Waves in Plasmas

[Part I]

Plasma - Wave interaction fundamentals

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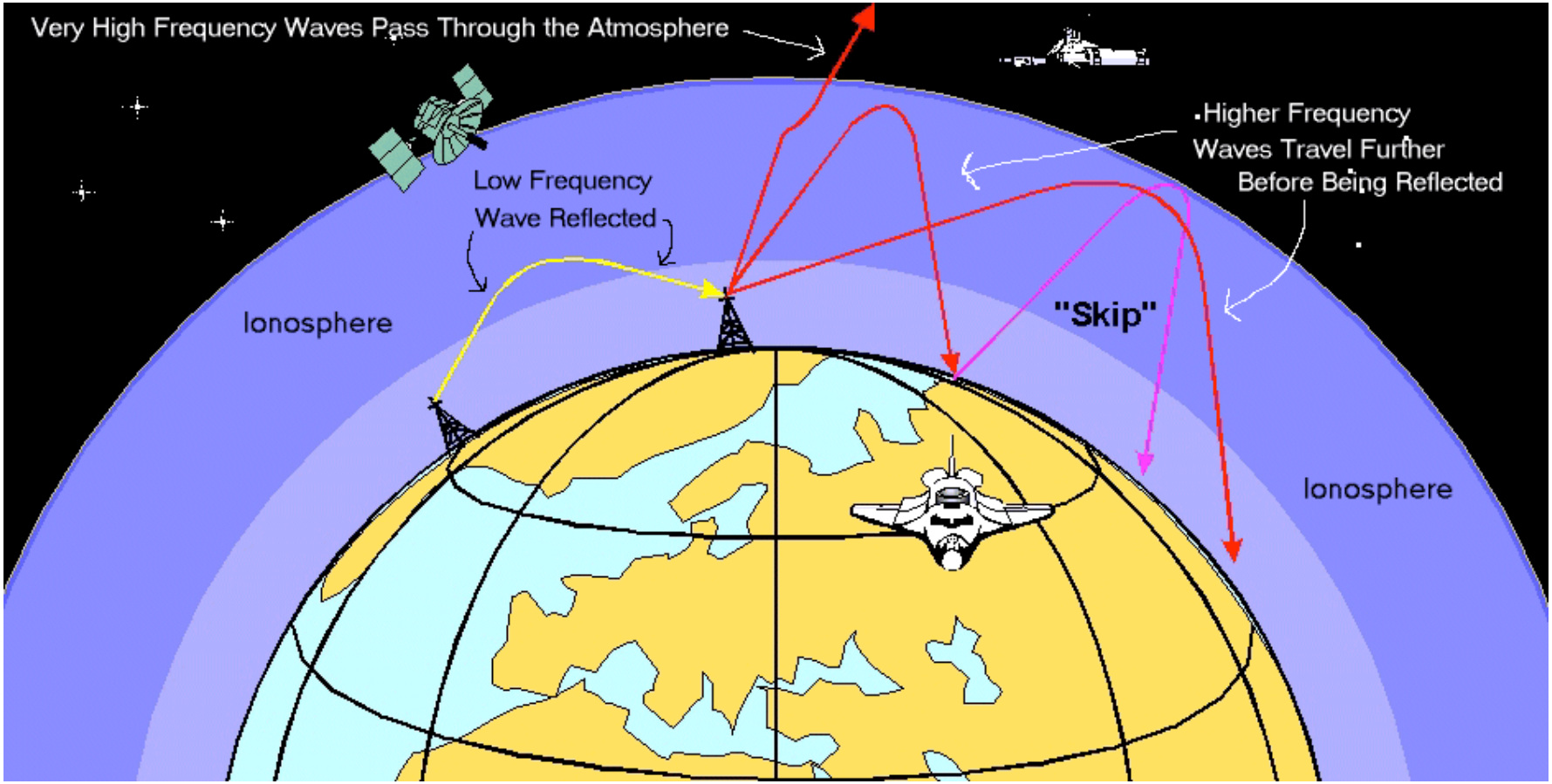
Spring school: “Hot Plasmas” - Cadarache - 03-04/2004

