A unified theory of tokamak transport via the generalized Balescu–Lenard collision operator

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A unified basis from which to study the transport of tokamaks at low collisionality is provided by specializing the "generalized Balescu–Lenard" collision operator to toroidal geometry. Explicitly evaluating this operator, ripple, turbulent, and neoclassical transport coefficients are obtained, simply by further specializing the single operator to different particular classes of fluctuation wavelength and mode structure. For each class of fluctuations, the operator possesses a diffusive, test-particle contribution **D**, and in addition a dynamic drag term **F**, which makes the operator self-consistent, and whose presence is accordingly essential for the resultant fluxes to possess the appropriate conservation laws and symmetries. These properties, well known for axisymmetric transport, are demonstrated for one type of turbulent transport, chosen for definiteness, by explicit evaluation of both the "anomalous diffusion" term arising from **D**, as well as the closely related "anomalous pinch" term coming from **F**. The latter term is neglected by test-particle calculations, but is shown to have an important impact on the predicted fluxes.

I. INTRODUCTION

In previous work,[1,2] the action-angle formalism[3] was used to generalize the Balescu–Lenard (BL) collision operator to its (fully electromagnetic) analog in the space of invariant actions $\mathbf{J}\equiv(J_1,J_2,J_3)$ of the unperturbed motion. Assuming only that the unperturbed particle motion is integrable, this "generalized-BL" (gBL) operator describes the process of diffusion and drag in action space **J**, just as the standard BL operator describes diffusion and drag in momentum space **p**, the invariant actions for the especially simple case of unmagnetized motion. The gBL operator is thus easily specialized to a wide range of geometries, including the unmagnetized case, as well as inhomogeneous slabs, cylindrical plasmas, and axisymmetric configurations, simply by assigning a specific physical significance to the actions **J** and conjugate angles **θ**. As for the standard BL operator in the homogeneous, unmagnetized case, the effects of perturbations of all wavelengths appear in the gBL operator in a uniform way. The gBL operator thus provides a common framework from which to view the effects on transport of perturbations of very short wavelength ($\lambda<\lambda_D$), which give rise to collisional symmetric ("classical" and "neoclassical") transport, of the longer wavelengths in the turbulent range, and of wavelengths in the macroscopic range of ripple perturbations. In Ref. 2 it was argued accordingly that, along with the quasilinear diffusion tensor[3] $\mathbf{D}(\mathbf{J})$ to describe transport induced by non-self-consistent perturbations, the gBL operator developed there (and by different means in Ref. 1) provides a unifying basis from which to view these three basic mechanisms of tokamak transport, which are normally regarded as different and distinct.

The principal purpose of the present work is to make contact between the formal expressions of Refs. 1 and 2 and tokamak physics by specializing the gBL operator to the case of tokamak geometry, and showing that the formal operator may in fact be explicitly evaluated, how the mechanics of this evaluation proceeds, and what the explicit results look like. In so doing, we more concretely demonstrate two general aspects of the unifying character of the gBL operator. First, it will be seen that, within the action-angle framework, symmetric, turbulent, and ripple transport coefficients emerge as specializations to different particular perturbing spectra of the same general expression. Second, the implications for transport of important properties shown by the gBL operator to be shared by the different mechanisms can be studied.

One of the most important of these common properties is the effect of "self-consistency," i.e., of including the back-reaction of the particles on the fluctuations, in addition to the transport effects of the fluctuations on the particles. As for the standard BL operator, for the gBL operator inclusion of self-consistency means retaining the "dynamic friction" term **F**, in addition to the diffusive term **D** in the operator. The operator may be written

$$\begin{aligned} Cf &= -\partial_{\mathbf{J}}\cdot\boldsymbol{\Gamma} \\ &\equiv \partial_{\mathbf{J}}\cdot(\mathbf{D}\cdot\partial_{\mathbf{J}}f-\mathbf{F}f). \end{aligned} \tag{1}$$

[In general, we shall use the convention that ∂_x denotes the partial derivative with respect to any variable x, and if x is a vector (e.g., $x\rightarrow\mathbf{J}$), a gradient in the space of that vector is denoted; $\boldsymbol{\Gamma}$ is a flux in action space, due to the perturbing fields.] As shown in Ref. 2, **F** is essential for the operator to possess the correct conservation laws, and, in particular, the conservation of toroidal angular momentum p_ζ. As observed in Ref. 2, and as will be demonstrated here in more concrete form, the constancy of p_ζ implies intrinsic ambipolarity of the transport induced by any portion of the fluctuation spectrum, and thus this property holds for the turbulent as well as the collisional spectrum. For a two species plasma, the turbulent transport is accordingly characterized by properties that are well known for symmetric transport[4]:

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(i) Interactions between particles of the same species do not produce any net particle transport; (ii) the particle fluxes of the two species are equal; and (iii) the transport is independent of the radial electrostatic potential.

For symmetric transport, it is well known that violating self-consistency by dropping **F** in C leads to a prediction for the ion particle flux much larger than the electron flux, in contradiction to property (ii) here. Similarly, it will be seen that doing the analogous test-particle calculation for turbulent transport results in a prediction for the particle flux of one species that is much larger than the self-consistent result. (Which species depends upon the particular transport mechanism.) As noted from the abstract form in Ref. 2, the turbulent contribution to **F** represents an "anomalous pinch" term, closely related in form to the "anomalous diffusion" term coming from **D**, which cancels the like-particle portion of the test-particle flux. In this paper, an explicit expression for this anomalous pinch term will be given for one particular turbulent transport mechanism.

The manner in which self-consistency of the turbulent transport is included in the present work requires some clarification. Entering into the expressions for **D** and **F** in Eq. (1) is the spectrum of the perturbations inducing the transport. As for the standard BL operator, the spectrum that appears in both **D** and **F** in the gBL operator is a *thermal* fluctuation spectrum, which will not properly predict the spectrum of a fully turbulent tokamak. However, the important properties of the gBL operator follow not from the specific *form* of the spectrum, but from the related way in which the spectrum appears in **D** and **F**. Thus in this work we adopt a "pseudothermal" model for the turbulent fluctuation spectrum, taking literally the thermal *structure* of the spectrum [cf. Eqs. (76) and (77)], but replacing the *form* of the thermal spectrum with a model spectrum that better represents the spectra of realistic experiments. In so doing we obtain an operator containing a realistic spectrum, while at the same time maintaining the desired conservation laws, H theorem, etc.

Performing this replacement, but retaining only the term in **D** in Eq. (1) is equivalent to doing the quasilinear, test-particle approach followed by a good deal of existing work in turbulent transport. This approach has the virtues of being analytically manageable, and of using fluctuation spectra that one regards as properly modeling experiment. However, as already noted, all such approaches are non-self-consistent, with the loss of important properties this implies. On the other hand, more complete theories of turbulence retain self-consistency, but result in formidable complexity of the equations to be solved. (Numerous examples of both approaches may be found in the review article by Liewer.[5]) The present treatment may be regarded as an intermediate approach, retaining the advantages of mathematical simplicity and ease of physical interpretation of the former, while also acquiring important self-consistency properties of the latter, without actually having to compute the turbulent spectrum.

The structure of the remainder of the paper is as follows. In Sec. II, we summarize the formal results that will be needed for the applications to follow. In Sec. III the notation and additional physics necessary to specialize the general expressions of Sec. II to toroidal geometry are introduced. In particular, central to the formalism are the "coupling coefficients" $h(\mathbf{l},\mathbf{J},\omega)$, which succinctly describe the characteristics of the interaction between particles and the perturbations of the system. Abstractly defined in Eq. (6), $h(\mathbf{l},\mathbf{J},\omega)$ specialized to toroidal geometry is given in Eq. (33). This single expression contains all three of the basic tokamak transport mechanisms for both electrostatic and magnetic perturbations.

Studying a particular transport mechanism in this formalism amounts to specializing $h(\mathbf{l},\mathbf{J},\omega)$ to the relevant class of perturbations inducing the transport, inserting it into the general expressions for **D** or **F**, and performing the necessary summations. In Sec. IV we illustrate this, and with it, the first of the unifying aspects of the formalism, by evaluating **D** for three classes of perturbations, one representing each of the three general types of tokamak transport.

Evaluating **D** alone brings out a good portion of the mechanics involved in the evaluation of the full flux $\mathbf{\Gamma}$, but in a somewhat simpler context. In order to further aid clarity, we evaluate **D** in this section in the "Lorentz limit," where the mass M_1 of diffusing particles is negligible in comparison with the mass M_2 of the scattering species. This causes substantial simplifications in the evaluation process. In Sec. V, both of these simplifications are removed, evaluating $\mathbf{\Gamma}$ for the turbulent mechanism of Sec. IV B for the cases $M_1 \ll M_2, M_1 \gg M_2$, as well as the case $M_1 \sim M_2$. This permits us to derive explicit expressions for the radial fluxes [cf. Eqs. (84), (87), and (102)] due to all combinations (1-1, 1-2, 2-1, and 2-2) of species–species interactions, and in so doing, to demonstrate explicitly properties (i)–(iii) of self-consistent transport already cited for the chosen turbulent transport mechanism. The two terms in the factor $A_1/e_1 - A_2/e_2$ in each of these expressions for the flux correspond to the diffusive and frictional contributions to $\mathbf{\Gamma}$. Thus the second of these terms is an explicit expression for the "anomalous pinch," whose existence was pointed out in Ref. 2 from consideration of the abstract form of the gBL operator. We conclude Sec. V by demonstrating that the results obtained obey the Onsager symmetries.

In Sec. VI, we provide some further physical interpretation of the transport results of Sec. V, considering the relative size and scaling of the transport contributions from different species–species interactions. This discussion completes the analogy that exists between symmetric transport and this representative turbulent mechanism. We conclude the section with some summarizing comments.

II. THE FORMALISM

Here, we briefly review the abstract results from the action-angle formalism which bear on the transport problem, without reference to any particular magnetic configuration. The essence of the action-angle formalism is the reparametrization of the phase point z of a particle from the more directly physical set $(\mathbf{r},\mathbf{p})$ of the real-space position $\mathbf{r}$ and its conjugate momentum $\mathbf{p}$, to the mathematically more convenient set $(\boldsymbol{\theta},\mathbf{J})$. Because the $\mathbf{J}$ are constants of the unperturbed motion, the unperturbed Hamiltonian H_0 is independent of $\boldsymbol{\theta}$:

$$H(z,t) = H_0(\mathbf{J}) + h(\boldsymbol{\theta},\mathbf{J},t), \tag{2}$$

where

$$h(z,t) = -(e/c)\mathbf{v}(z)\cdot\mathbf{A}_1[\mathbf{r}(z),t] + e\phi_1[\mathbf{r}(z),t] = -(e/c)v^\mu A_{1\mu} \tag{3}$$

is the perturbing Hamiltonian. In it, $\mathbf{A}_1$ and ϕ_1 are the perturbing parts of the vector and electrostatic potentials, respectively. In the second form here, h is written in covariant four-vector notation for compactness. From Hamilton's equations, we have

$$\dot{\boldsymbol{\theta}} = \partial_{\mathbf{J}} H = \boldsymbol{\Omega}(\mathbf{J}) + \partial_{\mathbf{J}} h \tag{4}$$

and

$$\dot{\mathbf{J}} = -\partial_{\boldsymbol{\theta}} h = -i\sum_{\mathbf{l}} \mathbf{l} h(\mathbf{l},\mathbf{J},t)\exp(i\mathbf{l}\cdot\boldsymbol{\theta}). \tag{5}$$

The time transform $h(\mathbf{l},\mathbf{J},\omega)$ of the Fourier coefficients of $h(\mathbf{l},\mathbf{J},t)$ here are the "coupling coefficients,"

$$h(\mathbf{l},\mathbf{J},\omega) \equiv \oint \frac{d\theta}{(2\pi)^3} e^{i\mathbf{l}\cdot\boldsymbol{\theta}} h(z,\omega), \tag{6}$$

which play a central role in the formalism. In Eq. (4), $\boldsymbol{\Omega} \equiv \partial_{\mathbf{J}} H_0$ is the unperturbed time rate of change of $\boldsymbol{\theta}$. Thus, in the absence of h, the $\boldsymbol{\theta}$ evolve linearly in time at rate $\boldsymbol{\Omega}$. In Eq. (5) $\mathbf{l} \equiv (l_1, l_2, l_3)$ is a three-component vector index specifying the Fourier harmonic.

When no ambiguity results, we shall simply abbreviate species label s_1, s_2 with the subscripts 1,2. Unless otherwise indicated, we adopt the convention that a subscript "1" refers to the particle or phase point being scattered, and subscript "2" refers to scattering particles. The quasilinear diffusion tensor in action space may then be written[3]

$$\mathbf{D}_{\mathrm{ql}}(1) = \sum_a \sum_{\mathbf{l}_1} \mathbf{l}_1 \mathbf{l}_1 \pi\delta(\mathbf{l}_1\cdot\boldsymbol{\Omega}_1 - \omega_a)|h(\mathbf{l}_1,\mathbf{J}_1,\omega_a|a)|^2. \tag{7}$$

Here, a is a mode index labeling each of the coherent perturbations to the system. The diffusive term $\mathbf{D}$ in the gBL operator is given by

$$\mathbf{D}(1) = \sum_2 \mathbf{D}(1|2),$$

where the diffusion of species 1 induced by species 2 is given by

$$\mathbf{D}(1|2) = \sum_a \sum_{\mathbf{l}_1} \mathbf{l}_1 \mathbf{l}_1 \sum_{\mathbf{l}_2} (2\pi)^3 \times \int d\,\mathbf{J}_2 f(2) \pi\delta(\mathbf{l}_1\cdot\boldsymbol{\Omega}_1 - \omega_1)|4\pi\alpha|^2|_{\omega_1 = \mathbf{l}_2\cdot\boldsymbol{\Omega}_2}. \tag{8}$$

Here, $f(2) \equiv f(\mathbf{J}_2)$ is the $\boldsymbol{\theta}_2$ average of the scattering distribution, and the coefficients α, defined by

$$\alpha(\mathbf{l}_1,\mathbf{J}_1,\mathbf{l}_2,\mathbf{J}_2,\omega,a) \equiv h(\mathbf{l}_1,\mathbf{J}_1,\omega|a) h^*(\mathbf{l}_2,\mathbf{J}_2,\omega|a)/N_a\Delta_a(\omega), \tag{9}$$

measure the effectiveness of mode a in coupling particles 1 and 2. In α, $N_a \equiv \int dx |\mathbf{E}_a(\mathbf{x})|^2$ is a normalizing factor, $\mathbf{E}_a(\mathbf{x})$ is the electric field for normal mode a, and $\Delta_a(\omega)$ is the eigenvalue for mode a of the Maxwell operator,[2] the generalization to inhomogeneous, electromagnetically interacting plasmas of the dielectric function $\epsilon(\mathbf{k},\omega)$ whose square appears in the denominator of the standard BL operator. In Eq. (8), $\mathbf{D}(1|2)$ is written in a form where its structural resemblance to $\mathbf{D}_{\mathrm{ql}}$ is most apparent; the spectrum $|h|^2$ in $\mathbf{D}_{\mathrm{ql}}$ is replaced in $\mathbf{D}(1|2)$ by the sum $\Sigma_{\mathbf{l}_2}(2\pi)^3\int d\,\mathbf{J}_2 f(2)|4\pi\alpha|^2$ over contributions to the spectrum from all particles in the scattering distribution. It may also be written in a form more transparently resembling its analog in the standard BL operator,

$$\mathbf{D}(1|2) = (2\pi)^3 \int d\,\mathbf{J}_2\, \mathbf{Q}_D(1,2) f(2), \tag{10}$$

where the diffusive kernel $\mathbf{Q}_D$ is given by

$$\mathbf{Q}_D(1,2) \equiv \sum_{\mathbf{l}_1,\mathbf{l}_2} \mathbf{l}_1 \mathbf{l}_1 Q(1,2), \tag{11}$$

with scalar kernel

$$Q(1,2) \equiv \sum_a \pi\delta(\mathbf{l}_1\cdot\boldsymbol{\Omega}_1 - \mathbf{l}_2\cdot\boldsymbol{\Omega}_2)|4\pi\alpha|^2|_{\omega = \mathbf{l}_2\cdot\boldsymbol{\Omega}_2} = Q(2,1). \tag{12}$$

