index i (or j) and using this in (16), the expression in angular brackets there involves

$$\sum_{a,b}\sum_{\mathbf{l}_2,\mathbf{l}_4}\sum_{i,j}\langle e^{i\mathbf{l}_2\cdot\boldsymbol{\theta}_i(0)}\,e^{-i\mathbf{l}_4\cdot\boldsymbol{\theta}_j(0)}\,\delta(\omega_1-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_i)\,\delta(\omega_3-\mathbf{l}_4\,.\,\boldsymbol{\Omega}_j)\rangle.$$

The δ functions may be taken outside the ensemble average: $\langle ee\delta\delta\rangle = \langle ee\rangle\,\delta\delta$. The average over $\boldsymbol{\theta}_{i,j}(0)$ yields zero result unless $i=j$ and, since the particles determine the wave phases, unless mode indices a and b are equal, i.e. $\langle ee\rangle = \delta_{ij}\,\delta_{ab}\,\delta(\mathbf{l}_2-\mathbf{l}_4)$. This leaves $\delta\delta = \delta(\omega_1-\mathbf{l}_2\,.\,\Omega_i)\,\delta(\omega_3-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_i) = \delta(\omega_1-\omega_3)\,\delta(\omega_1 -\mathbf{l}_2\,.\,\boldsymbol{\Omega}_i)$. Thus the same factor $\delta(\omega_1-\omega_3)$ provided in quasi-linear theory by the random-phase assumption, necessary for the step from (16) to (17), holds here as well. Finally, the remaining single sum over particles i is replaced by an integral over phase space, $\Sigma_i \to \int dz_2 f_0(z_2) = (2\pi)^3\int d\mathbf{J}_2 f_0(\mathbf{J}_2)$, resulting in

$$\langle h^*(\mathbf{l}_1,\mathbf{J}_1,\omega_1)\,h(\mathbf{l}_1,\mathbf{J}_1,\omega_3)\,e^{i(\omega_1-\omega_3)t}\rangle$$
$$= 2\pi\delta(\omega_1-\omega_3)\,(4\pi)^2(2\pi)^3\sum_{\mathbf{l}_2}\int d\mathbf{J}_2 f_0(\mathbf{J}_2)\,2\pi\delta(\omega_1-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_2)\sum_a|\alpha(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_2,\omega_1,a)|^2. \tag{33}$$

Inserting this expression for the spectrum into (16), we obtain, finally,

$$\mathbf{D}(\mathbf{J}_1) = \int d\mathbf{J}_2\,\mathbf{Q}_D(\mathbf{J}_1,\mathbf{J}_2) f_0(\mathbf{J}_2). \tag{34}$$

Here

$$\mathbf{Q}_D(\mathbf{J}_1,\mathbf{J}_2) \equiv \sum_{\mathbf{l}_1,\mathbf{l}_2}\mathbf{l}_1\mathbf{l}_1\,Q(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_2), \tag{35}$$

where

$$Q(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_2) \equiv (4\pi)^2(2\pi)^3\pi\delta(\mathbf{l}_1\,.\,\boldsymbol{\Omega}_1-\mathbf{l}_2\,.\,\boldsymbol{\Omega}_2)\sum_a|\alpha|^2\big|_{\omega_1=\mathbf{l}_1\,.\,\boldsymbol{\Omega}_1}. \tag{36}$$

3.2. *Calculation of* F

We now generalize the expression for the dynamic friction $\mathbf{F}$ in similar fashion. From (3) and (14), we have

$$\mathbf{F}(\mathbf{J}_1) = \langle -i\sum_{\mathbf{l}_1}\mathbf{l}_1\,e^{i\mathbf{l}_1\cdot\boldsymbol{\theta}_1(t)}\,h^{(p)}(\mathbf{l}_1,\mathbf{J}_1,t)\rangle. \tag{37}$$