Introduction

Ionosphere / Atmosphere

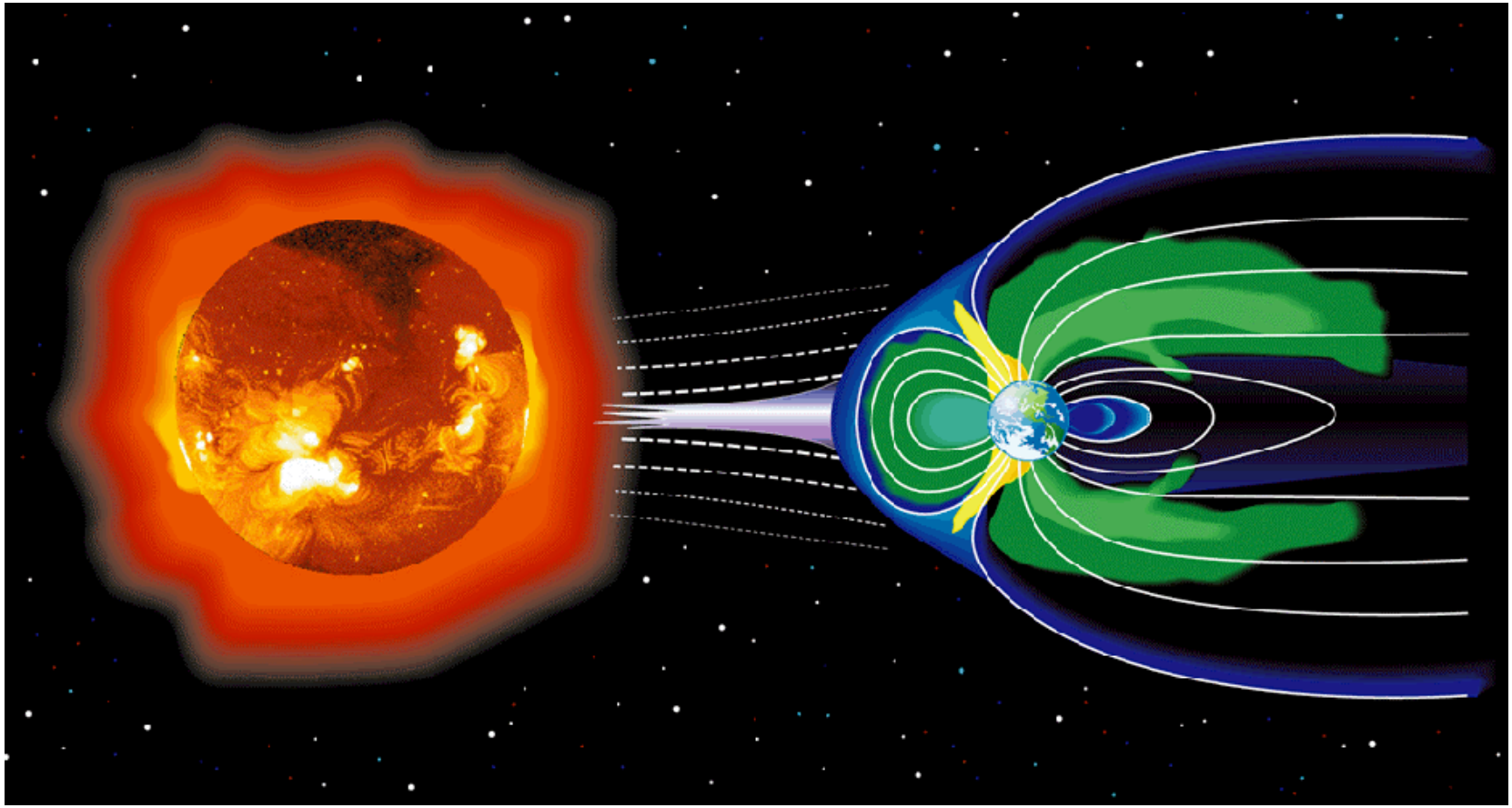


[Source: Nasa - GSFC]

Density gradient, cut-offs :