Similarly, the dynamic friction term $\mathbf{F}$ is given by

$$\mathbf{F}(1) = \sum_2 \mathbf{F}(1|2),$$

where

$$\mathbf{F}(1|2) = (2\pi)^3 \int d\,\mathbf{J}_2\, Q_F(1,2)\cdot\partial_{\mathbf{J}_2} f(2), \tag{13}$$

and with the frictional kernel $\mathbf{Q}_F$ defined by

$$\mathbf{Q}_F(1,2) \equiv \sum_{\mathbf{l}_1\mathbf{l}_2} \mathbf{l}_1 \mathbf{l}_2 Q(1,2), \tag{14}$$

differing from $\mathbf{Q}_D$ only in the replacement $\mathbf{l}_1\mathbf{l}_1 \to \mathbf{l}_1\mathbf{l}_2$. Using expressions (10) and (13) in the expression in Eq. (1) for the $\mathbf{J}$-space flux $\boldsymbol{\Gamma}$, we find

$$\boldsymbol{\Gamma}(1) = \sum_2 \boldsymbol{\Gamma}(1|2),$$

with

$$-\boldsymbol{\Gamma}(1|2) = \sum_{\mathbf{l}_1\mathbf{l}_2} \int d^6 z_2\, Q(1,2)\mathbf{l}_1(\mathbf{l}_1\cdot\partial_{\mathbf{J}_1} - \mathbf{l}_2\cdot\partial_{\mathbf{J}_2}) f(1) f(2). \tag{15}$$

Here, $\int d^6 z_2 \equiv (2\pi)^3 \int d\,\mathbf{J}_2$ denotes an integration over the full phase space of species 2, and so may also be written as $\int d\mathbf{r}_2 \int d\mathbf{p}_2$, when this parametrization is convenient.

Closely related to Eq. (15) for $\boldsymbol{\Gamma}$ is the expression derived in Ref. 2 for the time rate of change of the total entropy:

$$\dot{S} \equiv \sum_1 \dot{S}(1) \equiv -\sum_1 \frac{d}{dt}\int d^6 z_1 f(1) \ln f(1) = \frac{1}{2}\sum_{1,2}\sum_{\mathbf{l}_1,\mathbf{l}_2} \int d^6 z_1 \int d^6 z_2\, Q(1,2) f(1) f(2) \times \{(\mathbf{l}_1\cdot\partial_{\mathbf{J}_1} - \mathbf{l}_2\cdot\partial_{\mathbf{J}_2})\ln[f(1)f(2)]\}^2, \tag{16}$$

which is manifestly positive definite, giving zero only when $f(1)$ and $f(2)$ are Maxwellians, $f \sim \exp(-H_0/T)$. Equation (16) will be useful later in Sec. V in considering the (Onsager) symmetries of the transport.

III. SPECIALIZATION TO TOROIDAL GEOMETRY

A. Toroidal variables

We now specialize the general formalism of Sec. II to the axisymmetric geometry of a tokamak. We parametrize real space $\mathbf{r}$ by the flux coordinates (r,θ,ζ), with minor-radial variable r constant on a flux surface. It is useful to use the contravariant basis vectors $\mathbf{e}^i$, given by $\mathbf{e}^r \equiv \nabla r \simeq \hat{r}$, $\mathbf{e}^\theta \equiv \nabla\theta \simeq \hat{\theta}/r$, $\mathbf{e}^\zeta \equiv \nabla\zeta \simeq \hat{\zeta}/R$, dual to the covariant set $\mathbf{e}_r \simeq \hat{r}$, $\mathbf{e}_\theta \simeq r\hat{\theta}$, $\mathbf{e}_\zeta \simeq R\hat{\zeta}$. Thus, one may write wave vector $\mathbf{k}$ as

$$\mathbf{k} = k_i \mathbf{e}^i = k_r \nabla r + m \nabla\theta + n \nabla\zeta,$$

and the unperturbed vector potential $\mathbf{A}_0$ as

$$\mathbf{A}_0 = A_{0i}\mathbf{e}^i = \psi(r)\nabla\theta - \chi(r)\nabla\zeta,$$

i.e., $\mathbf{A}_0$ has covariant components $(A_{0r}, A_{0\theta}, A_{0\zeta}) = [0,\psi(r), -\chi(r)]$. The unperturbed magnetic field is thus given by

$$\mathbf{B} = \mathbf{B}_t + \mathbf{B}_p = \nabla\psi\times\nabla\theta + \nabla\zeta\times\nabla\chi,$$

and the safety factor is $q(r) \equiv d\psi/d\chi$. It will also be useful to employ the right-hand triad of unit vectors $(\hat{r}, \hat{q}, \hat{b})$, where $\hat{b} \equiv \mathbf{B}/B$ and $\hat{q} \equiv \hat{b}\times\hat{r}$, in terms of which we may write $\mathbf{k} = \mathbf{k}_\parallel + \mathbf{k}_\perp$, $\mathbf{k}_\parallel \equiv \hat{b}k_\parallel$, and $\mathbf{k}_\perp \equiv \hat{r}k_r + \hat{q}k_q$.

For toroidal geometry, an appropriate specialization of the actions $\mathbf{J}$ is[2,3,6,7] $(J_1,J_2,J_3) \to (J_g,J_b,J_\zeta \equiv p_\zeta)$, with J_g the gyroaction (equal to Mc/e times the usual magnetic moment μ), J_b the bounce action (equal to the toroidal flux ψ enclosed by a drift orbit), and p_ζ the toroidal angular momentum,

$$p_\zeta \equiv \mathbf{e}_\zeta\cdot\mathbf{p} = (e/c)A_{0\zeta} + Mv_\zeta. \tag{17}$$

The conjugate angles $\boldsymbol{\theta}$ are $(\theta_g,\theta_b,\zeta_0)$, the gyrophase, the bounce phase, and the bounce-averaged toroidal azimuth, respectively. The set of subscripts (g,b,ζ) used here in designating the actions $\mathbf{J}$ will also be used for related triplet quantities, e.g., $\mathbf{l} = (l_g,l_b,l_\zeta)$.

In $\mathbf{J}$ space, we will have use for the (contravariant) basis vectors $\boldsymbol{\epsilon}^i \equiv \partial_{\mathbf{J}} J_i$ $(i = g,b,\zeta)$, which are unit vectors. For any function $F(\mathbf{J})$, we also define $\boldsymbol{\epsilon}^F \equiv \partial_{\mathbf{J}} F$, which is useful in extracting more directly physical information from action-space quantities.

One useful physical function of $\mathbf{J}$ is the particle "banana center" r_b, the average value of r that a particle has over a bounce period. By considering transport in r_b rather than in r, we eliminate from the problem the complexities of the unperturbed motion, such as finite gyroradius ρ_g and banana width r_1, which one knows are irrelevant to the net radial step taken per bounce in the diffusive motion of the particle, no matter how large they are. Bounce averaging Eq. (17) and using the constancy of p_ζ, we have

$$p_\zeta = (e/c)\overline{A}_{0\zeta} + M\overline{v}_\zeta. \tag{18}$$

It will be useful to define a trapping-state index τ, equal to 0 (1) for toroidally trapped (passing) particles. For $\tau = 0$, one has $\overline{v}_\zeta = b_t R\overline{v}_\parallel = 0$ (where $b_{t,p} \equiv B_{t,p}/B$), so we may define r_b for this case by

$$p_\zeta \equiv (e/c)\overline{A}_{0\zeta}(r_b). \tag{19}$$

For $\tau = 1$, p_ζ in Eq. (18) acquires a kinetic portion, while J_b becomes a purely minor radial variable.[3,6,7] The appropriate definition of r_b in this case is

$$J_b \equiv (e/c)\overline{A}_{0\theta}(r_b). \tag{20}$$

Thus we have

$$\boldsymbol{\epsilon}^r \equiv \partial_{\mathbf{J}} r_b = \begin{cases} -\boldsymbol{\epsilon}^\zeta/M\Omega_g Rb_p & (\tau = 0), \\ \boldsymbol{\epsilon}^b/M\Omega_g rb_t & (\tau = 1). \end{cases} \tag{21}$$

For $\tau = 0$, J_b is $v_\parallel$-like,

$$J_b = (2\pi)^{-1}\oint ds_\parallel\, Mv_\parallel \simeq (2\pi)^{-1}\oint d\theta\, qMv_\zeta \equiv qM\overline{v}_\zeta,$$

while for $\tau = 1$, one readily shows from Eqs. (20) and (18) that[7]

$$J_v \equiv J_b - (e/c)\psi[\chi = -(c/e)p_\zeta] = qM\overline{v}_\zeta. \tag{22}$$

Thus by extending this $\tau = 1$ definition for J_v to $\tau = 0$ by $J_v(\tau = 0) \equiv J_b$, we can extract bounce-averaged $v_\parallel$ information from the $\mathbf{J}$ using

$$\boldsymbol{\epsilon}^{J_v} \equiv \partial_{\mathbf{J}} J_v = \begin{cases} \boldsymbol{\epsilon}^b & (\tau = 0), \\ \boldsymbol{\epsilon}^b + q\boldsymbol{\epsilon}^\zeta & (\tau = 1). \end{cases} \tag{23}$$

B. The coupling coefficients

Fundamental to the calculation of $\mathbf{D}$ are the coupling coefficients $h(\mathbf{l},\mathbf{J},\omega|a)$. These have been worked out previously for various cases.[6,8] Here, we give a more complete evaluation. The $h(\mathbf{l},\mathbf{J},\omega|a)$ require a description of the spatial structure of each contributing mode a, and of the unperturbed particle motion. Fully describing the structure of the modes in a tokamak is a field in itself. We shall content ourselves with a model description, chosen to satisfy a number of the general important characteristics that we know the modes should possess. Our model for the mode structure for all components $A_{a\mu}$ $(\mu = 1,\ldots,4)$ of mode a is the eikonal form

$$A_{a\mu}(\mathbf{x}) = \overline{A}_{a\mu}(r)\exp i\left(\int^r dr' k_r(r') + m\theta + n\zeta\right) \tag{24}$$

The toroidal "quantum number" n is rigorously constant, because of axisymmetry, while taking m constant is only approximate, because of toroidal effects. Here $k_r(r)$ is the radial wavenumber, and $\overline{A}_{a\mu}(r)$ is the mode amplitude. We shall assume here that k_r and $\overline{A}_{a\mu}(r)$ may be assumed about constant over the minor-radial excursion of the unperturbed orbit of the particle in question. (We emphasize that situations for which this assumption is not valid present no difficulty for the basic formalism; the integrals to be performed are simply somewhat different.) Thus, over a particle's orbit, mode a is characterized by a local wave vector $\mathbf{k}(r \simeq r_b)$. For externally imposed perturbations, such as ripple from the toroidal-field coils, $\overline{A}_{a\mu}$ typically falls off as one moves radially inward. For internally generated perturbations, a reasonable model is taking $\overline{A}_{a\mu}$ localized about some minor radius r_a, with localization width w_a. We choose the simplified form

$$\overline{A}_{a\mu} = \overline{\overline{A}}_{a\mu}\, s(w_a/2, r - r_a), \tag{25}$$

where $s(x,y)$ is a steplike localizing function, defined as

$$s(x,y)\equiv\begin{cases}1 & (x>|y|),\\ 0 & (x<|y|).\end{cases} \tag{26}$$

(Here, and in what follows, the double overbar will be used to denote the amplitude of quantities with this radial localization explicitly displayed.) In addition to being radially localized, the set of modes a represents a full set of eigenmodes of the Maxwell operator, which, owing to the (near) Hermiticity of that operator, we expect to be orthogonal and complete.[9] We incorporate this general property into our model by envisioning each mode a to be specified by both wave vector $\mathbf{k}$, and by localization radius r_a. Thus modes with the same $\mathbf{k}$ are localized within sequential nested toroidal shells, each of thickness w_a, centered at radii $r_{a1}, r_{a2},\ldots$, and having volume $V_a \simeq (2\pi r_{ai})(2\pi R)w_a$, jointly comprising the plasma volume $V_p = \Sigma_{r_a} V_a$. Therefore the sum over a in $Q(1,2)$ becomes a sum $\Sigma_{r_a}\Sigma_{\mathbf{k}}$.

We describe the particle motion as in Ref. 6. We make the usual separation of $\mathbf{r}$ into contributions from the guiding-center motion and the gyromotion,

$$\mathbf{r}(z) = \mathbf{R} + \boldsymbol{\rho}_g. \tag{27}$$

The gyromotion is given by

$$\boldsymbol{\rho}_g(\theta_g) = \rho_g(\hat{r}\cos\theta_g - \hat{q}\sin\theta_g), \tag{28}$$

with $\rho_g \equiv v_\perp/\Omega$ the gyroradius. The guiding-center motion is described by

$$\mathbf{R}(\theta_b,\zeta_0) = (\mathbf{e}_r r_b + \mathbf{e}_\theta \tau\theta_b + \mathbf{e}_\zeta\zeta_0) + \boldsymbol{\rho}_b, \tag{29}$$

where the oscillatory portion of the bounce motion is given by

$$\boldsymbol{\rho}_b(\theta_b) \equiv \mathbf{e}_r r_1\cos\theta_b + (\mathbf{e}_\theta\theta_1 + \mathbf{e}_\zeta\zeta_1)\sin\theta_b. \tag{30}$$

Here, r_b is the particle banana center, constant in time. The canonical phases $\boldsymbol{\theta} = (\theta_g, \theta_b, \zeta_0)$ evolve linearly in time, as discussed in Sec. II. The particle velocity, needed in Eq. (3), is thus obtained from Eq. (27) by

$$\mathbf{v}(z) \equiv \dot{\mathbf{r}}(z) = \Omega_i\,\partial_{\theta_i}\mathbf{r}(z). \tag{31}$$

The time-independent amplitudes r_1, θ_1, and ζ_1 measure the size of the particle excursions in the r, θ, and ζ directions in the course of a bounce (or transit) period τ_b. The trapping-state index τ, already defined, provides the secular contribution to θ_b for passing particles, and none for trapped particles, as is appropriate.

The use of only the fundamental harmonics $\cos\theta_b$ and $\sin\theta_b$ in expression (30) makes it strictly valid only for particles not too near the trapped/passing boundary. The coupling coefficients for particles near this boundary involve integrals that yield functions less standard than those from "harmonic approximation" of Eq. (30). However, the qualitative behavior is not much changed.