Here $h^{(p)}$ is the 'polarization' portion of h, i.e. that part of h generated by the particle 1 being acted on. Thus, in the notation of (25) or (32), $h^{(p)}(\mathbf{l}_1,\mathbf{J}_1,t) \equiv$

$h(\mathbf{l}_1, \mathbf{J}_1, t \,|\, i=1)$. Reading off the Fourier transform $h(\mathbf{l}_1, \mathbf{J}_1, \omega_1 \,|\, i=1)$ of this from (32), (37) becomes

$$\mathbf{F}(\mathbf{J}_1) = \Big\langle -i \sum_{\mathbf{l}_1} \mathbf{l}_1 \, 4\pi \sum_a \sum_{l_3} \int \frac{d\omega_1}{2\pi} 2\pi\delta(\omega_1 - \mathbf{l}_3 \,.\, \mathbf{\Omega}_1)$$

$$\times\, e^{i(\mathbf{l}_1 . \mathbf{\Omega}_1 - \omega_1)t} \, e^{i(\mathbf{l}_1 - \mathbf{l}_3) . \boldsymbol{\theta}_1(0)} \frac{h(\mathbf{l}_1, \mathbf{J}_1, \omega_1 \,|\, a)\, h^*(\mathbf{l}_3, \mathbf{J}_1, \omega_1 \,|\, a)}{N_a \Delta_a(\omega_1)} \Big\rangle$$

$$= -4\pi i \sum_a \sum_{\mathbf{l}_1} \mathbf{l}_1 \frac{|h(\mathbf{l}_1, \mathbf{J}_1, \omega_1 \,|\, a)|^2}{N_a \Delta_a(\omega_1)} \bigg|_{\omega_1 = \mathbf{l}_1 . \mathbf{\Omega}_1}. \tag{38}$$

This may be compared with the corresponding expression for the unmagnetized electrostatic case:

$$\mathbf{F}(\mathbf{p}_1) = -4\pi i \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{k} \frac{e_1^2}{k^2 \epsilon(\mathbf{k}, \omega_1)} \bigg|_{\omega_1 = \mathbf{k} . \mathbf{v}_1},$$

to which (38) reduces with the appropriate specializations (cf. §3.3).

In (38) we divide Δ_a into its real and imaginary parts, $\Delta_a \equiv \Delta_a' + i\Delta_a''$. As usual, only the imaginary part contributes to $\mathbf{F}$:

$$\mathbf{F}(\mathbf{J}_1) = -4\pi \sum_a \sum_{\mathbf{l}_1} \mathbf{l}_1 \left| \frac{h(\mathbf{l}_1, \mathbf{J}_1, \omega_1 \,|\, a)}{N_a \Delta_a(\omega_1)} \right|^2 N_a \Delta_a''(\omega_1) \,|_{\omega_1 = \mathbf{l}_1 . \mathbf{\Omega}_1}. \tag{39}$$

As outlined in Kaufman (1971, 1972), we correspondingly divide $\mathbf{\Delta}$ into its Hermitian (reactive) and anti-Hermitian (dissipative) parts, $\mathbf{\Delta} = \mathbf{\Delta}' + i\mathbf{\Delta}''$, and assume that $\mathbf{\Delta}''$ may be treated as a perturbation on $\mathbf{\Delta}'$. Then the eigenfrequencies ω_a are nearly real, and $\mathbf{\Delta}''$ comes from the dissipative part $\boldsymbol{\chi}''$ of $\boldsymbol{\chi}$ in (20). Thus

$$N_a \Delta_a''(\omega) \equiv \int d\mathbf{x}\, d\mathbf{x}'\, \mathbf{E}_a^*(\mathbf{x}) \,.\, \mathbf{\Delta}''(\omega) \,.\, \mathbf{E}_a(\mathbf{x}')$$

$$= \int d\mathbf{x}\, d\mathbf{x}'\, \mathbf{E}_a^*(\mathbf{x}) \,.\, \boldsymbol{\chi}''(\mathbf{x}, \mathbf{x}', \omega) \,.\, \mathbf{E}_a(\mathbf{x}')$$