Radio Waves reflected / transmitted

Magnetosphere

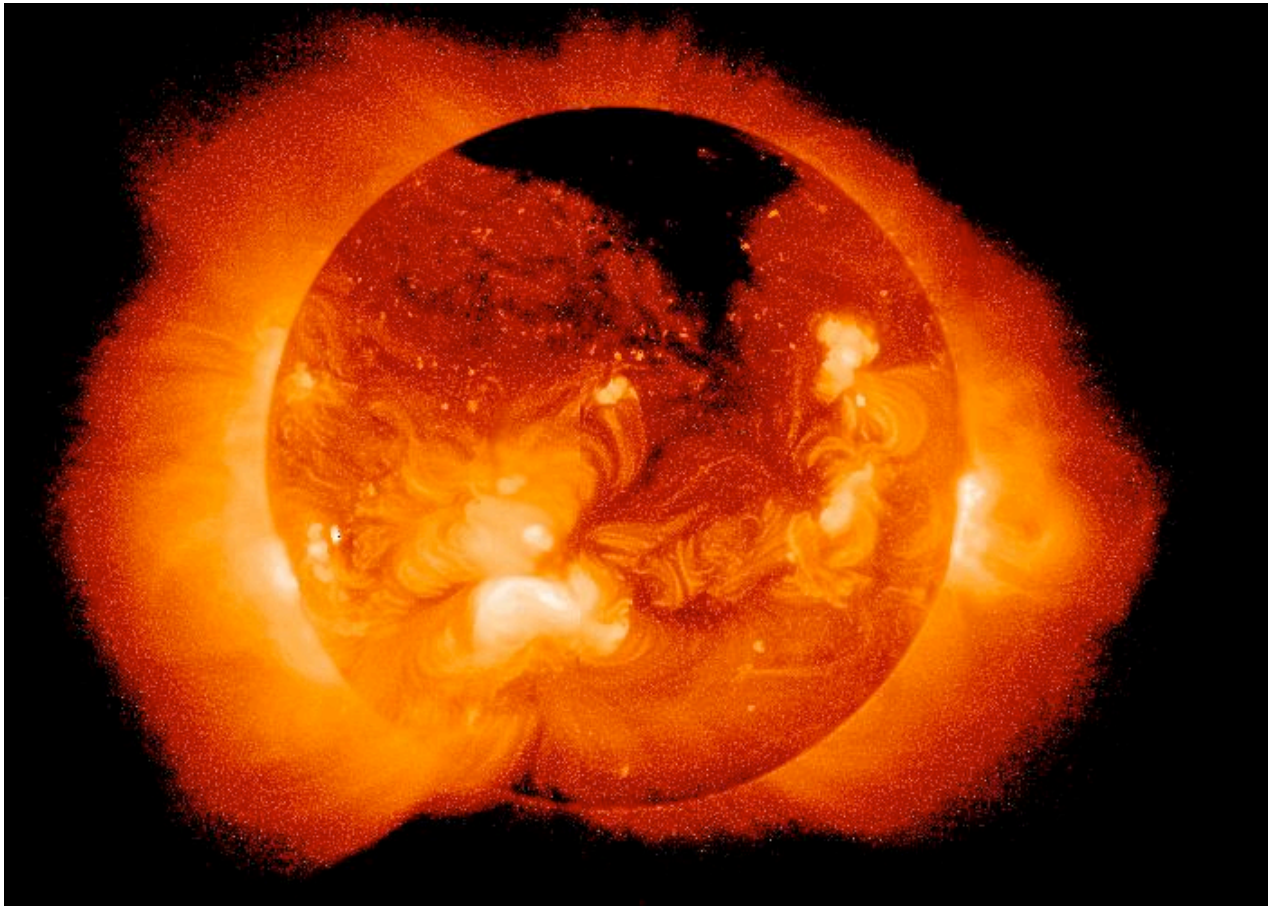


[Source: Nasa - GSFC]

Presence of whistlers:

Particle precipitation in loss cones enhanced

Solar Astrophysics

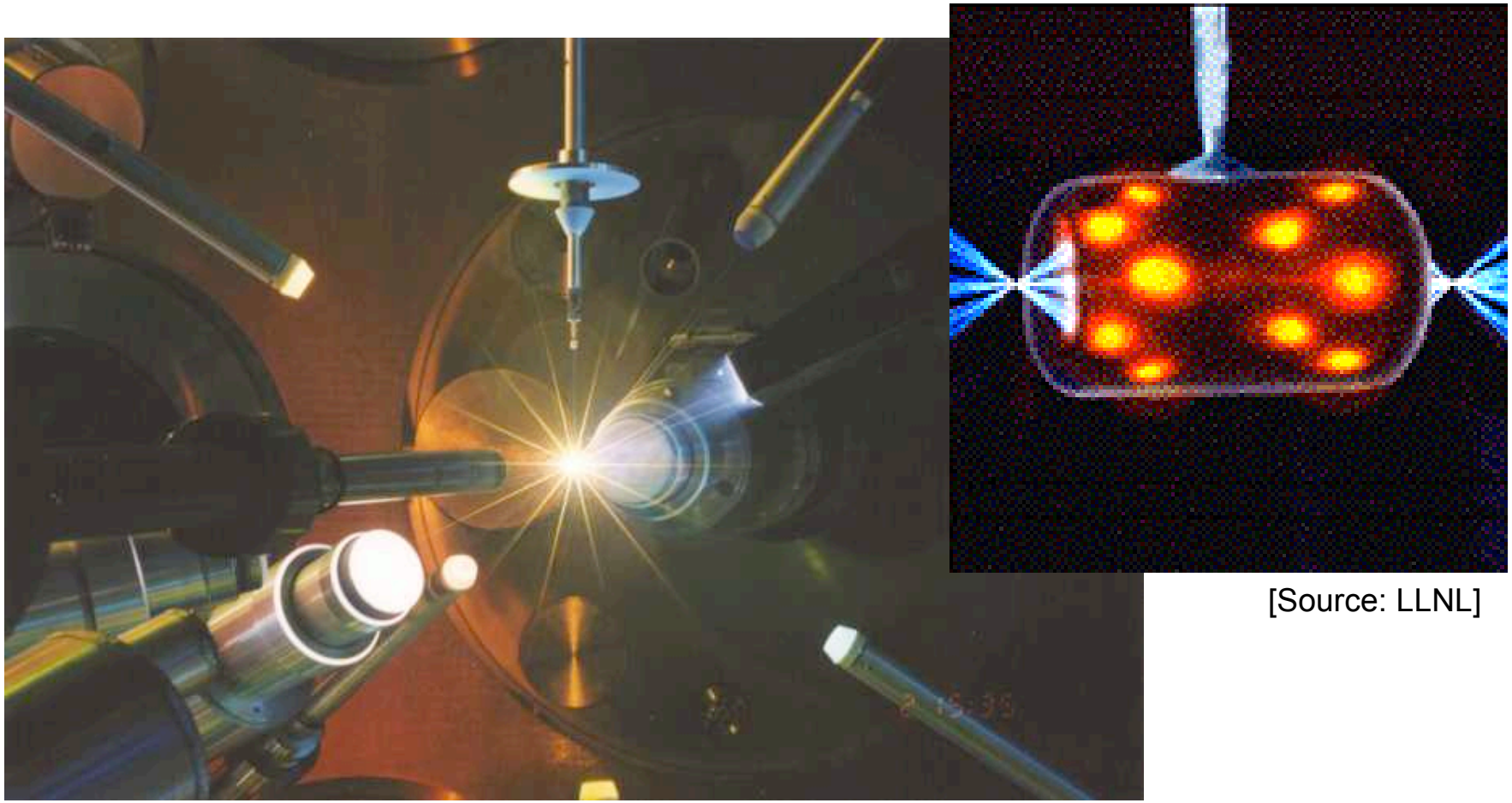


[Source: Nasa - MFSC]

Alfvén Waves:

Anomalous heating of the solar corona (?)