Using expressions (24)–(30) and Eq. (6), the evaluation of $h(\mathbf{l},\mathbf{J},\omega|a)$ is straightforward, using the Bessel identity

$$J_l(z) = \oint \frac{d\theta}{2\pi}\, e^{-il\theta} e^{iz\sin\theta}. \tag{32}$$

We find

$$\begin{aligned} h(\mathbf{l},\mathbf{J},\omega) = {}& \delta(l_\zeta - n)\, e^{-i(l_b - \tau m)\theta_{bk} - il_g\theta_{gk}}\Big[e\phi J_{l_g} J_{l_b-\tau m} \\ & - \Big(\frac{e}{c} v_\perp A_\perp\Big)\frac{1}{2}\big(J_{l_g-1} e^{i\theta_{gk} - i\theta_{gA}} + J_{l_g+1} e^{-i\theta_{gk} + i\theta_{gA}}\big) \\ & \times J_{l_b-\tau m} - \Big(\frac{e}{c} u_1 A\Big) J_{l_g} J_{l_b-\tau m} \Big(\frac{e}{c} u_0 A\Big) J_{l_g}\frac{1}{2} \\ & \times \big(J_{l_b-\tau m-1} e^{i\theta_{bk} - i\theta_{bA}} + J_{l_b-\tau m-1} e^{-i\theta_{bk} + i\theta_{bA}}\big)\Big]. \end{aligned} \tag{33}$$

First, we define some terms in this expression, and then provide some interpretation. In it, mode label a has been suppressed. All of the perturbing potentials ϕ or A appearing in (33) denote the amplitudes $A(r_b,m,n)$, obtained by setting $r = r_b$, $\theta = 0$, and $\zeta = 0$ in $A(\mathbf{x})$ in Eq. (24). Each Bessel function $J_l(z)$ with l_g as its index l has argument $z = z_g \equiv k_\perp\rho_g$, while each J_l having bounce harmonic l_b in its index has argument $z = z_b \equiv [(k_r r_1)^2 + (m\theta_1 + n\zeta_1)^2]^{1/2}$. (In Ref. 6, z_b is called y_1.) In Eq. (33) $A_\perp$ is that component of $\mathbf{A}$ normal to $\hat{b}$, while $A = |A|$, and $A_{\theta,\zeta} \equiv \mathbf{e}_{\theta,\zeta}\cdot\mathbf{A}$; θ_{gA} is the gyrophase at which $\mathbf{v}_\perp$ is parallel to $\mathbf{A}_\perp$, i.e., $\mathbf{v}_\perp\cdot\mathbf{A}_\perp = v_\perp A_\perp\cos(\theta_g - \theta_{gA})$, and θ_{gk} is defined by $\mathbf{k}\cdot\boldsymbol{\rho}_g = z_g\sin(\theta_g - \theta_{gk})$, the gyrophase where $\mathbf{v}_\perp$ is parallel to $\mathbf{k}_\perp$. Analogously, we define the phase θ_{bA} and velocity u_0 by $\dot{\boldsymbol{\rho}}_b\cdot\mathbf{A} = u_0 A\cos(\theta_b - \theta_{bA})$, and θ_{bk} by $\mathbf{k}\cdot\boldsymbol{\rho}_b = z_b\sin(\theta_b - \theta_{bk})$. The velocity u_1 is defined as $u_1 A = \tau\Omega_b A_\theta + \Omega_\zeta A_\zeta$. The term in $u_{0,(1)}$ dominates for particles with $\tau = 0\ (1)$.

We first note the overall factor $\delta(l_\zeta - n)$ multiplying the factor in square brackets in Eq. (33), a consequence of axisymmetry, used in Ref. 2 in the demonstration of p_ζ conservation. The factor in square brackets is a sum of four terms, each coming from a different portion of the inner product $v^\mu A_\mu$ giving h; the first is the electrostatic contribution, from the $\mu = 4$ component, the second is the contribution $\mathbf{v}_\perp\cdot\mathbf{A}_\perp$ resulting from the gyromotion, the third is from that part of the guiding-center contribution $\dot{\mathbf{R}}\cdot\mathbf{A}$ evolving secularly in time, and the fourth is from the part of $\dot{\mathbf{R}}\cdot\mathbf{A}$ which is oscillatory in time. The structure of each term is the same; each has an amplitude with units of $(e/c)vA$ (hence energy), times a factor involving Bessel functions $J_l(z_g)$ coming from the θ_g integration, times a factor involving $J_l(z_b)$ arising from the θ_b integration. The gyro- and bounce-related Bessel functions are the strength of that portion of mode a that is oscillatory at $\exp i(l_g\theta_g + l_b\theta_b)$, just as the overall factor $\delta(l_\zeta - n)$ gives that portion (namely, all or none) of the mode that is oscillatory at $\exp il_\zeta\zeta_0$.

In contrast to the all-or-none dependence on l_ζ of the contribution of a given mode, one notes that a mode contributes over a range of l_g and l_b. We recall the asymptotic forms for the J_l,

$$J_l(z) \simeq (z/2)^{|l|}/|l|! \quad (|l| > z), \tag{34}$$

$$J_l(z) \simeq (2/\pi z)^{1/2}\cos(z - l\pi/2 - \pi/4) \quad (|l| < z). \tag{35}$$

We consider J_l as a function of l. We see that for $|l| < z$, J_l has an l-independent amplitude $(2/\pi z)^{1/2}$, times a factor oscillatory in l. For $|l|>z$, J_l falls off rapidly to zero. Thus the gyro-related Bessel functions contribute over a range $\Delta l_g \sim 2z_g$ about $l_g=0$, while the bounce-related ones contribute over a range $\Delta l_b \sim 2z_b$ about $l_b=\tau m$.

Because it is helpful in physically interpreting these expressions, and also because it will be useful in making analytic progress in what follows, we pursue this examination of the asymptotic forms of the J_l a bit further. Applying the method of stationary phase to approximately evaluate Eq. (32), we obtain Eq. (35) for $|l|<z$ and $J_l \simeq 0$ in place of Eq. (34) for $|l|>z$. The points θ_0 of stationary phase, from which the dominant contributions to the integral come, are given by

$$l = z\cos\theta_0. \tag{36}$$

From this, we may ascribe a particular position θ_{g0} and θ_{b0} on a particle's orbit from which each $h(\mathbf{l},\mathbf{J},\omega)$ arises. Thus, in contrast to the unmagnetized case, where a particle is resonant or nonresonant with a given mode for all time, for the time-varying velocities in magnetized geometries, a particle passes through a series of local regions at which the variation of a given mode is as $\exp i\mathbf{l}\cdot\mathbf{\Omega}t$, for a succession of values of $\mathbf{l}$. This is the physical significance of the range of $\mathbf{l}$ contributing to $\mathbf{D}$.

From Eq. (33), one discerns the common origin of the three types of tokamak transport. Axisymmetric (collisional) transport is caused by short-wavelength electrostatic perturbations. Thus it arises from the first term in square brackets in Eq. (33). The electrostatic portion of turbulent transport also comes from this term, while the magnetic portion, arising from modes which tend to have $A_\parallel \gg A_\perp$, comes from the third and fourth terms. Finally, as will be seen for longer-wavelength modes, which leave μ invariant, the second term yields just the μB_1 ripple perturbation, which is the origin of magnetic ripple transport. (Here B_1 is the perturbation of the magnetic field strength B.) The first term provides the electrostatic contribution to ripple transport. In the following section, we use this single coupling coefficient to derive expressions for transport coefficients for each of the three basic tokamak transport mechanisms.

IV. EVALUATION OF D

In this section, we illustrate the first aspect of the unifying character of the action-angle formalism referred to in the Introduction, evaluating the diffusion tensor $\mathbf{D}$ for each of the symmetric, turbulent, and ripple transport mechanisms, simply by specializing the same expressions for $\mathbf{D}$ and $h(\mathbf{l},\mathbf{J},\omega)$ to the context relevant to the mechanism of interest.

A. Axisymmetric collisional transport (banana regime)

We begin with transport induced by the shortest wavelengths ($\lambda<\lambda_D$), viz., axisymmetric "banana" transport. It is instructive to consider first the transport of electrons (species 1) scattering off ions (species 2) in the Lorentz limit $M_1/M_2 \to 0$. From the symmetry of Cf under interchange of species label, performing this calculation is essentially the same as treating the opposite limit $M_1/M_2 \to \infty$ (cf. Sec. V). The evaluation in the more general case $M_1 \sim M_2$ can also be performed analytically, as will be seen in Sec. V. A fuller treatment of the non-Lorentz case will be given elsewhere.[10] Because the $\delta(l_\zeta - n)$ in the $h(\mathbf{l},\mathbf{J},\omega)$ ensures that $l_{1\zeta}=l_{2\zeta}=n$, and because the $E\times B$ drift is species independent, the $E\times B$ contribution $\Omega_{\zeta E}$ to the toroidal precession frequency in the argument $\Omega_{\rm res} \equiv \mathbf{l}_1\cdot\mathbf{\Omega}_1 - \mathbf{l}_2\cdot\mathbf{\Omega}_2$ of the delta function in Eq. (12) drops out. Referring to the remaining portion of the particle frequencies $\mathbf{\Omega}$ as $\mathbf{\Omega}'$, in the Lorentz limit, we may take

$$\mathbf{l}_2\cdot\mathbf{\Omega}_2' \to 0$$

in Eq. (12) for those $\mathbf{l}_2$ having appreciable $h(\mathbf{l},\mathbf{J},\omega)$. Keeping only the first (electrostatic) term in Eq. (33), $|\alpha|^2$ may be written

$$|\alpha|^2 = |\overline{\alpha}|^2\delta(l_{1\zeta}-n)\delta(l_{2\zeta}-n)J^2_{l_{1g}}J^2_{l_{1b}-\tau_1 m}J^2_{l_{2g}}J^2_{l_{2b}-\tau_2 m},$$
$$|\overline{\alpha}|^2 = |\overline{\overline{\alpha}}|^2 s(w_a/2,r_{b1}-r_a)s(w_a/2,r_{b2}-r_a), \tag{37}$$
$$|\overline{\overline{\alpha}}|^2 = |e_1e_2/N_a\Delta_a|^2,$$

and taking $\Delta_a(\omega)$ to its short wavelength, unshielded limit $\Delta_a \to 1$,

$$N_a\Delta_a = k^2V_a.$$

In (37), $r_{b1,2}\equiv r_b(\mathbf{J}_{1,2})$. Using this in Eqs. (10)–(12), we find

$$\mathbf{D}(1|2) = \int d^6z_2 f(2)\sum_a\sum_{l_{1g},l_{1b}}\sum_{l_{2g},l_{2b}} \mathbf{l}_1\mathbf{l}_1\pi\delta(\mathbf{l}_1\cdot\mathbf{\Omega}_1')|4\pi\overline{\overline{\alpha}}|^2$$
$$\times s(w_a/2,r_{b1}-r_a)s(w_a/2,r_{b2}-r_a)$$
$$\times J^2_{l_{1g}}J^2_{l_{1b}-\tau_1 m}J^2_{l_{2g}}J^2_{l_{2b}-\tau_2 m}\big|_{l_{1\zeta}=l_{2\zeta}=n}$$
$$=\sum_a\sum_{l_{1g},l_{1b}}\mathbf{l}_1\mathbf{l}_1\pi\delta(\mathbf{l}_1\cdot\mathbf{\Omega}_1')|\overline{h}_{\rm th}(1/a,2)|^2J^2_{l_{1g}}J^2_{l_{1b}-\tau_1 m}, \tag{38}$$

the same as yielded by the quasilinear expression (7), with thermal spectrum

$$|\overline{h}_{\rm th}(1|a,2)|^2 \equiv \int d^6z_2 f(2)|4\pi\overline{\alpha}|^2$$
$$= s(w_a/2,r_{b1}-r_a)|4\pi e_1e_2/k^2V_a|^2$$
$$\times\int_{V_a} d^6z_2 f(2). \tag{39}$$

The arguments $(1|a,2)$ here mean "the spectrum felt by species 1 from that portion of fluctuations a which are driven by species 2," and $\int_{V_a} d^6z_2(\cdots)\equiv\int d^6z_2\, s(r_{b2}-r_a)(\cdots)$ is the phase space integral over the toroidal shell V_a. Hence the integration $\int_{V_a} d^6z_2 f(2) \equiv \int_{V_a} d\mathbf{r}_2\int d\mathbf{p}_2$ over phase space in Eq. (39) yields a factor $V_a n_2(r_a)$, with $n_2(r_a)$ the density of species 2, averaged over the volume V_a around r_a. In moving from the first to the second form given for $\mathbf{D}$ in Eq. (38), we note that the only dependence on $\mathbf{l}_2$ in the first form lies in the Bessel functions $J_{l_{2g}}$ and $J_{l_{2b}-\tau_2 m}$. Therefore the sums over l_{2g} and l_{2b} may be performed exactly using the important identity

$$1=\sum_{l=-\infty}^{\infty}J_l^2(z). \tag{40}$$

Performing the sum over r_a in Eq. (38), which yields a nonzero contribution from only that volume V_a in which r_{b1} lies, we obtain

$$\mathbf{D}(1|2) = V_a^{-1} \sum_{\mathbf{k}} \sum_{l_{1g},l_{1b}} \mathbf{l}_1\mathbf{l}_1 \pi\delta(\mathbf{l}_1\cdot\mathbf{\Omega}'_1) \times \left|\frac{4\pi e_1 e_2}{k^2}\right|^2 n_2(r_{b1}) J^2_{l_{1g}} J^2_{l_{1b}-\tau_1 m}. \qquad (41)$$

As usual, the delta function present in Eq. (41) is to be interpreted not as strictly singular, but as broadened about the resonant surfaces (where $\mathbf{l}_1\cdot\mathbf{\Omega}'_1 = 0$), with resonance widths large enough that, given the density of resonances from the sums over l_{1g}, l_{1b}, and mode index a (or $\mathbf{k}$) in (41), the resonances overlap. Then these sums may be converted to integrals.

We perform the integrations, first over the (l_g, l_b) plane, and then over $\mathbf{k}$. (Since all the $\mathbf{l}_2$ dependence has been eliminated, for brevity we drop the subscript on $\mathbf{l}_1$ here.) It is convenient to reparametrize the (l_g, l_b) plane with $\omega_g \equiv \Omega_g l_g$, $\omega_b \equiv \Omega_b l_b$. The integration (summation) to be performed is illustrated in Fig. 1. As seen here, over the $(\omega_g, \dot\omega_b)$ plane, the sum (or integral) in (41) is nonzero along a line with slope -1, given by the resonance condition

$$0 = \mathbf{l}\cdot\mathbf{\Omega}'_1 = \omega_g + \omega_b + n\Omega'_\zeta, \qquad (42)$$

and is appreciable only where the J_l are appreciable; hence, in a rectangle centered about $(\omega_g, \omega_b) = (0, \tau m \Omega_b)$, of width $\Delta\omega_g$ and height $\Delta\omega_b$, defined by

$$\Delta\omega_g/\pi \equiv z_g \Omega_g \equiv k_\perp v_\perp, \quad \Delta\omega_b/\pi \equiv z_b \Omega_b \simeq k_\parallel \delta v_\parallel + k_\perp v_B. \qquad (43)$$

Here, $v_B \simeq (\rho_g/2R) v_\perp$ is the amplitude of the "grad-B" drift, normal to $\hat b$, and $\delta v_\parallel$ is the amplitude of the oscillations in $v_\parallel$ over a bounce period. Physically, $\Delta\omega_{g,(b)}$ represents the range of frequencies a particle moving through perturbation a encounters due to its gyro (bounce) motion over the course of its orbit. (The reason for the factor of π in the definition will become apparent shortly.) For modes having $k_\perp \sim k_\parallel$, typical of this section, the last term in Eq. (43), proportional to v_B, is negligible. However, when $k_\perp \gg k_\parallel$, as is typical of the turbulent spectrum to be considered in Sec. IV B, this term is important.