$$= -(4\pi)(2\pi)^3 \sum_{\mathbf{l}_2} \int d\mathbf{J}_2\, \mathbf{l}_2 \,.\, \partial_{\mathbf{J}_2} f_0\, \pi\delta(\omega - \mathbf{l}_2 \,.\, \mathbf{\Omega}_2)\, |h(\mathbf{l}_2, \mathbf{J}_2, \omega \,|\, a)|^2, \tag{40}$$

where (21) has been used in obtaining the final, explicit, form in (40). Using this in (39), we find, finally,

$$\mathbf{F}(\mathbf{J}_1) = \int d\mathbf{J}_2\, \mathbf{Q}_F(\mathbf{J}_1, \mathbf{J}_2) \,.\, \partial_{\mathbf{J}_2} f_0(\mathbf{J}_2), \tag{41}$$

where
$$\mathbf{Q}_F(\mathbf{J}_1, \mathbf{J}_2) \equiv \sum_{\mathbf{l}_1, \mathbf{l}_2} \mathbf{l}_1 \mathbf{l}_2\, Q(\mathbf{l}_1, \mathbf{J}_1, \mathbf{l}_2, \mathbf{J}_2), \tag{42}$$

and Q is given by (36). Using (34) and (41) in (12) yields the principal result of this paper:

$$Cf_0(\mathbf{J}_1) = \partial_{\mathbf{J}_1} \,.\, \int d\mathbf{J}_2 \,(\mathbf{Q}_D \,.\, \partial_{\mathbf{J}_1} - \mathbf{Q}_F \,.\, \partial_{\mathbf{J}_2}) f_0(\mathbf{J}_1) f_0(\mathbf{J}_2), \tag{43}$$

with $\mathbf{Q}_D$ and $\mathbf{Q}_F$ given by (35) and (42). Species indices s_1 and s_2 of the scattered and scattering distributions may be restored in the obvious manner.

3.3. *Recovery of the standard BL operator*

Insight into the physical content of the generalized operator in (43) is obtained by specializing it to specific physical situations. The standard BL operator $C(\mathbf{p})$ is derived in a uniform unmagnetized plasma, considering electrostatic fluctuations only. Thus we set $A_{1\mu} = (\mathbf{0}, \Phi_1)$ in (8), yielding $h(z, t) = e\Phi_1(\mathbf{r}, t)$, and make the replacements $z \equiv (\boldsymbol{\theta}, \mathbf{J}) \to (\mathbf{r}, \mathbf{p})$ already noted in §2 for the unmagnetized case. The normal-mode label a is replaced by a corresponding wave vector $\mathbf{k}, \Phi_a(\mathbf{x}) \to \Phi_{\mathbf{k}}(\mathbf{x}) = \bar{\Phi}_{\mathbf{k}} \exp i\mathbf{k}.\mathbf{x}$, with amplitude $\bar{\Phi}_{\mathbf{k}}$ having arbitrary normalization. Correspondingly, Σ_a is replaced by $\int d\mathbf{k}$. Then we easily find

$$N_{\mathbf{k}} \equiv \int d\mathbf{x}\, |\mathbf{E}_{\mathbf{k}}(\mathbf{x})|^2 = k^2 |\bar{\Phi}_{\mathbf{k}}|^2 (2\pi)^3 \delta(\mathbf{0}), \tag{44}$$

$$\Delta_{\mathbf{k}}(\omega) = \epsilon(\mathbf{k}, \omega) \equiv \hat{\mathbf{k}}.[\mathbf{I} + \boldsymbol{\chi}(\mathbf{k}, \omega)].\hat{\mathbf{k}}, \tag{45}$$

$$h(\mathbf{l}_1, \mathbf{p}_1, \omega_1 \,|\, \mathbf{k}) = e_1 \bar{\Phi}_{\mathbf{k}} \delta(\mathbf{l}_1 - \mathbf{k}). \tag{46}$$