Intertial fusion plasmas

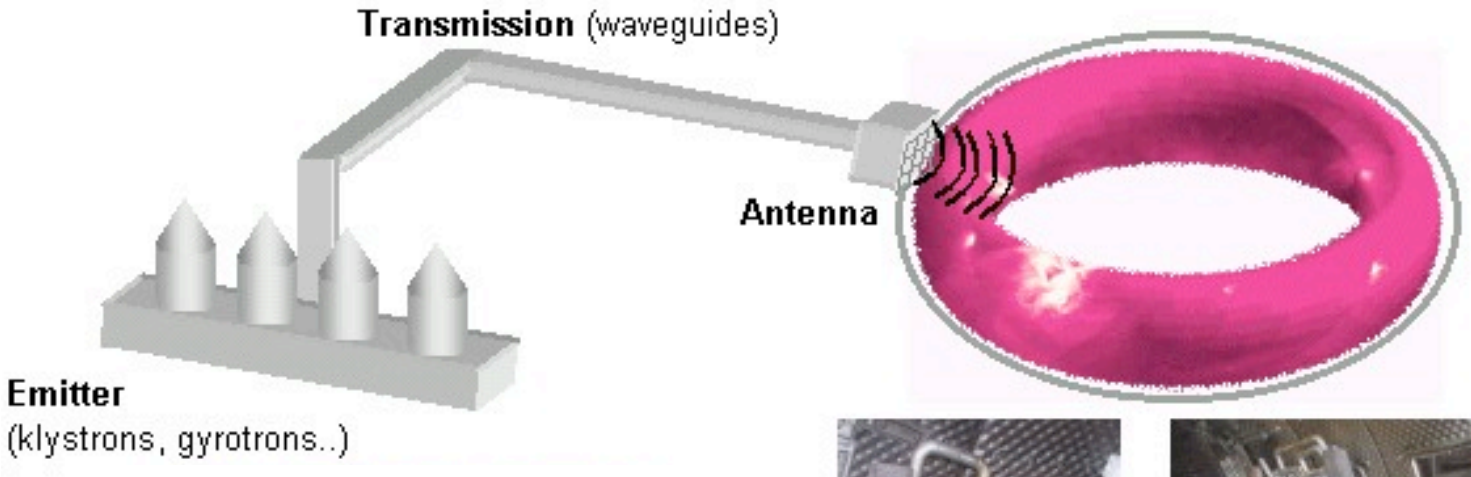


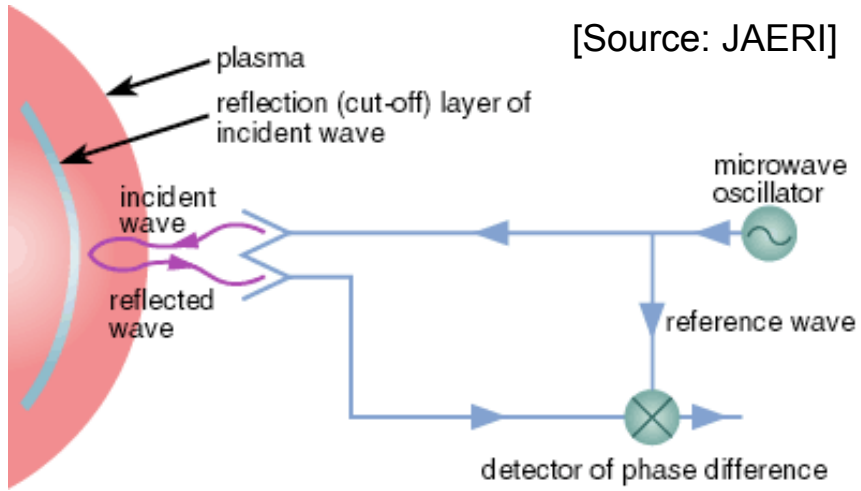
[Source: LLNL]

[Source: U. Rochester]

Non linear laser - plasma interaction

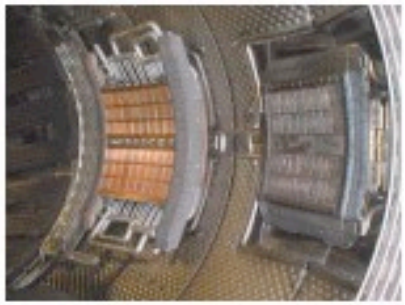
Magnetic fusion plasmas





Diagnostics

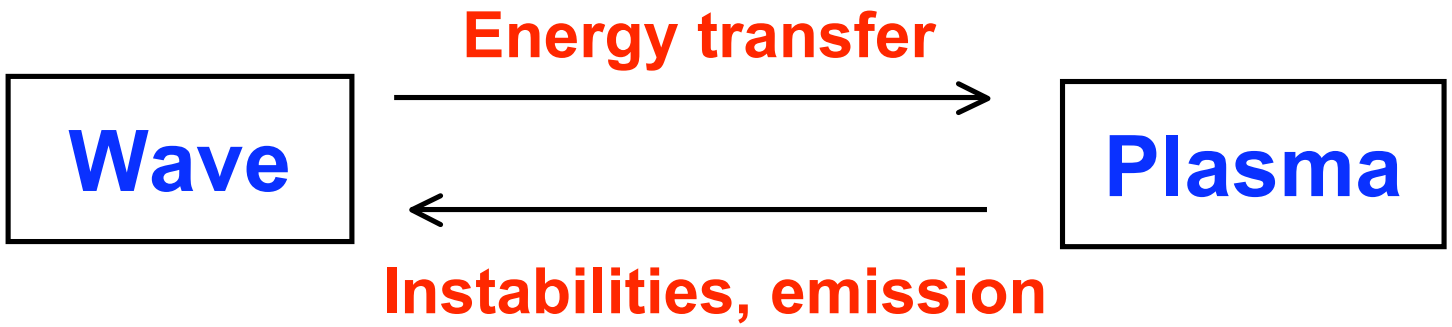
Ion cyclotron heating antenna (Tore Surpa)



Hybrid heating antenna (Tore Surpa)