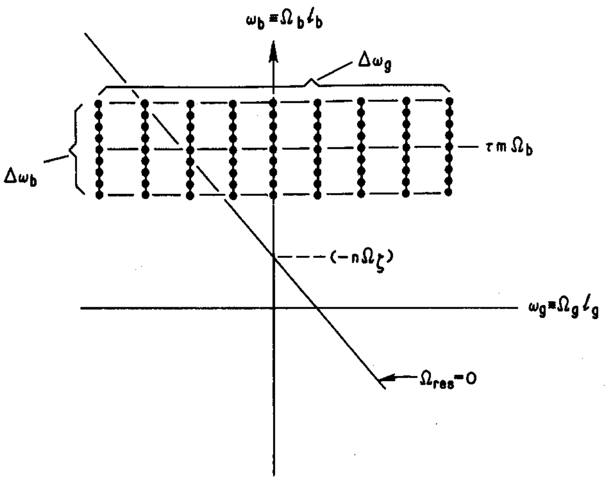

FIG. 1. Illustration of the summation/integration to be performed over the (l_g, l_b) or (ω_g, ω_b) plane, in the evaluation of the gBL operator. Indicated is the resonance line $\Omega_{res} = 0$, along which the integration contributes, and the rectangular box, of width $\Delta\omega_g$ and height $\Delta\omega_b$, within which the integrand is appreciable. The dots within this rectangle indicate the position of resonance with individual Fourier components $\mathbf{l}_1$ of the perturbing Hamiltonian of the scattering test particle.

For the short wavelengths of relevance to this section, we have $z_{b,g} \gg 1$, so many bounce and gyro harmonics are contained in this rectangle. We therefore approximate both the bounce- and gyro-related factors J_l^2 by the pairwise average of the asymptotic forms for J_l^2 discussed following Eq. (35), which eliminates the oscillatory character from the sums:

$$\bar J_l^2(z) \equiv \tfrac{1}{2}[J_l^2(z) + J_{l+1}^2(z)] = s(\pi z/2, l)/(\pi z). \qquad (44)$$

This gives the correct small- and large-l limits previously discussed, and the coefficient $\pi/2$ of z in (44), determining the precise transition point from the small- to large-l regimes, is fixed by requiring that (44) satisfy the averaged counterpart of identity (40).

We now use the approximation (44) in evaluating (41). With $\Sigma_{\mathbf{k}} \to V_a(2\pi)^{-3}\int d\mathbf{k}$, we have, for any component $D^{ij} \equiv \boldsymbol{\epsilon}^i\cdot\mathbf{D}\cdot\boldsymbol{\epsilon}^j$ of $\mathbf{D}$,

$$\begin{aligned} D^{ij}(1|2) &= \int \frac{d\mathbf{k}}{(2\pi)^3} \left|\frac{4\pi e_1 e_2}{k^2}\right|^2 n_2(r_{b1}) \times \int_{-\infty}^{\infty} d\omega_b \int_{-\infty}^{\infty} d\omega_g l_i l_j \pi\delta(\mathbf{l}\cdot\mathbf{\Omega}'_1) \times \frac{s(\Delta\omega_b/2, \omega'_b)}{\Delta\omega_b} \frac{s(\Delta\omega_g/2, \omega_g)}{\Delta\omega_g} \\ &= \int \frac{d\mathbf{k}}{(2\pi)^3} \pi \left|\frac{4\pi e_1 e_2}{k^2}\right|^2 n_2(r_{b1}) \int_{-\infty}^{\infty} d\omega_b l_i l_j \times \frac{s(\Delta\omega_b/2, \omega'_b)}{\Delta\omega_b} \frac{s(\Delta\omega_g/2, \omega_{g0})}{\Delta\omega_g}, \end{aligned} \qquad (45)$$

where $\omega'_b \equiv \omega_b - \tau m \Omega_b$, and $\omega_{g0}(l_b, n) \equiv -(\omega_b + n\Omega'_\zeta)$. For both $\tau = 0$ and 1, $\Delta\omega_b/\Delta\omega_g \lesssim \epsilon^{1/2} k_\parallel/k_\perp$, i.e., the contributing rectangle in the (ω_g, ω_b) plane is short and broad, for most $\mathbf{k}$. Thus we may approximately replace l_b in Eq. (45) by its value $\bar l_b \equiv \tau m$ at the vertical midpoint of the rectangle. This replacement in ω_{g0} makes the second (gyro-related) s factor in (45) independent of ω_b, rendering the remaining ω_b integration trivial, $\int_{-\infty}^{\infty} d\omega_b, s(\Delta\omega_b/2, \omega'_b)/\Delta\omega_b = 1$. [This corresponds to using the identity (40).] This leaves

$$D^{ij}(1|2) = \int \frac{d\mathbf{k}}{(2\pi)^3} \pi \bar l_i \bar l_j \left|\frac{4\pi e_1 e_2}{k^2}\right|^2 \times n_2(r_{b1}) \frac{s[\Delta\omega_g/2, \omega_{g0}(\bar l_b, n)]}{\Delta\omega_g}, \qquad (46)$$

where

$$\bar{\mathbf{l}} \equiv (\bar l_g, \bar l_b, \bar l_\zeta) \equiv [-(\bar l_b \Omega_b / + \bar l_\zeta \Omega'_\zeta)/\Omega_g, \tau m, n] \qquad (47)$$

is the averaged value of $\mathbf{l}$, having now performed the sum $\Sigma_{\mathbf{l}}$. The value of $\bar l_g$ is a consequence of (42). As a result of this, we note that there are really only three independent components of $\mathbf{D}$, namely, D^{bb},$D^{b\zeta}$, and $D^{\zeta\zeta}$, the rest being given in terms of these by

$$\begin{aligned} D^{gg} &= (\Omega_b^2 D^{bb} + 2\Omega_b \Omega'_\zeta D^{b\zeta} + \Omega'^2_\zeta D^{\zeta\zeta})/\Omega_g^2, \\ D^{gb} = D^{bg} &= -(\Omega_b D^{bb} + \Omega'_\zeta D^{b\zeta})/\Omega_g, \\ D^{g\zeta} = D^{\zeta g} &= -(\Omega_b D^{b\zeta} + \Omega'_\zeta D^{\zeta\zeta})/\Omega_g. \end{aligned} \tag{48}$$

From Eqs. (48) it follows that

$$\mathbf{\Omega}'_1 \cdot \mathbf{D} = \mathbf{D} \cdot \mathbf{\Omega}'_1 = 0. \tag{49}$$

Thus since $\epsilon^{E'} \equiv \partial_J H'_0 \equiv \mathbf{\Omega}'$ (where $H'_0 \equiv H_0 - p_\zeta \Omega_{\zeta E}$ is the particle energy in the frame precessing at $\Omega_{\zeta E}$), the diffusive portion alone of C conserves energy in this precessing frame, in the Lorentz approximation. (We shall see that the contribution $\mathbf{F}$ to C also conserves E' separately in the Lorentz approximation. For the non-Lorentz case only the contributions from $\mathbf{D}$ and $\mathbf{F}$ together, summed over species, conserve energy.[10])

We note from relations (48) the symmetric way in which diffusion in velocity space and real space enter the theory in the action-angle framework. This is in some contrast with more standard treatments,[4,11] where radial diffusion appears as a secondary *consequence* of diffusion in $\mathbf{v}$.

We now complete the evaluation of D^{bb}, $D^{\zeta\zeta}$, and $D^{b\zeta}$ by performing the integration over $\mathbf{k}$. Using

$$n \equiv Rk_t = R(k_\parallel b_t - k_q b_p), \quad m \equiv rk_p = r(k_\parallel b_p + k_q b_t),$$

the remaining integrals are easily evaluated. For $\tau = 0$, $\omega_{g0} = -n\Omega_\zeta \ll k_\perp v_\perp$, so that the gyro-related s function $s(\pi k_\perp v_\perp/2, \omega_{g0}) \simeq s(\pi k_\perp v_\perp/2, 0) = 1$, for essentially all $\mathbf{k}$. For $\tau = 1$, $\omega_{g0} = -(m\Omega_b + n\Omega_\zeta) = -k_\parallel \bar{v}_\parallel$ ($\bar{v}_\parallel$ is the bounce-averaged parallel velocity), so that $s(\pi k_\perp v_\perp/2, -k_\parallel \bar{v}_\parallel) = s(\pi v_\perp/2\bar{v}_\parallel, k_\parallel/k_\perp)$ in Eq. (46) selects contributions only from those $\mathbf{k}$ having $|\alpha_k| < \alpha_v$, where

$$\tan \alpha_k \equiv k_\parallel/k_\perp, \quad \tan \alpha_v \equiv \pi v_\perp/2\bar{v}_\parallel. \tag{50}$$

As a passing particle approaches the trapped/passing boundary, $\bar{v}_\parallel$ goes to 0, and the domain prescribed by the s function for $\tau = 1$ goes smoothly to that for $\tau = 0$, making the D^{ij} continuous across the $\tau = 0 \to 1$ transition.

Physically, for the local resonance discussed around Eq. (36), we need $0 = \mathbf{k}\cdot\mathbf{v} = k_\parallel v_\parallel + k_\perp v_\perp \cos\theta_g$, or

$$\tan \alpha_k = -v_\perp \cos\theta_g / v_\parallel. \tag{51}$$

Because $v_\parallel$ goes through 0 for trapped particles, resonance occurs for all values of α_k in the interval $[-\pi/2, \pi/2]$ for some θ_g. For passing particles, however, (51) implies $|\tan \alpha_k|$ is always smaller than some maximum $|v_\perp / v_\parallel^{\min}|$, hence such particles cannot resonate with modes having $|k_\parallel/k_\perp|$ too large.

Using the required $\mathbf{k}$ integrals

$$\int d\mathbf{k}\, \frac{s(\alpha_v, \alpha_k)}{k_\perp k^4} \begin{bmatrix} k_\parallel^2 \\ k_\perp^2 \\ k_\parallel k_\perp \end{bmatrix} = \pi^2 \ln \Lambda \begin{bmatrix} I_-(\alpha_v) \\ I_+(\alpha_v) \\ 0 \end{bmatrix}, \tag{52}$$

where $I_\pm(\alpha_v) \equiv (2/\pi)(\alpha_v \pm \frac{1}{2}\sin 2\alpha_v)$, we find

$$\begin{bmatrix} D^{bb} \\ D^{b\zeta} \\ D^{\zeta\zeta} \end{bmatrix} = \nu_{12}(v) \left(\frac{v}{v_\perp}\right)^3 (M_1 v_\perp)^2 \begin{bmatrix} r^2(b_p^2 I_- + \frac{1}{2} b_t^2 I_+) \\ rRb_p b_t (I_- - \frac{1}{2} I_+) \\ R^2(b_t^2 I_- + \frac{1}{2} b_p^2 I_+) \end{bmatrix}, \tag{53}$$

with

$$\nu_{12}(v) \equiv 2\pi e_1^2 e_2^2 \ln \Lambda n_2(r_{b1})/M_1^2 v^3$$

the collision frequency for species 1 on species 2. For trapped particles, one has $\bar{v}_\parallel = 0$, hence $\alpha_v = \pi/2$, so that $I_\pm = 1$. For deeply passing particles, $\alpha_v \to 0$. In this limit, we have $I_+ \to (4/\pi)\alpha_v$, $I_- \to (4/3\pi)\alpha_v^3$.

From Eqs. (53) and (21), we compute the radial diffusion coefficient $D^{rr} \equiv \epsilon^r \cdot \mathbf{D} \cdot \epsilon^r$:

$$D^{rr} = \begin{cases} \nu_{12}\rho_g^2 [(q/\epsilon)^2 + \frac{1}{2}] & (\tau = 0), \\ \nu_{12}\rho_g^2 (v/v_\perp)^3 [\frac{1}{2} I_+ + (\epsilon/q)^2 I_-] & (\tau = 1). \end{cases} \tag{54}$$

The first term in the $\tau = 0$ expression here is the dominant, neoclassical term, yielding banana diffusion $D_{bn} \simeq \nu_{12}\rho_g^2 q^2/\epsilon^{3/2}$ when averaged over pitch angle. The second term for $\tau = 0$, smoothly joining the first term for $\tau = 1$ at the $\tau = 0 \to 1$ transition, represents classical diffusion, $D_{\rm cl} \simeq \nu_{12}\rho_g^2$. The second term for $\tau = 1$ is negligible, down from $D_{\rm cl}$ by $(\epsilon/q)^2$.

Diffusion in velocity space is also described by Eqs. (53). For example, for $\tau = 1$, using Eqs. (23) and (53), we have

$$\begin{aligned} D^{\bar{v}_\parallel \bar{v}_\parallel} &\simeq \frac{1}{(qMR)^2} D^{J_v J_v} = \frac{1}{(qMR)^2} (D^{bb} + 2qD^{b\zeta} + q^2 D^{\zeta\zeta}) \\ &= \nu_{12}(v) v_\perp^2 \left(\frac{v}{v_\perp}\right)^3 I_-. \end{aligned} \tag{55}$$

Similarly, using Eqs. (48) and (53) along with $\partial J_g/\partial v_\perp = Mv_\perp/\Omega_g$ and the fact that $\Omega_\zeta \simeq q\Omega_b$ for $\tau = 1$, we find

$$\begin{aligned} D^{v_\perp v_\perp} &= \left(\frac{\Omega_g}{Mv_\perp}\right)^2 D^{gg} = \left(\frac{\Omega_b}{Mv_\perp}\right)^2 D^{J_v J_v} \\ &= \nu_{12}(v) \bar{v}_\parallel^2 \left(\frac{v}{v_\perp}\right)^3 I_-. \end{aligned} \tag{56}$$

The factor $(v/v_\perp)^3 I_-$ appearing in Eqs. (55) and (56) is a weak function of pitch angle, varying from 1 at $\alpha_v = \pi/2$ to $\pi^2/6$ at $\alpha_v = 0$. For both $D^{\bar{v}_\parallel \bar{v}_\parallel}$ and $D^{v_\perp v_\perp}$, one notes that only I_-, and not I_+, appears. This is because $\mathbf{D}$ is a particular $\mathbf{k}$ average $\langle \cdots \rangle_{\mathbf{k}}$ of $\overline{\overline{\mathbf{I}}}$, and so

$$D^{J_v J_v} \propto \langle \epsilon^{J_v} \cdot \overline{\overline{\mathbf{I}}} \epsilon^{J_v} \rangle_{\mathbf{k}} = \langle (m + nq)^2 \rangle_{\mathbf{k}} = (qR)^2 \langle k_\parallel^2 \rangle_{\mathbf{k}} \propto I_-,$$

noting the definition of I_- in (52). For the turbulent spectrum, treated in the next section, we have $\langle k_\parallel^2 \rangle_{\mathbf{k}} \simeq 0$. Thus, as concluded from earlier quasilinear calculations,[7,12] turbulence causes very little velocity-space diffusion ($\propto I_-$), even while quite effectively inducing radial transport.

These results may be related to the more familiar Lorentz collision operator, as shown in Appendix A.

B. Turbulent transport

We now move to somewhat longer wavelengths ($\lambda \sim \rho_i \gg \lambda_{\rm D}$) characterizing plasma microturbulence, and perform an analogous calculation of **D**, using the same expression (33) for the coupling coefficients used in the preceding subsection, but specialized to this different regime.

Specifically, we consider the transport induced by magnetic microturbulence at vanishing collisionality. For thermal electrons, whose gyroradius and drifts across field lines may be neglected ($z_{g,b} \to 0$), this mechanism was studied by Rechester and Rosenbluth,[13] who found an anomalous diffusion coefficient for passing electrons

$$D_{RR}^{rr} \simeq (\pi/|\Omega_b|)(u\tilde{b})^2, \qquad (57)$$

where $\tilde{b} \equiv \tilde{B}/B$ is the average amplitude of the perturbing radial field B_r, normalized to B. (Here, $u \equiv \overline{|v_\parallel|}$ is the bounce average of the absolute value of the parallel velocity, nonzero for trapped as well as passing particles.) The modification of this mechanism by the large z_g and z_b of energetic electrons and ions was estimated in Ref. 6. Here, we do a more complete evaluation of **D** than was done there, for comparison with the collisional and ripple mechanisms considered in this section, and in preparation for the inclusion of the contribution from **F** in Sec. V.