In (44), $\delta(\mathbf{0}) \equiv \delta(\mathbf{k})|_{\mathbf{k}=0}$, and in (45), $\epsilon(\mathbf{k}, \omega)$ is the usual longitudinal plasma dielectric. Thus the $|\alpha|^2$ term in (36) becomes

$$|\alpha|^2 = (2\pi)^{-6} \delta(\mathbf{k} - \mathbf{l}_1)\, \delta(\mathbf{k} - \mathbf{l}_2) \left| \frac{e_1 e_2}{k^2 \epsilon(\mathbf{k}, \omega_1)} \right|^2 \left[\frac{\delta(\mathbf{k} - \mathbf{l}_1)\, \delta(\mathbf{k} - \mathbf{l}_2)}{\delta^2(\mathbf{0})} \right]. \tag{47}$$

The term in square brackets here is zero unless $\mathbf{k} = \mathbf{l}_1 = \mathbf{l}_2$, in which case it is unity. Thus the value of the expression (47) is the same in both cases if the term in brackets is omitted. (These manipulations with δ functions may be more rigorously performed using the familiar limiting arguments in which a finite plasma volume is allowed to approach infinity.) Putting (47) into (36) for Q, and using this in (35) and (42), replacing the sums over $\mathbf{l}_1$ and $\mathbf{l}_2$ there by integrals, we obtain

$$\begin{aligned} \mathbf{Q}_D(\mathbf{p}_1, \mathbf{p}_2) &= \mathbf{Q}_F(\mathbf{p}_1, \mathbf{p}_2) \\ &= (4\pi)^2 \int \frac{d\mathbf{k}}{(2\pi)^3} \mathbf{k}\mathbf{k}\, \pi \delta(\mathbf{k}.\mathbf{v}_1 - \mathbf{k}.\mathbf{v}_2) \left| \frac{e_1 e_2}{k^2 \epsilon(\mathbf{k}, \mathbf{k}.\mathbf{v}_1)} \right|^2, \end{aligned} \tag{48}$$

the standard result. One notes the degenerate nature, already mentioned, of the natural basis sets for the modes and particles in the homogeneous unmagnetized case, through the δ function in expression (46) for the wave–particle coupling coefficients. These δ functions eliminate the extra sums over $\mathbf{l}_1$ and $\mathbf{l}_2$ that exist in the general form, and cause $\mathbf{Q}_D$ to equal $\mathbf{Q}_F$, a relation that is not true in general.

4. Properties of $C(\mathbf{J})$

We now consider some of the standard properties that one expects collision operators to possess, namely the appropriate set of conservation laws and an H theorem.

As usual, particle conservation for C holds simply because C in (12) or (43) is a divergence in action space.

The proof of energy conservation is a bit more involved. We have

$$\langle \dot{H}_0 \rangle_J \equiv \partial_t \int d\mathbf{J}_1 H_0(\mathbf{J}_1) f_0(1) = \int d\mathbf{J}_1 H_0(\mathbf{J}_1) C f_0(1)$$

$$= -\int d\mathbf{J}_1 \mathbf{\Omega}_1 . [\mathbf{D} . \partial_{\mathbf{J}_1} - \mathbf{F}] f_0(1)$$

$$= -\int d\mathbf{J}_1 d\mathbf{J}_2 \sum_{\mathbf{l}_1, \mathbf{l}_2} Q(1,2)\, \mathbf{\Omega}_1 . [\mathbf{l}_1 \mathbf{l}_1 . \partial_{\mathbf{J}_1} - \mathbf{l}_1 \mathbf{l}_2 . \partial_{\mathbf{J}_2}] f_0(1) f_0(2), \qquad (49)$$