Heating
Current Drive

The problem



Maxwell equations

Statistical description

Basic equations

Maxwell equations

- The heart of the problem:

Electric displacement

Charge density

$$\nabla \cdot \mathbf{D} = \rho$$

Magnetic induction

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Electric field

$$\nabla \times \mathbf{H} = \mathbf{j}_s + \frac{\partial \mathbf{D}}{\partial t}$$

Magnetic intensity

Source current

- Continuity equation for the sources:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j}_s = 0$$

Constitutive relation

- Isotropic medium: $\begin{cases} \mathbf{D} = \epsilon \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \end{cases}$

- Anisotropic medium: $\begin{cases} \mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} \\ \mathbf{B} = \bar{\bar{\mu}} \cdot \mathbf{H} \end{cases}$

- Plasma: $\begin{cases} \mathbf{D} = \bar{\bar{\epsilon}} \cdot \mathbf{E} \\ \mathbf{B} = \mu_0 \mathbf{H} \end{cases}$

- Link electric field / displacement:

$$\mathbf{j}_p + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_s + \frac{\partial \mathbf{D}}{\partial t}$$

Total current **"Response" current**

- Coherence of current:

$$\mathbf{j}_p \equiv \mathbf{j}_s + \mathbf{j} \longrightarrow \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\partial \mathbf{D}}{\partial t}$$

- Harmonic fields: $(\mathbf{E}, \mathbf{B}) \propto \exp(-i\omega t)$

$$\mathbf{D} = \epsilon_0 \left(\mathbf{E} + \frac{i}{\omega \epsilon_0} \mathbf{j} \right)$$

- Constitutive relation: $\mathbf{j} \stackrel{?}{=} \mathbf{j}(\mathbf{E})$

Linear approximation

- Statistical description of the plasma:

$$\rho = \sum_s q_s \int d^3\mathbf{v}\, f_s(\mathbf{r},\mathbf{v},t) \qquad \mathbf{j} = \sum_s q_s \int d^3\mathbf{v}\, \mathbf{v} f_s(\mathbf{r},\mathbf{v},t)$$

- Kinetic equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v}\cdot\frac{\partial f_s}{\partial \mathbf{r}} + \underbrace{\frac{q_s}{m_s}\Big[(\mathbf{E}_0+\mathbf{E}) + \mathbf{v}\times(\mathbf{B}_0+\mathbf{B})\Big]\cdot\frac{\partial f_s}{\partial \mathbf{v}}}_{\text{Non-linearity}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll.}$$

- Linear approximation:

$$f_s(\mathbf{r},\mathbf{v},t) \equiv F_s(\mathbf{r},\mathbf{v},t) + \tilde{f}_s(\mathbf{r},\mathbf{v},t)$$

Equilibrium solution

Perturbation

$\tilde{f}_s \propto \mathbf{E}$

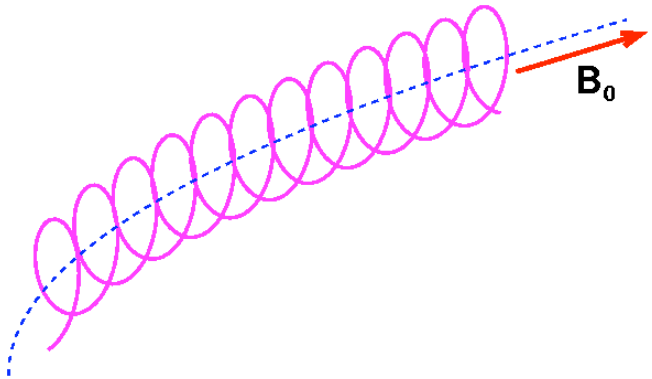
- Linear response of the plasma:

$$(\rho, \mathbf{j}) \propto \mathbf{E}$$

The plasma medium

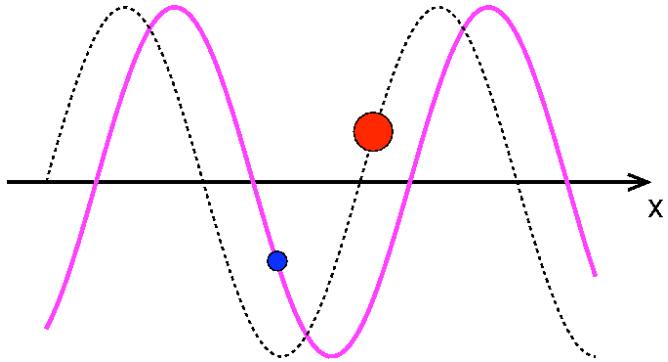
- Anisotropy

$$\mathbf{j}(\mathbf{r},t) = \overline{\overline{\boldsymbol{\sigma}}} \cdot \mathbf{E}(\mathbf{r},t)$$



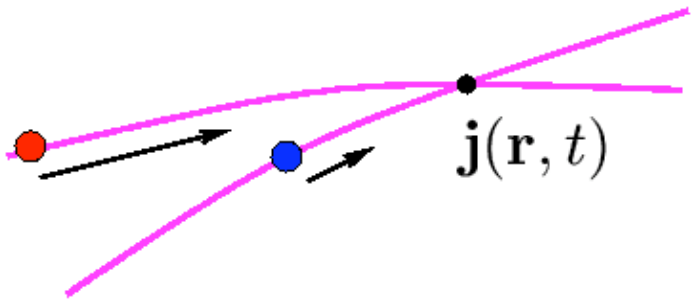
- Time dispersion

$$\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \, \overline{\overline{\boldsymbol{\sigma}}}(\mathbf{r},t,t') \cdot \mathbf{E}(\mathbf{r},t')$$