Since, using the present framework, the derivation for the turbulent **D** is quite similar to that just given for symmetric collisional transport, the details of the present derivation are relegated to Appendix B. The coupling coefficients, and so $|\alpha|^2$, may be written in forms [cf. Eqs. (B5) and (B6)] analogous to those for the symmetric case. Performing the sum over $\mathbf{l}_2$ as done in Eq. (38), we find the closely analogous expression

$$\mathbf{D}(1|2) = \sum_a \sum_{l_{1g} l_{1b}} \mathbf{l}_1 \mathbf{l}_1 \pi\delta(\mathbf{l}_1 \cdot \boldsymbol{\Omega}_1') |\bar{h}_{\rm th}(1|a,2)|^2 J_{l_{1g}}^2 \overline{J_{l_{1b}-\tau_1 2m}}, \qquad (58)$$

again the same as yielded by the quasilinear expression (7), with thermal spectrum

$$|\bar{h}_{\rm th}(1|a,2)|^2 \equiv \int d^6 z_2 f(2) |4\pi\bar{\alpha}|^2$$
$$= |\bar{h}(1|a)^2 \int d^6 z_2 f(2) \left| \frac{4\pi\bar{h}(2|a)}{N_a \Delta_a} \right|^2. \qquad (59)$$

As discussed in the Introduction, the thermal fluctuations given by Eq. (59) do not properly represent the turbulent spectrum of realistic experiments. Thus we replace $|\bar{h}_{\rm th}|^2$ in Eq. (58) by a model spectrum, $|\bar{h}(1|a,2)|^2$, satisfying the general characteristics of the turbulent spectrum described in Appendix B. We assume that Δ_a in expression (59) is nonlinearly modified from its thermal value, so that the fields $A(a,2)$ driven by species 2, given by the z_2 integration in (59), are given by

$$|A(a,2)|^2 \equiv |\bar{A}(a,2)|^2 s(w_a/2, r_{b1} - r_a),$$

$$|\bar{A}(a,2)|^2 = \frac{V_A (2\pi)^{3/2} \tilde{B}^2}{V_a (\Delta k_\perp)^2} \times \exp\left(-\frac{k_\perp^2}{2(\Delta k_\perp)^2} - \frac{k_\parallel^2}{2(\Delta k_\parallel)^2} \right). \qquad (60)$$

Here, $\Delta k_\perp \sim \rho_i^{-1}$ and $\Delta k_\parallel \sim L_s^{-1}$ are the spectrum widths in the perpendicular and parallel directions, $V_A^{-1} \equiv (\Delta k_\perp)^2 \Delta k_\parallel$ measures the volume in **k** space over which $A^2(\mathbf{k})$ is appreciable, and $\tilde{B}^2$ measures the overall strength of the turbulent fluctuations. The normalization is chosen so that

$$\langle B_r^2 \rangle \equiv V_p^{-1} \int d\mathbf{x} |k_q A(\mathbf{x})|^2 = \tilde{B}^2.$$

Thus, $|\bar{h}(1|a,2)|^2$ has the same form as $|\bar{h}(1|a)|^2$ in Eq. (B5), but with $|A|^2$ there replaced by $|A(a,2)|^2$ in Eq. (60). [More detailed use will be made of the structure of the spectrum in Eq. (59) in Sec. V, where we consider the relative contributions to the spectrum from different species.]

Because of the longer wavelengths of the turbulent spectrum, and because in addition we have $k_\parallel \ll k_\perp$, in Appendix B it is shown that the 3×3 matrix **D** has only a single independent nonzero component to be evaluated. It is convenient to choose $D^{\zeta\zeta}$. Performing the evaluation, as detailed in Appendix B, we find

$$D^{\zeta\zeta}(1|2) = \left(\frac{\partial p_\zeta}{\partial r_{b1}} \right)^2 D_{RR}^{rr}(1|2) \langle \overline{J_{l_{1g}}^2 J_{l_{1b}-\tau_1 m}^2} \rangle_{\mathbf{k}}, \qquad (61)$$

with $\partial p_\zeta / \partial r_b \equiv -(M\Omega_g R b_p) = -(eBRb_p/c)$. Here $D_{RR}^{rr}(1|2)$ is as given in (57), with $u = u(1)$, $\tilde{b}$ due to species 2 [via Eq. (60)], and

$$\langle \overline{J_{l_{1g}}^2} \; \overline{J_{l_{1b}-\tau_1 m}^2} \rangle_{\mathbf{k}} \equiv \frac{1}{\pi \Delta k_\perp \rho_g} \frac{1}{\pi \Delta k_\perp r_1} \begin{bmatrix} \pi^2/32 \\ 1/2 \end{bmatrix}, \qquad (62)$$

where the upper (lower) component applies for $\tau = 0$ (1).

Parallel to Eqs. (54)–(56) of the preceding section, from Eqs. (21) and (23), we have

$$D^{rr} = D_{RR}^{rr} \langle \overline{J_{l_{1g}}^2} \; \overline{J_{l_{1b}-\tau_1 m}^2} \rangle_{\mathbf{k}} \qquad (63)$$

for $\tau = 0$ and 1, and

$$D^{uu} \propto D^{v_\parallel v_\parallel} \propto D^{J_g J_g} \propto \langle k_\parallel^2 \rangle_{\mathbf{k}} \simeq 0. \qquad (64)$$

The reduction of the Rechester–Rosenbluth result due to finite z_g and z_b, discussed in Ref. 6, is contained in the factor $\langle \cdots \rangle_{\mathbf{k}}$ in Eqs. (61) and (63). As discussed in more general terms around Eq. (36), this factor measures the (square of) the fraction of each gyro and bounce time that a particle spends locally resonant with a given perturbing mode a. In the $z_{g,b} \to 0$ limit, where this fraction becomes unity (or zero), the J_l^2 there become Kronecker δ functions $\delta(l)$, and the Rechester–Rosenbluth result is recovered.

C. Ripple transport (stochastic regime)

We complete the series of illustrations for this section by computing from the same expressions for $h(\mathbf{l},\mathbf{j},\omega)$ and **D** a ripple transport result. Specifically, we generalize the expression for the "stochastic regime"[14] to include an electrostatic component, and to ripple perturbations having $m \neq 0, \omega \neq 0$, and to allow $qN \equiv qn + m$ of order unity, as well as the limit $qN \gg 1$ previously assumed. These generalizations make the theory applicable to internally generated modes, including low-n MHD modes, as well as to the TF-coil ripple with which the theory was originally concerned. This is the same generalization for the stochastic regime that Ref. 8 achieved for the more collisional "banana drift" regimes, and is thus relevant for very energetic ions, such as alpha particles.

The modes we consider have a still longer wavelength than those of the previous subsections, long enough so that δl_g discussed around Eq. (B8) is small compared with unity (even for alphas). Then only terms $\mathbf{l}$ with $l_g = 0$ contribute to $\mathbf{D}$, so that Eq. (B11), which held for almost all particles in turbulence, holds for all particles here. Thus J_g is constant, and a guiding-center description of the particle motion is valid.

It has been noted that the second term in Eq. (33) yields the μB_1 perturbation in the guiding-center Hamiltonian H_G. Taking the small-z_g limit (34) of the J_{l_g}'s there, and setting $l_g = 0$, this second term reduces to (momentarily restoring mode index $\mathbf{k}$ for clarity)

$$\begin{aligned} h(\mathbf{l},\mathbf{J},\omega|a) &= -\tfrac{1}{2}(e/c)v_\perp \rho_g i k_\perp A_{\perp\mathbf{k}} \sin(\theta_{gA} - \theta_{gk}) \\ &\quad \times \delta(l_\zeta - n) J_{l_b - \tau m} \times e^{-i(l_b - \tau m)\theta_{bk} - i l_g \theta_g k} \quad (65)\\ &= (\mu B_{1\mathbf{k}})\delta(l_\zeta - n) J_{l_b - \tau m} e^{-i(l_b - \tau m)\theta_{bk} - i l_g \theta_g k}, \end{aligned}$$

where $B_{1\mathbf{k}} \equiv \hat{b}\cdot i\mathbf{k}\times\mathbf{A}_\mathbf{k} \sin(\theta_{gA} - \theta_{gk})$ is the perturbation to $B \equiv |\mathbf{B}|$ due to $A_\mathbf{k}$. A more fundamental derivation of this form, which does not rely on the eikonal representation (24), comes from recognizing that the $l_g = 0$ Fourier component of $h(\mathbf{l},\mathbf{J},\omega)$ simply involves a line integration around a gyro orbit:

$$\begin{aligned} \oint \frac{d\theta_g}{2\pi} h(\boldsymbol{\theta},\mathbf{J}) &= -\frac{e\Omega_g}{2\pi c} \oint d\mathbf{l}_\perp \cdot \mathbf{A}(\mathbf{R} + \boldsymbol{\rho}_g) \\ &= \frac{e\Omega_g}{2\pi c} \int d\mathbf{S}_\parallel \cdot (\nabla \times \mathbf{A}) \\ &= \frac{e\Omega_g}{2\pi c} (\pi \rho_g^2) B_1 = \mu B_1, \quad (66) \end{aligned}$$

where $d\mathbf{l}_\perp \equiv \mathbf{v}_\perp\, dt$ is an incremental line element, and $d\mathbf{S}_\parallel \equiv \hat{b}\, dS_\parallel$ is an element of area on the disk formed by the gyro orbit. Expression (65) has the same form as the first term in (33), to which it may be added to yield the full coupling coefficient for the ripple problem,

$$h(\mathbf{l},\mathbf{J},\omega) = \bar{h}_G \delta(l_\zeta - n) J_{l_b - \tau m} e^{-i l_b - \tau m \theta_{bk} - i l_g \theta_g k}, \quad (67)$$

where $\bar{h}_G \equiv e\phi(r_b,m,n) + \mu B_1(r_b,m,n)$ is the amplitude of the perturbing portion of H_G, from which ripple transport calculations normally begin.

As usual in ripple calculations, we consider the effects of a single mode a (or $\mathbf{k}$). This perturbation may be internally or externally induced. For either, the quasilinear expression (7) applies. Thus, using Eq. (67) in (7), we have (suppressing species subscript "1" for simplicity of notation)

$$\mathbf{D}(1) = \sum_{l_b} \mathbf{l}\mathbf{l}\delta(\mathbf{l}\cdot\boldsymbol{\Omega}_1 - \omega)\bar{h}_G^2 J^2_{l_b - \tau m}|_{l_\zeta = n}. \quad (68)$$

This parallels Eqs. (38) or (58) of the previous sections, except that it lacks the sums over both l_g and mode index a, and so is substantially simpler to evaluate. Diffusion now occurs only because of overlap of the resonances of successive bounce harmonics l_b, equivalent to the overlap criterion given in Ref. 16. When overlap exists, the sum over l_b may be converted to an integral, as in the previous subsections, yielding

$$D^{ij}(1) = (\pi/|\Omega_b|) l_i l_j \bar{h}_G^2\, \overline{J^2_{l_b - \tau m}}(z_b)|_{\mathbf{l}=\bar{\mathbf{l}}}, \quad (69)$$

where now

$\bar{\mathbf{l}} = [0,(\omega - n\Omega_\zeta)/\Omega_b, n]$.

Thus Eqs. (B11) again hold, as already noted. Specializing to $\tau = 0$ for comparison with the previous theory, from Eqs. (69) and (21) we find

$$D^{rr} = (\pi/4|\Omega_b|)\hat{v}^2\, \overline{J^2_{l_b}}(z_b)|_{\bar{l}_b}. \quad (70)$$

Here, $\hat{v} \equiv 2qn\bar{h}_G/(M\Omega_g r)$ is the amplitude of the radial drift $\dot{r}_b$ due to the ripple, given by

$$\dot{r}_b = \mathbf{e}^r\cdot\dot{\mathbf{J}} = -\hat{v}\sum_{l_b} J_{l_b} \sin(n\zeta_0 + l_b\theta_b - \omega t), \quad (71)$$

following from Eq. (5). The physics of the result (70) may be described as follows [almost the same description may be applied to the turbulent result (63)]: a particle resonant with bounce harmonic l_b performs a random walk, taking a radial step at velocity $\hat{v}J_{l_b}$, for coherence time τ_b. The factor J_{l_b} represents the fraction of the full bounce period when the radial motion is nonoscillatory. Contrary to the usual lore, one notes that the point on the bounce orbit during which this radial step is taken is *not* at the particle's turning points, in general. From Eq. (36), the step is taken at the turning point only for $\bar{l}_b/z_b \simeq 0$, a condition holding only for particles precessing at almost the same frequency as the ripple perturbation (i.e., $\omega \simeq n\Omega_\zeta$). This condition does not hold, for example, for a typical alpha particle precessing in ripple due to TF coils, for which $\bar{l}_b/z_b$ can be comparable to unity. We also note the dependence of $\bar{l}_b$ on mode frequency ω. For $(\omega - n\Omega_\zeta)$ large enough, the factor $\overline{J^2_{l_b}}$ can be made to move into the $\bar{l}_b > z_b$ limit of the Bessel function, where $\overline{J^2_{l_b}}$, and so D^{rr}, fall off rapidly.

For the longer-wavelength modes of this section, one has $z_b \simeq m\theta_1 + n\zeta_1 \simeq qN\theta_1$. The result of Ref. 16 is recovered by letting $\bar{h}_G = \mu B_1$, $\omega = 0 = m$, and assuming $z_b \simeq qn \gg 1$, so that the large-z form in (44) may be used for $\overline{J^2_{l_b}}$. For $z_b \sim 1$, the point on a bounce orbit where the particle receives a radial kick is no longer localized, and correspondingly, the stationary-phase form (44) fails. Then the full form of the Bessel function must be retained.

V. EVALUATION OF Γ FOR THE TURBULENT CASE

Having gained experience through the evaluation of $\mathbf{D}$ with much of the mathematical mechanics, we can proceed to consider the effect of reinstating self-consistency, by computing the full flux $\boldsymbol{\Gamma}$ in $\mathbf{J}$ space, retaining both the $\mathbf{D}$ and $\mathbf{F}$ contributions.

We will perform this evaluation for the same turbulent mechanism as examined in Sec. IV B, because this yields results which are new, and that can thus be compared with the analogous results for collisional symmetric transport, which are already well-established.

A. Preliminaries

Up to this point, nothing has been said about the specific form of the distribution functions $f(\mathbf{J})$. We now adopt the near-equilibrium form for both species (suppressing species label)

$$f_0(\mathbf{J}) \equiv [n/(2\pi MT)^{3/2}] \exp(-K_0/T), \tag{72}$$

where n, Φ, and T are functions of $r_b(\mathbf{J})$, $K_0 \equiv H_0 - e\Phi$ is the (unperturbed) kinetic energy when a particle is at $r = r_b$, and for simplicity we take equal temperature distributions, $T_1 = T_2 = T$. If r_b in (72) were replaced by r, f_0, would be of the "local Maxwellian" form f_M used as the lowest-order distribution function in more standard approaches to transport. Because it is a function of $\mathbf{J}$ alone, f_0 is an exact solution of the unperturbed ($h = 0$) Liouville equation, and thus contains, in addition to f_M, the collisionless correction (in banana width to minor radius) to f_M, which is what produces the radial fluxes[11,15] We thus substitute Eq. (72) into expression (15) for $\mathbf{\Gamma}(1|2)$ to compute the transport.