where for the moment we abbreviate $Q(\mathbf{l}_1, \mathbf{J}_1, \mathbf{l}_2, \mathbf{J}_2)$ by $Q(1,2)$ and $f_0(\mathbf{J}_1)$ by $f_0(1)$, and we have used the Hamilton equation $\partial_{\mathbf{J}_1} H_0(\mathbf{J}_1) = \mathbf{\Omega}(\mathbf{J}_1) \equiv \mathbf{\Omega}_1$. We now interchange the dummy variables 1 and 2 in the second term (arising from $\mathbf{F}$) in (49), and use the symmetry, evident from (36), that $Q(1,2) = Q(2,1)$, finding

$$\langle \dot{H}_0 \rangle_J = -\int d\mathbf{J}_1 d\mathbf{J}_2 \sum_{\mathbf{l}_1, \mathbf{l}_2} Q(1,2)\, (\mathbf{\Omega}_1 . \mathbf{l}_1 - \mathbf{\Omega}_2 . \mathbf{l}_2)\, \mathbf{l}_1 . \partial_{\mathbf{J}_1} f_0(1) f_0(2)$$

$$= 0. \qquad (50)$$

The final equality holds because $\delta(\mathbf{l}_1 . \mathbf{\Omega}_1 - \mathbf{l}_2 . \mathbf{\Omega}_2)$ in $Q(1,2)$ is non-zero only where the factor $\mathbf{l}_1 . \mathbf{\Omega}_1 - \mathbf{l}_2 . \mathbf{\Omega}_2$ explicitly appearing in (50) is zero.

We defer treatment of conservation of the canonical momenta $\mathbf{J}$ to the end of this section, as it is less standard, and take up the H theorem now. The entropy is defined in the usual manner as

$$S \equiv -\int d\mathbf{J}_1 f_0 \ln f_0. \qquad (51)$$

Following standard manipulations, involving the interchange of indices 1 and 2 (this time for *both* the $\mathbf{F}$ and $\mathbf{D}$ terms), we find the following positive-definite form for $\dot{S}$:

$$\dot{S} = \tfrac{1}{2} \int d\mathbf{J}_1 d\mathbf{J}_2 \sum_{\mathbf{l}_1, \mathbf{l}_2} Q(1,2) f_0(1) f_0(2) \{(\mathbf{l}_1 . \partial_{\mathbf{J}_1} - \mathbf{l}_2 . \partial_{\mathbf{J}_2}) \ln [f_0(1) f_0(2)]\}^2. \qquad (52)$$

For the special case of a Maxwellian $f_0, f_0 \propto e^{-H_0/T}$ we have

$$(\mathbf{l}_1 . \partial_{\mathbf{J}_1} - \mathbf{l}_2 . \partial_{\mathbf{J}_2}) \ln [f_0(1) f_0(2)] = (\mathbf{l}_2 . \mathbf{\Omega}_2 - \mathbf{l}_1 . \mathbf{\Omega}_1)/T.$$

As in the proof of H_0 conservation, in conjunction with the δ function in $Q(1,2)$, this factor yields $\dot{S} = 0$ for Maxwellian distributions.

Finally, we turn to considering the conservation of $\mathbf{J}$. With manipulations similar to those used for computing $\langle \dot{H}_0 \rangle_J$ and $\dot{S}$, we find

$$\langle \dot{\mathbf{J}}_1 \rangle_J \equiv \partial_t \int d\mathbf{J}_1 \mathbf{J}_1 f_0(1)$$

$$= -\int d\mathbf{J}_1 d\mathbf{J}_2 \sum_{\mathbf{l}_1, \mathbf{l}_2} Q(1,2)\, (\mathbf{l}_1 - \mathbf{l}_2)\, \mathbf{l}_1 . \partial_{\mathbf{J}_1} f_0(1) f_0(2)$$

$$= -\tfrac{1}{2} \int d\mathbf{J}_1 d\mathbf{J}_2 \sum_{\mathbf{l}_1, \mathbf{l}_2} Q(1,2)\, (\mathbf{l}_1 - \mathbf{l}_2)\, (\mathbf{l}_1 . \partial_{\mathbf{J}_1} - \mathbf{l}_2 . \partial_{\mathbf{J}_2}) f_0(1) f_0(2). \qquad (53)$$