- Space dispersion

$$\mathbf{j}(\mathbf{r},t) = \int d^3\mathbf{r}' \, \overline{\overline{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t) \cdot \mathbf{E}(\mathbf{r}',t)$$



Dielectric tensor

- Non-local current: $\mathbf{j}(\mathbf{r},t) = \int_{-\infty}^{t} dt' \int d^3\mathbf{r}'\, \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t,t') \cdot \mathbf{E}(\mathbf{r}',t')$

- Stationary plasma: $\bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t,t') = \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t-t')$

- Uniform plasma: $\bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r},\mathbf{r}',t,t') = \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{r}-\mathbf{r}',t,t')$

- Plane waves: $(\mathbf{E},\mathbf{B}) \propto \exp\left(i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right)$

$$\longrightarrow \quad \mathbf{j}(\mathbf{k},\omega) = \bar{\bar{\boldsymbol{\sigma}}}(\mathbf{k},\omega)\cdot\mathbf{E}(\mathbf{k},\omega)$$

$$\longrightarrow \quad \mathbf{D} = \epsilon_0\left(\mathbf{E} + \frac{i}{\omega\epsilon_0}\mathbf{j}\right) = \underbrace{\epsilon_0\left(1 + \frac{i}{\omega\epsilon_0}\bar{\bar{\boldsymbol{\sigma}}}\right)}_{\bar{\bar{\boldsymbol{\epsilon}}} \equiv \epsilon_0 \bar{\bar{\mathbf{K}}}} \cdot \mathbf{E}$$

(Dielectric tensor)

Dispersion relation

- Wave equation (no source):

$$\nabla \times \nabla \times \mathbf{E} - \frac{\omega^2}{c^2} \overline{\overline{\epsilon}} \cdot \mathbf{E} = 0$$

- Plane waves - Homogeneous plasma:

$$\mathbf{n} \times \mathbf{n} \times \mathbf{E} + \overline{\overline{\epsilon}} \cdot \mathbf{E} = 0, \qquad \mathbf{n} \equiv \frac{c}{\omega}\mathbf{k}$$

$$\longrightarrow \quad \overline{\overline{\mathbf{M}}}_{\mathbf{k},\omega} \cdot \mathbf{E} = \mathbf{0}$$

- Non-trivial solutions for the electric field:

$$\det(\overline{\overline{\mathbf{M}}}_{\mathbf{k},\omega}) = 0 \quad \longrightarrow \quad \det\left[\left(n^2 \frac{k_i k_j}{k^2} - \delta_{ij}\right) + \epsilon_{ij}(\mathbf{k},\omega)\right] = 0$$

$\longrightarrow$
- Characteristic waves of the uniform plasma $\quad n = n(\mathbf{k},\omega)$
- Propagative / Evanescent modes
- Polarization

Plasma description

- Main goal of the plasma modeling: $\overline{\overline{\epsilon}}$
- Various available models:
  - Cold plasma
  - Cold plasma + thermal corrections

    $\longrightarrow$ Fluid modeling

  - Hot plasma - linear
  - Hot plasma - quasi-linear

    $\longrightarrow$ Kinetic modeling

Waves in a fluid plasma

Cold plasma model

- Current (fluid model): $\mathbf{j} = \sum_s \mathbf{j}_s = \sum_s n_s q_s \mathbf{v}_s$

Macroscopic velocity

- Momentum equation:

$$n_s m_s \frac{d\mathbf{v}_s}{dt} = n_s m_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \nabla)\mathbf{v}_s \right) = n_s q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nabla \cdot \overline{\overline{\mathbf{\Phi}}}_s$$

Pressure tensor

- Cold plasma: $\overline{\overline{\mathbf{\Phi}}}_s = \overline{\overline{\mathbf{0}}}$

$$\longrightarrow \quad \frac{d\mathbf{v}_s}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B})$$

- Linearization: $\mathbf{B} \equiv \mathbf{B}_0 + \delta\mathbf{B}, \quad \mathbf{E} \equiv \delta\mathbf{E}$

$\longrightarrow$ To first order and omitting the "δ"

$$-i\omega m_s \mathbf{v}_s = q_s(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0)$$

Cold plasma model (2)

- Static magnetic field: $\mathbf{B}_0$ // $\mathbf{e}_z$

$$\begin{cases} -i(v_x + iv_y)\cdot(\omega - \omega_{cs}) &= \dfrac{q_s}{m_s}(E_x + iE_y) \\ -i(v_x - iv_y)\cdot(\omega + \omega_{cs}) &= \dfrac{q_s}{m_s}(E_x - iE_y) \\ -iv_z\omega = \dfrac{q_s}{m_s}E_z \end{cases}$$

$$\omega_{cs} \equiv \frac{q_s B_0}{m_s}$$ **(Cyclotron frequency)**

- Rotating field and velocity:

$$E^{\pm} \equiv \frac{1}{2}(E_x \pm iE_y) \qquad v^{\pm} \equiv \frac{1}{2}(v_x \pm iv_y)$$

$$\longrightarrow \begin{cases} -iv^{\pm}\cdot(\omega \mp \omega_{cs}) &= \dfrac{q_s}{m_s}E^{\pm} \\ -iv_z\omega = \dfrac{q_s}{m_s}E_z \end{cases}$$

Cold plasma model (3)

- Current:

$$\begin{cases} j^{\pm} = i \sum_s \frac{n_s q_s^2}{m_s} \cdot \frac{E^{\pm}}{\omega \mp \omega_{cs}} \\ j_z = i \sum_s \frac{n_s q_s^2}{m_s} \cdot \frac{E_z}{\omega} \end{cases}$$