To evaluate $\mathbf{\Gamma}(1|2)$, $\mathbf{l}\cdot\partial_{\mathbf{J}} f$ is needed. Using f_0 in (72) for f, we find

$$\mathbf{l}\cdot\partial_{\mathbf{J}} f_0 = (\mathbf{l}\cdot\mathbf{e}^r A - \mathbf{l}\cdot\mathbf{\Omega}/T) f_0, \tag{73}$$

where

$$A \equiv A_{nK} + (e\Phi'/T), \quad A_{nK} \equiv A_n + A_K(K_0/T),$$
$$A_n \equiv [(n'/n) - \tfrac{3}{2}A_K], \quad A_K \equiv T'/T \tag{74}$$

describes the thermal forces. (The prime denotes derivatives with respect to r_b.) Using (B6) and (73) in Eq. (15), we find

$$-\mathbf{\Gamma}(1|2) = \int d^6 z_2 \sum_a \sum_{\mathbf{l}_1,\mathbf{l}_2} \pi\delta(\Omega_{\text{res}}) |4\pi\alpha|^2 \mathbf{l}_1 (\mathbf{l}_1\cdot\partial_{\mathbf{J}_1} - \mathbf{l}_2\cdot\partial_{\mathbf{J}_2}) f_0(1) f_0(2) = \int d^6 z_2 \sum_a \sum_{l_{1g},l_{1b}} \sum_{l_{2g},l_{2b}}$$
$$\times \pi\delta(\Omega_{\text{res}}) |4\pi\bar{\alpha}|^2 J^2_{l_{1g}} \overline{J^2_{l_{1b}-\tau_1 m}} J^2_{l_{2g}} \overline{J^2_{l_{2b}-\tau_2 m}}$$
$$\times \mathbf{l}_1 (\mathbf{l}_1\cdot\boldsymbol{\epsilon}^{r_1} A_1 - \mathbf{l}_2\cdot\boldsymbol{\epsilon}^{r_2} A_2) f_0(1) f_0(2), \tag{75}$$

where the contributions from $\mathbf{D}(\sim \mathbf{l}_1\mathbf{l}_1)$ and $\mathbf{F}(\sim \mathbf{l}_1\mathbf{l}_2)$ are still apparent. The contributions to these terms from the term in $\mathbf{l}\cdot\mathbf{\Omega}$ in (73) have canceled, due to the argument $\mathbf{l}_1\cdot\mathbf{\Omega}_1 - \mathbf{l}_2\cdot\mathbf{\Omega}_2$ of the δ function. (An analogous cancellation occurs in linearizing the standard BL operator.)

We note in Eq. (75) the same z_2 integral over $|4\pi\bar{\alpha}|^2$ that produced the scattering spectrum $\bar{h}(1|a,2)|^2$ in the calculation in Sec. IV B and Appendix B of $\mathbf{D}(1|2)$, and the same identification can be made here in computing $\mathbf{\Gamma}(1|2)$. Here, however, we wish to consider the symmetries that the retention of $\mathbf{F}$ creates between $\mathbf{\Gamma}(1|2)$ and $\mathbf{\Gamma}(2|1)$, and this requires that we make further use of the spectral structure given in Eq. (59) to strip away the z_2 integration present in the model spectrum (60). Dropping the subscript "th" on $\bar{h}_{\text{th}}$ in Eq. (59), an expression for $|\bar{\alpha}|^2$ yielding the spectrum (60) is

$$|4\pi\bar{\alpha}(1,2|a)|^2 = \left| \frac{4\pi e_1 e_2}{N_a \Delta_a} \frac{u_\tau(1)}{c} \frac{u_\tau(2)}{c} \right|^2, \tag{76}$$

with Δ_a given by

$$\left| \frac{4\pi e_1 e_2}{N_a \Delta_a} \right|^2 \equiv \frac{|e_1 \bar{A}(a,2)|^2}{[V_a n_2 \langle |u_\tau(2)/c|^2 \rangle]}. \tag{77}$$

Here, $V_a n_2 \langle \cdots \rangle \equiv \int_{V_a} d^6 z_2 f(2)(\cdots)$ defines the phase space average over the toroidal shell V_a. Thus the desired model spectrum $|\bar{A}|^2$ is achieved in Eq. (77) by modifying the dielectric function Δ_a from its form in a stable plasma, which would produce a thermal spectrum.

B. Lorentz case

As in Sec. IV, we simplify the summation over $\mathbf{l}_2$ by taking the Lorentz limit $\mathbf{l}_2\cdot\mathbf{\Omega}_2 \to n\Omega_{\zeta E}$, and then approximately perform the summation over $\mathbf{l}_1$, again using (44). Equation (75) then yields

$$-\mathbf{\Gamma}(1|2) = \int d^6 z_2 \sum_a \bar{Q}(1,2|a) \bar{\mathbf{l}}_1 (\bar{\mathbf{l}}_1\cdot\boldsymbol{\epsilon}^{r_1} A_1 - \bar{\mathbf{l}}_2\cdot\boldsymbol{\epsilon}^{r_2} A_2) f_0(1) f_0(2), \tag{78}$$

with the $\mathbf{l}$-averaged kernel $\bar{Q}$ given by

$$\bar{Q}(1,2|a) = \bar{\bar{Q}}(1,2|a) s(.\,, r_{b1} - r_a) s(.\,, r_{b2} - r_a),$$

$$\bar{\bar{Q}}(1,2|a) \equiv \pi\Omega_{g1} \frac{s(\Delta\omega_{g1}/2, \omega_g)}{\Delta\omega_{g1}} \times \frac{s(\Delta\omega_{b1}/2, \omega'_{b0})}{\Delta\omega_{b1}} |4\pi\bar{\alpha}|^2|_{\mathbf{l}=\bar{\mathbf{l}}} \tag{79}$$
$$= \pi\Omega_{g1} \frac{s(\Delta\omega_{g1}/2, \omega_g)}{\Delta\omega_{g1}} \frac{s(\Delta\omega_{b1}/2, \omega'_{b0})}{\Delta\omega_{b1}} \times \frac{|u_\tau(1)|^2 |e_1/c \bar{A}(a,2)|^2 |u_\tau(2)|^2}{V_a n_2 \langle |u_\tau(2)|^2 \rangle}\bigg|_{\bar{\mathbf{l}}}.$$

Here, as in Sec. IV B, we have assumed that only $l_{1g} = 0$ contributes, so that $\bar{\mathbf{l}}_{1,2}$ are again given by $\bar{\mathbf{l}} = (0, \tau m, n)$. With Eq. (79), one readily sees that the term in $\bar{\mathbf{l}}_1\bar{\mathbf{l}}_1$ in Eq. (78) yields expression (B9), as it should.

From Eq. (21), we have

$$e\boldsymbol{\epsilon}^r\cdot\bar{\mathbf{l}} = (-c/BRb_p) n, \tag{80}$$

for any species, and for $\tau = 0, 1$. Therefore the terms in $e\Phi'$ in the thermal force factor in Eq. (78) cancel, resulting in property (iii) of turbulent (or symmetric) transport noted in Sec. I:

$$(\bar{\mathbf{l}}_1\cdot\boldsymbol{\epsilon}^{r_1} A_1 - \bar{\mathbf{l}}_2\cdot\boldsymbol{\epsilon}^{r_2} A_2) = n\left(\frac{-c}{BRb_p}\right)\left(\frac{1}{e_1} A_1 - \frac{1}{e_2} A_2\right)$$
$$= n\left(\frac{-c}{BRb_p}\right)\left(\frac{1}{e_1} A_{nK1} - \frac{1}{e_2} A_{nK2}\right). \tag{81}$$

The $\mathbf{k}$ integration over $\bar{Q}$ in (78) yields the same $D^{\zeta\zeta}(1|2)$ as in Eq. (B9) or (61):

$$\sum_{\mathbf{k}} \bar{Q}(1,2|a) n^2 = \frac{\pi}{|\Omega_{b1}|} \sum_{\mathbf{k}} |\bar{h}(1|,\mathbf{k},2)|^2 \times \overline{J^2_{l_{1g}}}\ \overline{J^2_{l_{1b}-\tau_1 m}}\, n^2 \frac{|u_\tau(2)|^2}{V_a n_2 \langle |u_\tau(2)|^2 \rangle}$$
$$= D^{\zeta\zeta}(1|2) \frac{|u_\tau(2)|^2}{V_a n_2 \langle |u_\tau(2)|^2 \rangle}. \tag{82}$$

Using this in Eq. (78), we find

$$-\boldsymbol{\Gamma}(1|2)=\left(\frac{-c}{BRb_p}\right)D^{\zeta\zeta}(1|2)\left(\frac{1}{e_1}A_1-\frac{1}{e_2}A_2\right)\begin{bmatrix}0\\-q\tau_1\\1\end{bmatrix}\times f_0(1)|_{r_{b1}\simeq r_{b2}}. \tag{83}$$

Dotting this with $\boldsymbol{\epsilon}^{r_1}$, the radial flux is given by

$$-\Gamma^r(1|2)\equiv-\boldsymbol{\epsilon}^{r_1}\cdot\boldsymbol{\Gamma}(1|2)=e_1D^{rr}(1|2)\times[(1/e_1)A_1-(1/e_2)A_2]f_0(1)|_{r_{b1}\simeq r_{b2}}. \tag{84}$$

We now consider $\boldsymbol{\Gamma}(2|1)$. This is given by interchanging species indices 1 and 2 in Eq. (75). We evaluate the resulting expression in the same limit $\mathbf{l}_2\cdot\boldsymbol{\Omega}_2\rightarrow n\Omega_{\zeta E}$ used with $\boldsymbol{\Gamma}(1|2)$. This yields

$$-\boldsymbol{\Gamma}(2|1)=\int d^6z_1\sum_a\bar{Q}(1,2|a)\times\bar{\mathbf{l}}_2(\bar{\mathbf{l}}_2\cdot\boldsymbol{\epsilon}^{r_a}A_2-\bar{\mathbf{l}}_1\cdot\boldsymbol{\epsilon}^{r_a}A_1)f_0(1)f_0(2), \tag{85}$$

with $\bar{Q}$ the same as given in Eq. (79). Except for the integration over z_1 instead of z_2, the evaluation of this is essentially the same as for $\boldsymbol{\Gamma}(1|2)$ in Eq. (78). With (82), we have

$$-\boldsymbol{\Gamma}(2|1)=\int_{V_a}d^6z_1f_0(1)\left(\frac{-c}{BRb_p}\right)\times D^{\zeta\zeta}(1|2)\frac{|u_\tau(2)|^2}{V_an_2\langle|u_\tau(2)|^2\rangle}\times\left(\frac{1}{e_2}A_2-\frac{1}{e_1}A_1\right)\begin{bmatrix}0\\-q\tau_2\\1\end{bmatrix}f_0(2)|_{r_{b2}\simeq r_{b1}}, \tag{86}$$

and thus

$$\begin{aligned}-\Gamma^r(2|1)&\equiv-\boldsymbol{\epsilon}^{r_2}\cdot\boldsymbol{\Gamma}(2|1)\\&=e_2D^{rr}(2|1)[(1/e_2)A_2\\&\quad-(1/e_1)A_1]f_0(2)|_{r_{b2}\simeq r_{b1}},\end{aligned} \tag{87}$$

where

$$e_2^2D^{rr}(2|1)\equiv\int_{V_a}d^6z_1f_0(1)e_1^2D^{rr}(1|2)\frac{|u_\tau(2)|^2}{V_an_2\langle|u_\tau(2)|^2\rangle}. \tag{88}$$

Using Eqs. (84) and (87), we easily verify property (ii) cited in Sec. I:

$$\int_{V_a}d^6z_1\,e_1\Gamma^r(1|2)+\int_{V_a}d^6z_2\,e_2\Gamma^r(2|1)=0. \tag{89}$$

Taking like-species interactions ($1=2$, i.e., $s_1=s_2$), either of Eqs. (84) or (87) confirms property (i) of Sec. I, viz.,

$$\int_{V_a}d^6z_1\,\Gamma^r(1|2)|_{1=2}=0. \tag{90}$$

C. Non-Lorentz case

Since the results (89) and (90) were derived in the Lorentz approximation, neither is strictly valid for like-particle interactions. To prove property (90) correctly, therefore, we must return to expression (75), and reevaluate it in the non-Lorentz case, where $M_1\sim M_2$. We shall see that the essential feature of Eqs. (84) or (87) needed for Eq. (90) to hold, namely, the presence of the factor $(A_1/e_1-A_2/e_2)$, still holds for the case $M_1\sim M_2$.

Motivated by the form of Eqs. (89) and (90), therefore, we define the particle and energy fluxes, averaged over a toroidal shell V_a:

$$\begin{aligned}I_n(1|2)&\equiv\int_{V_a}d^6z_1\,\Gamma^r(1|2),\\I_K(1|2)&\equiv\int_{V_a}d^6z_1\frac{K_{01}}{T}\Gamma^r(1|2),\end{aligned} \tag{91}$$

where $K_{01}\equiv K_0(1)$. Using Eqs. (B6) and (75) in the definition of I_n, and again using the averaging effect of the z integrations on the factors $J^2_{l_g}$, we have

$$\begin{aligned}-I_n(1|2)=&\int_{V_a}d^6z_1\int_{V_a}d^6z_2\sum_{\mathbf{k}}\sum_{l_{1g},l_{1b}}\sum_{l_{2g},l_{2b}}\pi\delta(\Omega_{\rm res})\\&\times\overline{J^2_{l_{1g}}}\;\overline{J^2_{l_{1b}-\tau_1m}}\;\overline{J^2_{l_{2g}}J^2_{l_{2b}-\tau_2m}}\\&\times|4\pi\bar{\alpha}|^2\boldsymbol{\epsilon}^{r_1}\cdot\mathbf{l}_1(\mathbf{l}_1\cdot\boldsymbol{\epsilon}^{r_1}A_1-\mathbf{l}_2\cdot\boldsymbol{\epsilon}^{r_2}A_2)\\&\times f_0(1)f_0(2)|_{r_{b1}=r_{b2}=r_a}.\end{aligned} \tag{92}$$

The new feature of the non-Lorentz case is that the term $\mathbf{l}_2\cdot\boldsymbol{\Omega}_2'$ in the argument $\Omega_{\rm res}$ of the δ function cannot be neglected, thereby coupling the sums over $\mathbf{l}_1$ and $\mathbf{l}_2$. The full resonance condition is now

$$0=\Omega_{\rm res}=\omega_{g1}-\omega_{g2}+\omega_{b1}-\omega_{b2}+n(\Omega_{\zeta1}-\Omega_{\zeta2}), \tag{93}$$

and the relevant ordering is

$$\Delta\omega_{g1}\sim\Delta\omega_{g2}\gg\Delta\omega_{b1}\sim\Delta\omega_{b2}. \tag{94}$$

The new aspect of the problem is thus the evaluation of the $\mathbf{l}$ sums in the presence of the full $\Omega_{\rm res}$:

$$\begin{aligned}&\sum_{l_{1g},l_{1b}}\sum_{l_{2g},l_{2b}}\pi\delta(\Omega_{\rm res})\,\overline{J_l^2}\;\overline{J_l^2}\;\overline{J_l^2}\;\overline{J_l^2}\cdots\\&\simeq\sum_{l_{1g},l_{1g}}\overline{J^2_{l_{g1}}}\;\overline{J^2_{l_{g2}}}\\&\quad\times\int d\omega_{b2}\int d\omega_{b1}\,\pi\delta(\Omega_{\rm res})\frac{s(,\omega_{b2}')}{\Delta\omega_{b2}}\frac{s(,\omega_{b1}')}{\Delta\omega_{b1}}\cdots\\&=\pi\sum_{l_{1g},l_{2g}}\overline{J^2_{l_{g1}}}\;\overline{J^2_{l_{g2}}}\,I_b(\Delta\omega_{b1},\Delta\omega_{b2},\omega_{b0}')\cdots,\end{aligned} \tag{95}$$

where $\omega_{b0}'\equiv\omega_{g2}-\omega_{g1}+m(\tau_2\Omega_{b2}-\tau_1\Omega_{b1})+n(\Omega_{\zeta2}-\Omega_{\zeta1})$, and where the overlap integral I_b, given by

$$I_b(\Delta\omega_1,\Delta\omega_2,x)\equiv\int_{-\infty}^{\infty}d\omega_2\frac{s(\Delta w_2/2,\omega_2)}{\Delta\omega_2}\frac{s(\Delta\omega_1/2,\omega_2-x)}{\Delta\omega_1} \tag{96}$$

is even about $x=0$, and symmetric in $\Delta\omega_1$, $\Delta\omega_2$. With the definition $\Delta\omega_>\equiv\max(\Delta\omega_1,\Delta\omega_2)$, we have

$$I_b(\;)=\begin{cases}0, & |x|>|\Delta\omega_1+\Delta\omega_2|/2,\\1/\Delta\omega_>, & |x|<|\Delta\omega_1-\Delta\omega_2|/2,\end{cases} \tag{97}$$

and I_b varies linearly with x in the intervening interval $|\Delta\omega_1 - \Delta\omega_2|/2 < x < |\Delta\omega_1 + \Delta\omega_2|/2$.