While this vanishes for Maxwellians $f_0(1)$ and $f_0(2)$ of the same temperature, for the same reason that $\dot{S} = 0$ under these circumstances, the expression (53) does not vanish, in general. The reason is that the interaction Hamiltonian $h(1\,|\,2) = h(2\,|\,1)$ between two particles at z_1 and z_2 in general depends on $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ separately, and not only on their difference $\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2$. Unless this is the case, there is no momentum-conservation law for the two-particle system governed by the Hamiltonian

$$H(1,2) \equiv H_0(\mathbf{J}_1) + H_0(\mathbf{J}_2) + h(1\,|\,2). \tag{54}$$

This microscopic origin of the non-conservation of $\mathbf{J}$ becomes apparent in deriving $C(\mathbf{J})$ from the alternate BBGKY hierarchy method (Ichimaru 1973). C involves the two-point correlation function $G(1,2)$, whose evolution equation involves

$$[\partial_{\boldsymbol{\theta}_1} h(1\,|\,2)\,.\,\partial_{\mathbf{J}_1} + \partial_{\boldsymbol{\theta}_2} h(1\,|\,2)\,.\,\partial_{\mathbf{J}_2}] f(1) f(2), \tag{55}$$

where the first operator in brackets gives rise to $\mathbf{D}$ and the second to $\mathbf{F}$. The operator in brackets is just the perturbed portion of the Liouville operator $\{\ ,H\}$ for the two-particle system described by (54).

While $C(\mathbf{J})$ does not conserve $\mathbf{J}$ in general, it does conserve those momenta J_i conjugate to coordinates θ_i for which there is translational symmetry, since then $h(1\,|\,2)$ does depend only on $\theta_{1i} - \theta_{2i}$. For tokamaks, such a coordinate is the (bounce-averaged) toroidal azimuth ζ_0. The coupling coefficients $h(\mathbf{l}, \mathbf{J}, \omega\,|\,a)$ appropriate for a tokamak are given in Mynick & Krommes (1980) and in Mynick (1986), for both toroidally trapped and passing particles, and for fluctuations of both turbulent and ripple-like character. For present purposes, the essential feature of all these particular limits is that, analogous to the δ function appearing in the coupling coefficient in (46), the axisymmetry in a tokamak implies that

$$h(\mathbf{l}, \mathbf{J}, \omega\,|\,a) \propto \delta(l_\zeta - n_a),$$

where $\mathbf{n}_a$ is the toroidal mode number of mode a, and p_ζ is the third component of $\mathbf{l} \equiv (l_g, l_b, l_\zeta)$, corresponding to ζ_0. Thus, from (36),

$$Q \propto \sum_a \delta(l_{1\zeta} - n_a)\,\delta(l_{2\zeta} - n_a) = \delta(l_{1\zeta} - l_{2\zeta}) \sum_a \delta(l_{1\zeta} - n_a).$$

Therefore the ζ_0 component of (53) contains $(l_{1\zeta} - l_{2\zeta})\,Q(1,2) \propto (l_{1\zeta} - l_{2\zeta})\,\delta(l_{1\zeta} - l_{2\zeta}) = 0$, and p_ζ is accordingly conserved.

5. Discussion

The goal of this paper has been the derivation of the generalization $C(\mathbf{J})$ of the standard BL operator, and the demonstration of some of its important properties. The application of $C(\mathbf{J})$ to a range of tokamak transport problems seems indicated, but is outside the scope of this work. In this final section we discuss some of the general features of $C(\mathbf{J})$ that have significant implications for transport, and sketch some of the directions in which the theory developed here might be usefully applied.