- Conductivity tensor: $\mathbf{j} = \bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{E}$ $\longrightarrow$ **Cold dielectric tensor**

$$\bar{\bar{\boldsymbol{\epsilon}}} = \epsilon_0 \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

$$\omega_{ps}^2 \equiv \frac{n_s q_s^2}{m_s \epsilon_0}$$

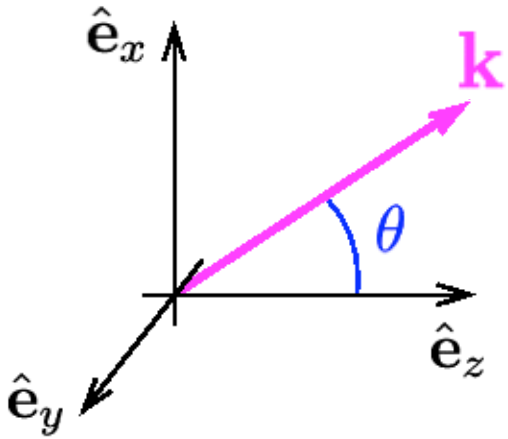
(Plasma frequency)

$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} \qquad L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})} \qquad P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

$$S \equiv \frac{1}{2}(R + L) \qquad D \equiv \frac{1}{2}(R - L)$$

N.B.: The cold plasma is not space dispersive

Dispersion relation



$$\mathbf{n} \times \mathbf{n} \times \mathbf{E} + \overline{\overline{\epsilon}} \cdot \mathbf{E} = 0$$

$$\begin{cases} n_x \equiv n\cos(\theta) \\ n_z \equiv n\sin(\theta) \end{cases}$$

$$\begin{pmatrix} S - n^2\cos^2(\theta) & -iD & n^2\cos(\theta)\sin(\theta) \\ iD & S - n^2 & 0 \\ n^2\cos(\theta)\sin(\theta) & 0 & P - n^2\sin^2(\theta) \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \mathbf{0}$$

- Dispersion relation:

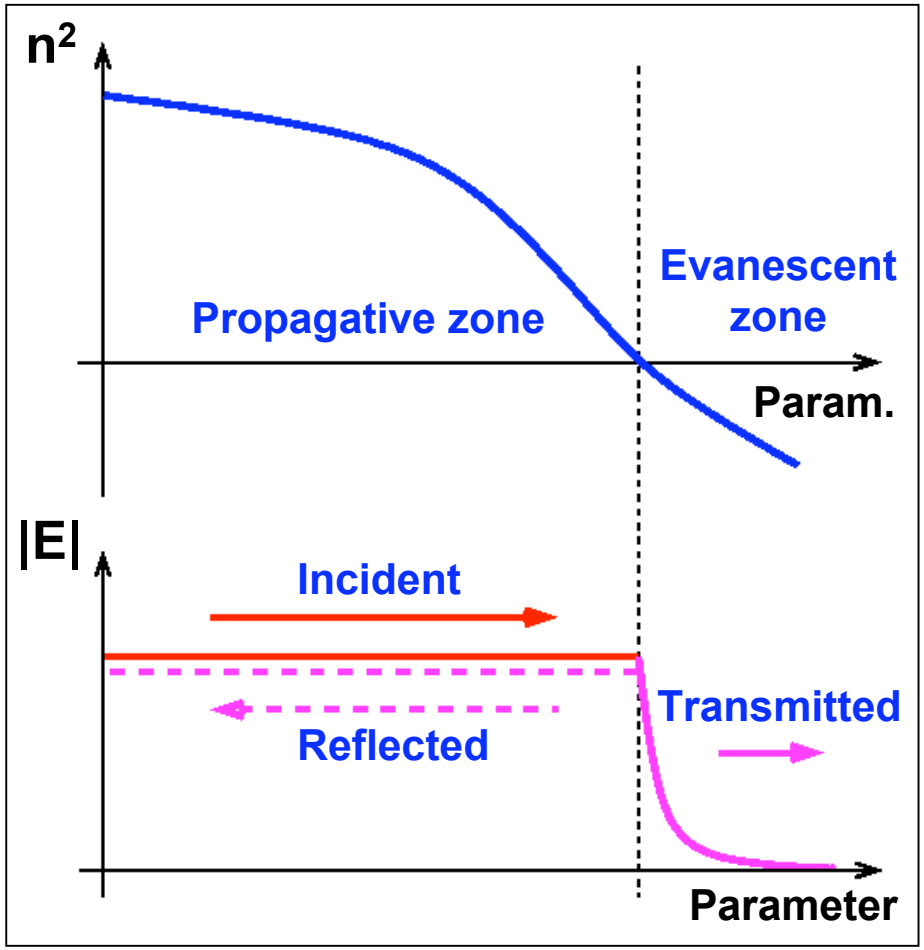
$$An^4 - Bn^2 + C = 0 \quad \longrightarrow \quad \textbf{2 propagation modes}$$

$$\begin{cases} A = S\sin^2(\theta) + P\cos^2(\theta) \\ B = RL\sin^2(\theta) + PS\big(1 + \cos^2(\theta)\big) \\ C = PRL \end{cases}$$