Owing to relations (94), the "short and broad" approximation used for the Lorentz case is still well satisfied, and thus, again, we have

$$l_{b\,1,2} \simeq \bar{l}_{b\,1,2} \equiv \tau_{1,2} m. \tag{98}$$

As indicated by the spectrum in Eq. (60), the modes considered here have nearly zero frequency in the frame precessing at $\Omega_{\zeta E}$. Thus, for $M_1 \sim M_2$, the dielectric function $\Delta_a(\omega = \mathbf{l}_2\cdot\mathbf{\Omega}_2)$ appearing in (92) again permits contributions only from terms with

$$l_{g1,2} = \bar{l}_g = 0, \tag{99}$$

causing the remaining sums over $l_{g\,1}$ and $l_{g\,2}$ in (95) to drop out. Using Eqs. (95), (96), (98), and (99) in (92), therefore, we find

$$\begin{aligned} -I_n(1|2) &= \int_{V_a} d^6z_1 \int_{V_a} d^6z_2 \sum_{\mathbf{k}} \bar{\bar{Q}}(1,2|a) \\ &\quad \times \boldsymbol{\epsilon}^{r_1}\cdot\bar{\mathbf{l}}_1(\bar{\mathbf{l}}_1\cdot\boldsymbol{\epsilon}^{r_1}A_1 - \bar{\mathbf{l}}_2\cdot\boldsymbol{\epsilon}^{r_2}A_2) f_0(1) f_0(2) \\ &= \int_{V_a} d^6z_1 \int_{V_a} d^6z_2 \sum_{\mathbf{k}} \bar{\bar{Q}}(1,2|a) n^2 \left(\frac{-c}{BRb_p}\right)^2 \\ &\quad \times \frac{1}{e_1}\left(\frac{1}{e_1}A_1 - \frac{1}{e_2}A_2\right) f_0(1) f_0(2), \end{aligned} \tag{100}$$

with kernel $\bar{\bar{Q}}$ given by

$$\bar{\bar{Q}}(1,2|a) \equiv \pi\, \overline{J^2_{l_{g1}}}\ \overline{J^2_{l_{g2}}} I_b(\Delta\omega_{b\,1}, \Delta\omega_{b\,2}, \omega'_{b0}) |4\pi\bar{\alpha}|^2|_{\mathbf{l}=\bar{\mathbf{l}}}, \tag{101}$$

extending the Lorentz expression (79). The expression for $I_K(1|2)$ is the same as given in Eq. (100) for $I_n(1|2)$, but with an additional factor of K_{01}/T in the integrand. The factor $(A_1/e_1 - A_2/e_2)$ in expression (100), present also in the Lorentz results, implies Eq. (90), as already noted. In the Lorentz limit, the factor I_b in (101) recovers the factor $\overline{J^2_{l_{b1} - \tau_1 m}}/\Omega_{b\,1} = s(,\omega'_{b0})/\Delta\omega_{b\,1}$ in Eq. (79). For the case $M_1 \sim M_2, I_b(,,\omega'_{b0})$ is a function of comparable size $(\sim \Delta\omega_{b\,1}^{-1})$ over a comparable range $(\sim \Delta\omega_{b\,1})$ of ω'_{b0} as in the Lorentz case, but with a somewhat modified functional form. The factor $\overline{J^2_{l_{g2}}}$ in (101) is not present in (79), since there all gyroharmonics $l_{g\,2}$ contributed, while only $l_{g\,2} = 0$ contributes in (101).

The $\mathbf{k}$ summation in (100) may be performed using Eq. (82), yielding

$$\begin{aligned} -I_n(1|2) &= \int_{V_a} d^6z_1 \int_{V_a} d^6z_2 D^{\zeta\zeta}(1|2) \\ &\quad \times \frac{|u_\tau(2)|^2}{V_a n_2 \langle |u_\tau(2)|^2 \rangle} \left(\frac{-c}{BRb_p}\right)^2 \\ &\quad \times \frac{1}{e_1}\left(\frac{1}{e_1}A_1 - \frac{1}{e_2}A_2\right) f_0(1) f_0(2), \end{aligned} \tag{102}$$

where the non-Lorentz kernel $\bar{\bar{Q}}$ in Eq. (101) now yields a slightly modified diffusion coefficient $D^{\zeta\zeta}(1|2)$ from that in (82) or (61):

$$\begin{aligned} D^{\zeta\zeta}(1|2) &= \pi \sum_{\mathbf{k}} |\bar{h}(1|\mathbf{k},2)|^2\, \overline{J^2_{l_{g1}}}\ \overline{J^2_{l_{g2}}} I_b n^2 \\ &= \left(\frac{\partial p_\zeta}{\partial r_{b\,1}}\right)^2 D^{\pi}_{RR}(1|2) \langle \overline{J^2_{l_{1g}}}\ \overline{J^2_{l_{2g}}} I_b \Omega_{b\,1} \rangle_{\mathbf{k}}, \end{aligned} \tag{103}$$

with

$$\langle \overline{J^2_{l_{1g}}}\ \overline{J^2_{l_{2g}}} I_b \Omega_{b\,1} \rangle_{\mathbf{k}} \equiv \frac{1}{\pi\Delta k_\perp \rho_{g1}} \frac{1}{\pi\Delta k_\perp \rho_{g2}} \frac{1}{\pi\Delta k_\perp r_1} \begin{bmatrix} \pi^2/32 \\ 1/2 \end{bmatrix} \sqrt{\frac{\pi}{2}}, \tag{104}$$

and where, as in (62), the upper (lower) component holds for $\tau = 0$ (1).

D. Onsager relations

We now turn to the Onsager symmetries of the transport coefficients. In the following, it is convenient to use indices p or q, which may take on values n and K in designating the components of the fluxes I_p and forces A_p. We define the coefficients of the thermal force terms, which can be read off from Eq. (100), and from its counterpart for $I_K(1|2)$:

$$\begin{aligned} L^{12}_{pq} &\equiv \frac{1}{e_1^2}\left(\frac{-c}{BRb_p}\right)^2 \int_{V_a} d^6z \int_{V_a} d^6z'\, f_{0s_1}(z) \left(\frac{K_0(z)}{T}\right)^{x_p + x_q} \\ &\quad \times f_{0s_2}(z') \sum_{\mathbf{k}} \bar{\bar{Q}}(z,z'|a) n^2, \end{aligned} \tag{105}$$

$$\begin{aligned} M^{12}_{pq} &\equiv \frac{1}{e_1 e_2}\left(\frac{-c}{BRb_p}\right)^2 \int_{V_a} d^6z \int_{V_a} d^6z'\, f_{0s_1}(z) \left(\frac{K_0(z)}{T}\right)^{x_p} \\ &\quad \times f_{0s_2}(z') \left(\frac{K_0(z')}{T}\right)^{x_q} \sum_{\mathbf{k}} \bar{\bar{Q}}(z,z'|a) n^2. \end{aligned} \tag{106}$$

The L^{12}_{pq} are the coefficients arising from $\mathbf{D}$, and the M^{12}_{pq} are those arising from $\mathbf{F}$. We have used z and z' for the variables of integration here instead of z_1 and z_2 used previously to emphasize that the species labels 1 and 2 in the superscripts of L^{12}_{pq} and M^{12}_{pq} correspond to the species labels s_1, s_2 on the distribution functions, and not to the integration variables. This avoids ambiguity when $s_1 = s_2$. The exponent x_p of the energy-weighting equals 0 (1) for $p = n$ (K), and similarly for x_q. In terms of the L's and M's, one can write $I_n(1|2)$ and $I_K(1|2)$ succinctly as

$$-I_p(1|2) = \sum_{q=n,K} (L^{12}_{pq} A_{q1} - M^{12}_{pq} A_{q2}) \quad (p = n,K). \tag{107}$$

From inspection of Eqs. (105) and (106), we note the following symmetries:

$$L^{12}_{pq} = L^{12}_{qp}, \quad M^{12}_{pq} = M^{21}_{qp}, \quad e_1 L^{12}_{pn} = e_2 M^{12}_{pn}, \tag{108}$$

but

$$e_1 L^{12}_{pK} \neq e_2 M^{12}_{pK}.$$

These relations may be used to efficiently prove Eqs. (89) and (90), as well as the Onsager symmetries, now under consideration.

Following essentially the same steps as used to obtain I_n

in Eq. (100), we may evaluate expression (16) for $\dot{S}$. The result is

$$\dot{S} = \frac{1}{2}\sum_{r_a}\sum_{1,2}\int_{V_a} d^6 z_1 \int_{V_a} d^6 z_2 f_0(2)\left(\frac{-c}{BRb_p}\right)^2 \times\left(\frac{1}{e_1}A_1 - \frac{1}{e_2}A_2\right)^2 \times \sum_{\mathbf{k}} \overline{Q}(1,2|\alpha)n^2$$
$$= -\frac{1}{2}\sum_{r_a}\sum_{1,2}[A_{n1}I_n(1|2) + A_{n2}I_n(2|1) + A_{K1}I_K(1|2) + A_{K2}I_K(2|1)]$$
$$= -\sum_{r_a}\sum_{1,2}\sum_{p} A_{p1}I_p(1|2). \quad (109)$$

The first form given parallels expression (100). The four terms in the second form come from identifying the coefficients of each of the four terms in the first thermal-force factor $A_1/e_1 - A_2/e_2$ on the first line, and the final version presents the second form somewhat more compactly. The total particle and energy fluxes for species 1 are given by $I_p(1) \equiv \Sigma_2 I_p(1|2)$. Assuming the thermal forces A_q are independent of species, the total particle and energy fluxes (summed over species 1) are given by

$$-I_p \equiv -\sum_{1,2} I_p(1|2) = \sum_q L_{pq}A_q, \quad (110)$$

where from Eq. (107), $L_{pq} \equiv \Sigma_{1,2}(L_{pq}^{12} - M_{pq}^{12})$. Using (110) in (109), $\dot{S}$ may be written in the symmetric form[15]

$$\dot{S} = \sum_{r_a}\sum_{p,q} A_p L_{pq} A_q. \quad (111)$$

The Onsager symmetries are expressed in the relations

$$L_{pq} = L_{qp}. \quad (112)$$

Since we are considering only a 2×2 matrix L_{pq} here, the only nontrivial member of relation (112) is $L_{nK} = L_{Kn}$. This is easily demonstrated using the definition of L_{pq} and the first two terms of the symmetries (108).

VI. DISCUSSION

In previous sections, we have evaluated both the diffusive and frictional portions of the radial fluxes, for a particular turbulent transport problem of interest. The evaluation has been carried out for each of the three cases when the mass of the scattered species is much less than [Eq. (84)], much greater than [Eq. (87)], and equal to [Eq. (102)] that of the scattering species. For each of these, both portions of the flux are simply given in terms of the corresponding diffusion coefficient $D^{rr}(1|2)$. Here, we assess the physical implications of these expressions, by considering the relative sizes and scalings of $D^{rr}(1|2)$ for all four possible species–species interactions.

We use $\langle 1\rangle_{\mathbf{k}}$ as shorthand notation for the orbit-averaging factor $\langle \overline{J^2_{l_{1g}}}\ \overline{J^2_{l_{1b}-\tau_1 m}}\rangle_{\mathbf{k}}$ appearing in Eqs. (61)–(63). For simplicity, ignoring the factor $J^2_{l_{2g}}$ entering the comparable-mass expression $D^{\zeta\zeta}(1|2)$ in Eq. (103), as it is of secondary importance, the remaining angle-bracketed term is approximately equal to $\langle 1\rangle_{\mathbf{k}}$ as well. Then the expressions for $D^{rr}(1|2)$ for all cases may be approximately written

$$D^{rr}(1|2) \simeq (\pi/|\Omega_{b>}|)\langle > \rangle_{\mathbf{k}} u^2(1)\tilde{b}^2(2). \quad (113)$$

Consistent with the definition of $\Delta\omega_>$ in Eq. (97), the symbol "$>$" here refers to the species with larger thermal speed v_T (and so smaller mass). Expression (113) transparently reduces to Eq. (63) for the case $M_1 \ll M_2$ to which (84) pertains, and also approximately yields the equal-mass case (103), as already noted. For the case $M_2 \gg M_1$ described by Eq. (87), approximately evaluating expression (88), one finds $D^{rr}(2|1)$ given by Eq. (113), with $\tilde{b}^2(1)$, representing the perturbing fields driven by species 1, given by

$$[\tilde{b}(1)/\tilde{b}(2)]^2 \simeq (n_1 e_1^2 v_{T1}^2)/(n_2 e_2^2 v_{T2}^2). \quad (114)$$

Expression (114) arises from our adoption of Eqs. (76) and (77) to describe the coupling $\overline{\overline{\alpha}}(1,2)$ between any two species 1 and 2. Applying expression (113), one finds

$$D^{rr}(1|1){:}D^{rr}(1|2){:}D^{rr}(2|1){:}D^{rr}(2|2)$$
$$::u^2(1)\tilde{b}^2(1)\frac{\langle 1\rangle_{\mathbf{k}}}{\Omega_{b1}}{:}u^2(1)\tilde{b}^2(2)\frac{\langle 1\rangle_{\mathbf{k}}}{\Omega_{b1}}$$
$${:}u^2(2)\tilde{b}^2(1)\frac{\langle 1\rangle_{\mathbf{k}}}{\Omega_{b1}}{:}u^2(2)\tilde{b}^2(2)\frac{\langle 2\rangle_{\mathbf{k}}}{\Omega_{b2}}$$
$$::n_1e_1^2v_{T1}^4{:}n_2e_2^2v_{T1}^2v_{T2}^2{:}n_1e_1^2v_{T1}^2v_{T2}^2$$
$${:}n_2e_2^2v_{T2}^4\frac{\langle 2\rangle_{\mathbf{k}}\Omega_{b1}}{\langle 1\rangle_{\mathbf{k}}\Omega_{b2}}, \quad (115)$$

and thus, for $M_1 \ll M_2$,

$$n_1e_1^2D^{rr}(1|1) \gg n_1e_1^2D^{rr}(1|2) \simeq n_2e_2^2D^{rr}(2|1) \gg n_2e_2^2D^{rr}(2|2). \quad (116)$$

The statement $n_1e_1^2D^{rr}(1|2) \simeq n_2e_2^2D^{rr}(2|1)$ here is the approximate counterpart of statement (89) of intrinsic ambipolarity. Because $D^{rr}(1|1) \gg D^{rr}(1|2)$, the energy flux of species 1, which is dominated by $D^{rr}(1|1)$, will be much more rapid than the particle flux, to which only $D^{rr}(1|2)$, and not $D^{rr}(1|1)$, contributes. Both the particle and energy fluxes of species 2 caused by this mechanism will be dominantly governed by $D^{rr}(2|1)$. We note that the situation is quite analogous to the relations holding for collisional (neoclassical) transport, except that for that mechanism, the roles of the heavier and lighter species are interchanged. This is because collisional transport is an electrostatic mechanism, and thus lacks the velocity weighting in both the factors u^2 and $\tilde{b}^2$ in Eq. (113), which, in the present case, the magnetic mechanism enhances the transport and fluctuation spectrum of the higher-velocity species.