An obvious but advantageous feature of using $C(\mathbf{J})$ is that it may be trivially decomposed into a sum of 'binary' and 'turbulent' collision terms:

$$C(\mathbf{J}) = C_b(\mathbf{J}) + C_t(\mathbf{J}), \tag{56}$$

a decomposition achieved simply by dividing the full spectrum of contributing modes a into those with short wavelength ($\lambda < \lambda_D$) for C_b, and longer wavelength ($\lambda > \lambda_D$) for C_t. Both terms individually possess all the properties proved for the full C in §4.

The diffusion in p_ζ induced by C_b alone is what produces axisymmetric collisional transport, while that induced by C_t (sometimes in concert with C_b) produces turbulent and ripple transport. The identical structure of C_b and C_t provides a useful conceptual bridge between the different physical notions used to think about the transport arising from them. For example, it is usual to think of ripple transport as being due to the axisymmetry-breaking effect of perturbations of strength h, each causing p_ζ to vary in accordance with (3):

$$\dot{p}_\zeta = -inh\,(\mathbf{l}_1, \mathbf{J}, \omega_a)\exp(i\mathbf{l}_1 . \boldsymbol{\theta} - i\omega_a t). \tag{57}$$

From this follows the 3-3 or p_ζ-p_ζ component of the quasi-linear diffusion tensor. From (17), this tensor is (Kaufman 1972)

$$\mathsf{D}^{ql}(\mathbf{J}_1) = \hat{S}_{ql}\mathbf{l}_1\mathbf{l}_1\pi\delta(\omega_a - \mathbf{l}_1 . \boldsymbol{\Omega}_1)|h|^2, \tag{58}$$

where
$$\hat{S}_{ql} \equiv \sum_a \sum_{\mathbf{l}_1}$$

is the summation operator appropriate for quasi-linear theory. If the perturbations are internally generated, yielding transport from either self-consistent ripple, turbulence or collisions, then from (32) one sees that h in (57) and (58) is replaced by a sum over contributions $4\pi\alpha$ from scatterers at $(\mathbf{l}_2, \mathbf{J}_2)$. From (34)–(36), the BL diffusion tensor may be written in a form analogous to (58):

$$\mathsf{D}^{BL}(\mathbf{J}_1) = \hat{S}_{BL}\mathbf{l}_1\mathbf{l}_1\pi\delta(\mathbf{l}_2 . \boldsymbol{\Omega}_2 - \mathbf{l}_1 . \boldsymbol{\Omega}_1)|4\pi\alpha|^2, \tag{59}$$

where
$$\hat{S}_{BL} \equiv \sum_a \sum_{\mathbf{l}_1} (2\pi)^3 \sum_{\mathbf{l}_2} \int d\mathbf{J}_2 f_0(\mathbf{J}_2)$$

is the summation operator appropriate to D^{BL}. Thus collisional transport may be regarded as a particular ripple-transport problem, with a superposition $\hat{S}_{BL}$ of electrostatic ripple components, each of strength $4\pi\alpha \propto 4\pi e_1 e_2/k^2$, breaking the axisymmetry.

Following this structural correspondence in the opposite direction, just as performing the sum $\hat{S}_{BL}$ for short-wavelength perturbations yields the binary collision frequency $\nu_b \propto n_{02}\, e_1^2 e_2^2 \ln\Lambda$, performing the sum indicated by (58) or (59) for a turbulent or ripple spectrum will yield a turbulent collision frequency ν_t, where the particular form of ν_t depends upon the set of contributing modes considered.

One example where making explicit this connection between the 'ripple' versus the 'collisional' pictures should aid in unifying (and thereby simplifying) theory lies in the banana-drift 'branch' of ripple transport. It is well known that the diffusion coefficients of the collisionless (D_{st}) and highest-collisionality (D_{rp}) regimes (the 'stochastic' regime (Goldston, White & Boozer 1981) and the 'ripple-plateau' regime (Boozer 1980) respectively) are equal, up to a numerical constant of order unity. Moreover, the intuitive physical arguments used to describe the two transport mechanisms are similar, except that the decorrelation frequency in the random-walk process, approximately equal to the bounce frequency Ω_b in both cases, has a collisionless origin for D_{st}, but a collisional one for D_{rp}. Though the derivation methods in Goldston *et al.* (1981)

and Boozer (1980) are very different, it seems likely from this underlying physical commonality that, using the decomposition (56) of $C(\mathbf{J})$, these two regimes may be derived as a single result, with C_t providing the effective collisionality for D_{st} that C_b does for D_{rp}.