This completes the demonstration of the unity that exists between the different tokamak transport mechanisms, facilitated by use of the gBL operator. Extension of the properties demonstrated above for the case of magnetic turbulence (and already well established for symmetric transport) to other cases of interest (for example, electrostatic turbulence, or internally generated ripple) should be straightforward. For the sample turbulent mechanism chosen, we have obtained an explicit expression for the anomalous pinch term, and showed that, because of its close relation to the contribution from the diffusive term, the total flux possesses the appropriate conservation laws, which are lost by test-particle calculations. Moreover, we have seen that, just as for symmetric transport, neglect of this term can totally modify the expected particle flux for one species from the correct answer.

The theory developed here is not yet complete. Possible modifications in the transport results may result from using more realistic descriptions of the mode structure, including the appropriate combination of electrostatic versus electromagnetic components. The treatment of the non-Lorentz case given here in Sec. V is only for the particular turbulent mechanism studied, and applied principally toward treatment of the equal-mass case. The effect of the dc inductive electric field has been ignored here, resulting in only a 2×2 Onsager matrix, rather than the 3×3 matrix of a full theory. Some work in incorporating this extension into the action-angle transport framework has already been carried out in Ref. 7, for a test-particle calculation, and an extension of this to the present theory should be possible. Finally, we again note that the application of the thermal structure of the gBL operator to fully turbulent transport is somewhat *ad hoc*. Thus, for example, it is unclear that our demonstration of the Onsager relations for this theory, which depended on the use of the thermal form (76) of the particle–particle coupling, can be extended to a fully turbulent theory. However, our "pseudothermal" ansatz does yield an analytically manageable theory with the requisite symmetries, conservation laws, and features of self-consistency, and it does become fully valid in the limit of a stable plasma, where the dielectric function Δ_a reverts to its usual thermal form. Thus the theory represents an improvement upon the traditional quasilinear approach, adding to that approach several of the important properties required of a complete theory.

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APPENDIX A: CONNECTION TO THE LORENTZ OPERATOR

Here we relate the results of Sec. IV A to the Lorentz collision operator. Parametrizing velocity space by (J_g, E'), the bounce-averaged Lorentz operator may be written[8]

$$C_L f = \Omega_b\, \partial_{J_g} \frac{2\nu_{12}}{\Omega_g} J_g J_v\, \partial_{J_g} f. \tag{A1}$$

This is to be identified with the velocity-space portion of the diffusive contribution in Eq. (1), written in the noncanonical variable set $y \equiv \{y^i\}$ in which J_b in the canonical set $z \equiv (\boldsymbol{\theta}, \mathbf{J})$ has been replaced by E'. This diffusive term may be written

$$\partial_{\mathbf{J}}\cdot\mathbf{D}\cdot\partial_{\mathbf{J}} f = \mathscr{J}^{-1}\partial_{y^i}\mathscr{J} D^{y^i y^j}\partial_{y^j} f, \tag{A2}$$

where the Jacobian between the sets y and z is given by

$$\mathscr{J} \equiv \left|\frac{\partial(z)}{\partial(y)}\right| = \left|\frac{\partial J_b}{\partial E'}\right| = |\Omega_b^{-1}|.$$

Taking the restriction $C_v f$ of expression (A2) to the 2×2 velocity-space submatrix of the full $\mathbf{D}$, and using the fact, following from Eq. (49), that $D^{E'j} = D^{jE'} = 0$, one has

$$\begin{aligned} C_v f &= \Omega_b\, \partial_{J_g}\Omega_b^{-1} D^{gg}\partial_{J_g} f \\ &= \Omega_b\, \partial_{J_g}\left[\frac{2\nu_{12}}{\Omega_g} J_g qRM\bar{v}_\parallel \left(\frac{v}{v_\perp}\right)^3 I_-\right]\partial_{J_g} f, \end{aligned} \tag{A3}$$

where we have used Eq. (56) and $\bar{v}_\parallel \simeq qR\Omega_b$. We see that Eqs. (A1) and (A3) are in agreement, making the identification $J_v \simeq qRM\bar{v}_\parallel (v/v_\perp)^3 I_-$, in approximate agreement with (22).

APPENDIX B: DERIVATION OF D FOR TURBULENCE

Here we provide the details of the calculation of $\mathbf{D}$ due to a spectrum of magnetic microturbulent fluctuations, discussed in Sec. IV B. We thus consider modes having ϕ and $A_\perp$ negligible, and therefore retain only the last two terms in Eq. (33). Additionally, the spectrum is characterized by $k_\perp \sim \rho_i^{-1}$, $k_\parallel \lesssim L_s^{-1}$, with L_s the shear scale length. Thus, $k_\perp \gg k_\parallel$, as opposed to a typical contributing fluctuation for collisional transport, for which $k_\perp \sim k_\parallel$. For $\mathbf{A}_\perp = 0$, one has $\theta_{bA} = 0$ in Eq. (33). Then, again using the stationary-phase approximation for the bounce-associated J_l in Eq. (33), one finds

$$\begin{aligned} h(\mathbf{l},\mathbf{J},\omega) = {} & -(e/c)A\delta(l_\zeta - n)J_{l_g}\sqrt{(2/\pi z_b)} \\ & \times(u_1\cos\theta_z - iu_0\sin\theta_{bk}\sin\theta_z)e^{-i(l_b-\tau m)\theta_{bk}} \end{aligned} \tag{B1}$$

for $|l_b - \tau m| < z_b$, and

$$h(\mathbf{l},\mathbf{J},\omega) \simeq 0 \tag{B2}$$

for $|l_b - \tau m| > z_b$. Here, $\theta_z \equiv z_b - (l_b - \tau m)\pi/2 - \pi/4$, and u_0, defined following Eq. (33), now simplifies to $u_0 A = \Omega_b(\theta_1 A_\theta + \zeta_1 A_\zeta)$. From the definitions following Eq. (33), for $k_\perp \gg k_\parallel$, one has

$$\sin\theta_{bk} \equiv -k_r r_1/z_b \simeq -k_r/k_\perp, \tag{B3}$$

and

$$u \simeq \begin{cases} (2/\pi)u_0 & (\tau = 0), \\ |u_1| & (\tau = 1), \end{cases} \tag{B4}$$

with Eqs. (B4) valid not too near the trapped/passing boundary.

As in Sec. IV A, we may replace the contributions $\cos^2\theta_z$ or $\sin^2\theta_z$ in $|h|^2$, oscillatory in l_b, by their pairwise average. Thus, again using Eq. (44), we can write

$$|h(\mathbf{l},\mathbf{J},\omega|a)|^2 \simeq |\bar{h}(\mathbf{J}|a)|^2\delta(l_\zeta - n)J_{l_g}^2\ \overline{J_{l_b-\tau m}^2}\ , \tag{B5}$$

where $|\bar{h}(\mathbf{J}|a)|^2 = |(e/c)u_\tau A|^2$, with

$$u_\tau \equiv \begin{cases} u_0\sin\theta_{bk} & (\tau = 0), \\ u_1 & (\tau = 1). \end{cases}$$

Using (B5), we write $|\alpha|^2$ in a form paralleling the symmetric collisional expression (37):

$$\begin{aligned} |\alpha|^2 &= |\bar{\alpha}|^2\delta(l_{1\zeta} - n)\delta(l_{2\zeta} - n) \\ &\quad\times J_{l_{1g}}^2\ \overline{J_{l_{1b}-\tau_1 m}^2} J_{l_{2g}}^2\ \overline{J_{l_{2b}-\tau_2 m}^2}, \\ |\bar{\alpha}|^2 &= |\bar{\alpha}|^2 s(w_a/2, r_{b1} - r_a)s(w_a/2, r_{b2} - r_a), \\ |\bar{\alpha}|^2 &= |\bar{h}(1|a)\bar{h}^*(2|a)/N_a\Delta_a|^2, \end{aligned} \tag{B6}$$

where we have written $|\bar{h}|^2$ with its localizing factor $s(\ )$ explicitly displayed,

$$|\bar{h}|^2 = |\bar{\bar{h}}|^2 s(w_a/2, r_b - r_a).$$

These expressions are inserted into Eqs. (10)–(12) to yield Eq. (58).

Because $k_\parallel \ll k_\perp$ for the turbulent spectrum, $(m\theta_1 + n\zeta_1)$ in z_b is given by the perpendicular drift motion. Within a flux surface $(m\theta_1 + n\zeta_1) \simeq k_q q_1 \simeq k_q r_1$, and thus,

$$z_b \simeq k_\perp r_1 \simeq k_\perp v_B/\Omega_b. \tag{B7}$$

The summation over the (ω_g, ω_b) or (l_g, l_b) plane in Fig. 1 is conceptually the same as for collisional transport, but, because the wavelengths involved are longer, the characteristic frequencies are smaller. Two consequences of the fact that $k_\parallel \ll k_\perp$ are, first, that the resonance line in Fig. 1 passes through the contributing rectangle very nearly centered about $l_g = 0$, and second, the height $\Delta\omega_b$ of the contributing rectangle about $\omega_b = \tau m \Omega_b$ is so small that the resonance line crosses only a very few values of l_g. From Eq. (43), the change δl_g in l_g of the resonance line in crossing the rectangle is

$$\frac{\delta l_g}{\pi} = \frac{\Delta\omega_b}{\Omega_g} \simeq \epsilon^{1/2} k_\parallel \rho_g + \left(\frac{\rho_g}{2R}\right) z_g. \tag{B8}$$

For the turbulent spectrum, the term in $k_\parallel$ in (B8) (which dominated for a typical mode of Sec. IV A) is totally negligible. The term in z_g is negligible as well, except for extremely energetic ions, such as alpha particles. Even for these (using TFTR-like parameters $T_i \simeq 10$ keV, $R = 2.5$ m, $B = 5$ T), we have $\rho_g \simeq 5.2$ cm, $z_g \equiv k_\perp \rho_g \sim \rho_g/\rho_{gi} \simeq 26$, hence $\delta l_g \sim 0.8$. Therefore, for most particles, only the $l_g = 0$ term contributes, while for alphas the $l_g = \pm 1$ terms may or may not also contribute, depending upon the specifics of the turbulence.

In the absence of $l_g \neq 0$ contributions, we have $\dot{\mu} = \dot{J}_g = 0$, $D^{gj} = D^{ig} = 0$, and $F^g \equiv \mathbf{e}^g \cdot \mathbf{F} = 0$, i.e., no pitch-angle scattering can occur. Thus the turbulent contribution to pitch-angle scattering of these $l_g \neq 0$ terms might be of interest for very energetic ions. A similar problem has been treated by Putvinskii and Shurygin[16] who considered TF ripple as the perturbation, but the wavelength from this source is too long to produce significant effects.[17] Here, we treat the case where only the $l_g = 0$ harmonic contributes, which applies to most classes of particles and types of turbulence.

Whether a single or several terms are kept, the sum over l_g in (58) may not be converted into an integration, in contrast to the case of collisional transport. However, since the factor $J^2_{l_g}$ in Eq. (58) is rapidly oscillatory in the particle energy and pitch angle, we may replace it with the averaged form (44) as well. Making this replacement, and the replacement $|h_{\rm th}|^2 \to |h|^2$ just discussed, from (58) we write the analog of expression (45):

$$D^{ij}(1|2) = \int \frac{d\mathbf{k}}{(2\pi)^3} V_a |\bar{\bar{h}}(1|\mathbf{k},2)|^2 \times \int_{-\infty}^{\infty} d\omega_b\, \Omega_g \sum_{l_g} l_i l_j \pi \delta(\mathbf{l}\cdot\boldsymbol{\Omega}_1') \times \frac{s(\Delta\omega_b/2, \omega_b')}{\Delta\omega_b} \frac{s(\Delta\omega_g/2, \omega_g)}{\Delta\omega_g}$$

$$= \pi \int \frac{d\mathbf{k}}{(2\pi)^3} V_a |\bar{\bar{h}}(1|\mathbf{k},2)|^2 \Omega_g \times \sum_{l_g} l_i l_j \frac{s(\Delta\omega_b/2, \omega_{b0}')}{\Delta\omega_b} \frac{s(\Delta\omega_g/2, \omega_g)}{\Delta\omega_g}\bigg|_{\mathbf{l}=\bar{\mathbf{l}}}, \tag{B9}$$

where

$$\omega_{b0}'(l_g, m, n) \equiv -(\omega_g + n\Omega_\zeta' + \tau m \Omega_b) = \begin{cases} -(l_g\Omega_g + n\Omega_\zeta') & (\tau = 0), \\ -(l_g\Omega_g + k_\parallel \bar{v}_\parallel) & (\tau = 1), \end{cases} \tag{B10}$$

and $\bar{\mathbf{l}}$ is as given in Eq. (47). [Because $\Delta\omega_b \ll \Delta\omega_g$, the "short and broad" approximation used in Eq. (46) is still better satisfied here.]

Now, we impose the additional relation that $\bar{l}_g = 0$. Thus, Eqs. (48), while still true, are replaced by the simpler relations already mentioned,

$$D^{gj} = D^{ig} = 0. \tag{B11}$$

Since $k_\parallel \ll k_\perp$, we neglect $k_\parallel$ in m and n, hence $m \simeq r b_t k_q$, $n \simeq -R b_p k_q$, and thus

$$D^{bb} = -qD^{b\zeta} = -qD^{\zeta b} = q^2 D^{\zeta\zeta}, \tag{B12}$$

leaving only a single component D^{ij}, which we take as $D^{\zeta\zeta}$, to be determined.

Both ω_{b0}' and ω_g in the s functions in (B9) are approximately zero, so both of these may be replaced by unity. Keeping only the $l_g = 0$ term there, using (60) in (B9), the remaining $\mathbf{k}$ integrals needed are again easily evaluated:

$$\int \frac{d\mathbf{k}}{(2\pi)^{3/2}} \frac{\exp[-k_\perp^2/2(\Delta k_\perp)^2 - k_\parallel^2/2(\Delta k_\parallel)^2]}{(\Delta k_\perp)^2 \Delta k_\parallel} \times \frac{k_q^2}{k_\perp^2} \begin{bmatrix} \sin^2\theta_{bk} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{8} \\ \frac{1}{2} \end{bmatrix}, \tag{B13}$$

where we have used Eq. (B3). The upper (lower) component here is needed for $\tau = 0$ (1). Using Eqs. (B13) in (B9), we obtain Eq. (61).

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