It has already been noted in §1 that a consequence of the fact that $C(\mathbf{J})$ conserves p_ζ is that turbulent transport in an axisymmetric tokamak with a steady-state self-consistent spectrum should be intrinsically ambipolar; an electron can change its 'banana centre' $r_b(p_\zeta)$ by an amount dr_b only if an ion changes its r_b by $Z^{-1}\,dr_b$ (Z is the atomic number of the ion). As a result, as for collisional axisymmetric ('neoclassical') transport, calculations of turbulent transport in non-self-consistent fields (i.e. where only $\mathbf{D}$ is kept, and $\mathbf{F}$ dropped) may grossly overestimate the expected particle flux (but not the energy flux) of the species that would escape faster in the non-self-consistent theory. Thus, for example, the expression for the diffusive flux given by Rechester & Rosenbluth (1978), which has come to be a standard in discussions of turbulent transport, must be supplemented by the corresponding dynamic frictional term in considering particle transport, in order not to lead to erroneous conclusions.

Unlike the special case of collisional transport, however, it should be noted that the p_ζ conservation property proved in §4 is radially *non-local*; non-local turbulent fluctuations mediating the exchange of an increment dp_ζ of p_ζ can transfer dp_ζ from a particle on one flux surface to a second on a different surface.

The conservation of p_ζ, and the self-consistent character of the BL operator, depend critically on the presence of the dynamic-friction term $\mathbf{F}$ in C. As for axisymmetric transport, conservation of p_ζ holds because of the intimate structural relation between $\mathbf{F}$ and $\mathbf{D}$, resulting in an exact balance between the p_ζ component $F_\zeta f$ of $\mathbf{F}f$ and the diffusive flux $(\mathbf{D}\,.\,\partial_{\mathbf{J}} f)_\zeta = D_{\zeta i}\,\partial_{J_i} f$ (summation over i implied). In the action-angle formulation $\mathbf{F}$ represents a drag (i.e. a 'pinch') in action space, and in particular F_ζ is a pinch in the predominantly minor radial variable p_ζ or r_b. Since the turbulent portion of $\mathbf{D}$ is what is usually expected to produce anomalous diffusion in r_b, the turbulent portion of F_ζ represents an 'anomalous pinch' in r_b, with size and scalings closely related to those of the anomalous diffusive term. The relationship between this theoretically indicated anomalous pinch, and the anomalous pinch experimentally observed (Strachan *et al.* 1982; Gentle, Richards & Waelbroeck 1986) on a number of tokamaks awaits investigation.

As for the standard BL operator, the derivation given here for the generalized operator is limited by the assumption of a stable background plasma. As a result, the fluctuation spectrum (cf. (33)), driven solely by shielded test particles, will not correctly model the turbulent spectrum of realistic tokamaks. However, the validity of many of the properties of the operator $C(\mathbf{J})$ derived here are considerably more robust than the precise form of $C(\mathbf{J})$. For example, the conservation of p_ζ, and its implications for transport, should hold purely by virtue of the axisymmetry of the wave–plasma system, and the presence of the 'anomalous pinch' term $\mathbf{F}$ to make the wave–plasma interaction self-consistent should also be independent of the details of the spectrum. Moreover, as the work of this paper indicates, generalization of the present $C(\mathbf{J})$ to one valid for a fully turbulent spectrum should be achievable, with only slightly more difficulty than that accompanying previous turbulent generalizations (Dupree 1970) of the standard BL operator in the unmagnetized electrostatic context.

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