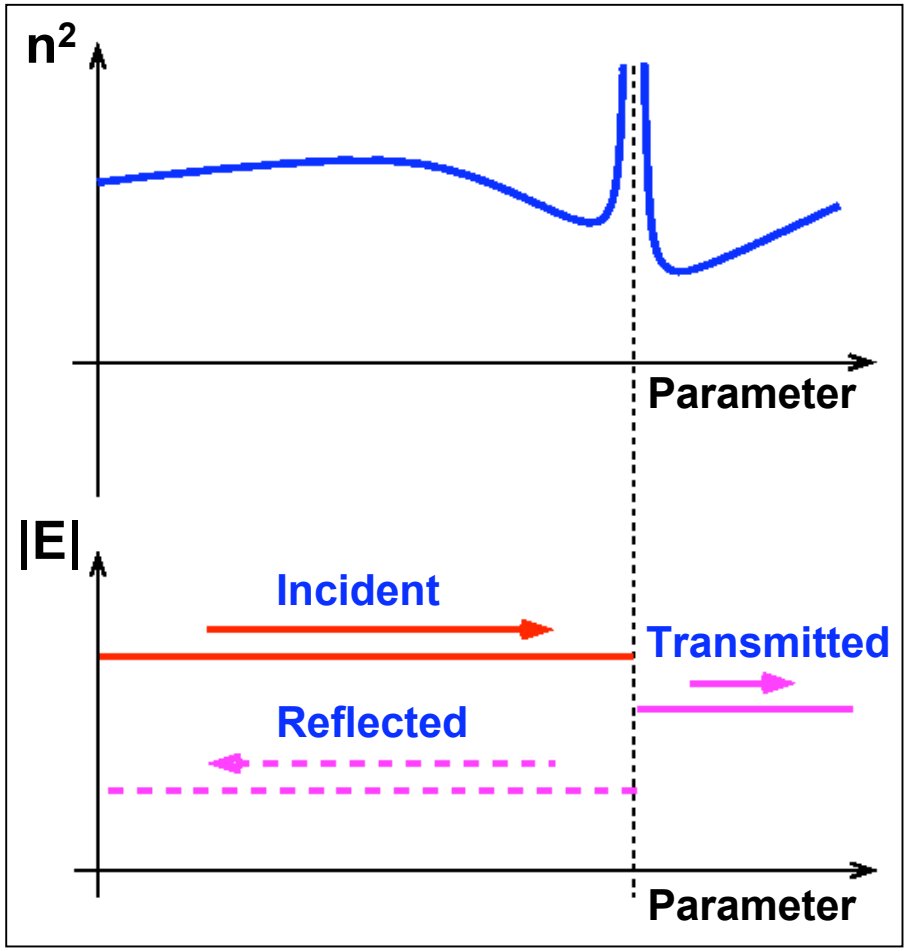
Cut-offs - Resonances

- Dispersion relation (alt. form): $\tan^2(\theta) = -\dfrac{P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}$

Cut-off: $n^2 = 0$



Resonance: $n^2 \to \infty$



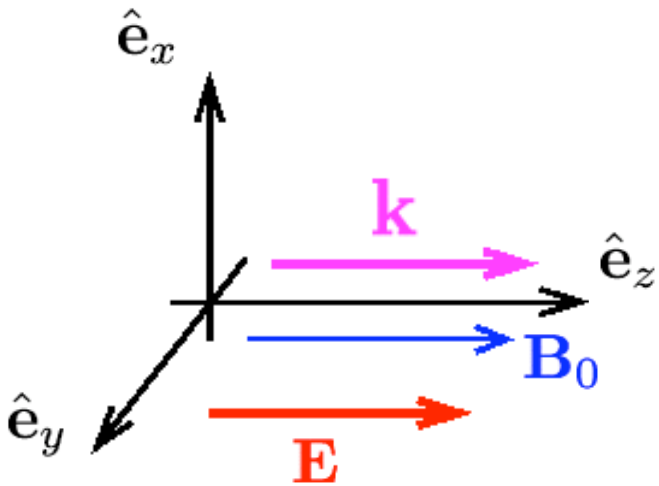
Parallel propagation

- Wave equation (□=0):

$$\left\{\begin{array}{l} (S-n^2)E_x - iDE_y = 0 \\ iDE_x + (S-n^2)E_y = 0 \\ PE_z = 0 \end{array}\right\} \begin{array}{l} \textbf{(2)} \\ \\ \textbf{(1)} \end{array}$$

1. Langmuir wave

- Dispersion relation: $P = 0 \longrightarrow \omega = \sqrt{\omega_{pe}^2 + \omega_{pi}^2}$
- Phase and group velocities undefined (non-propagative phenomenon)
- Oscillation parallel to $\mathbf{B}_0$

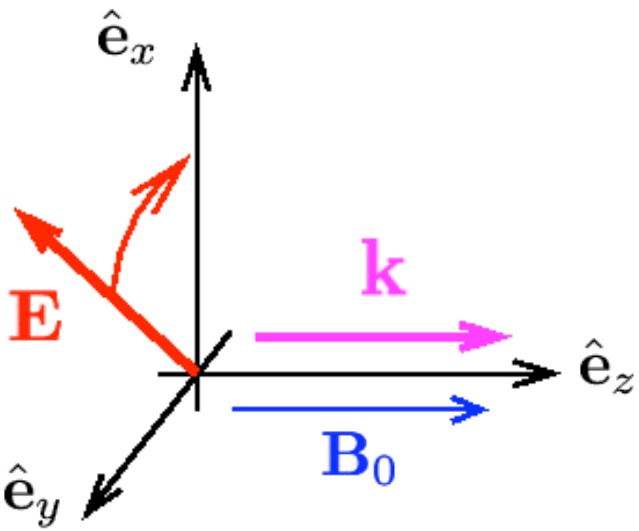


Parallel propagation (2)

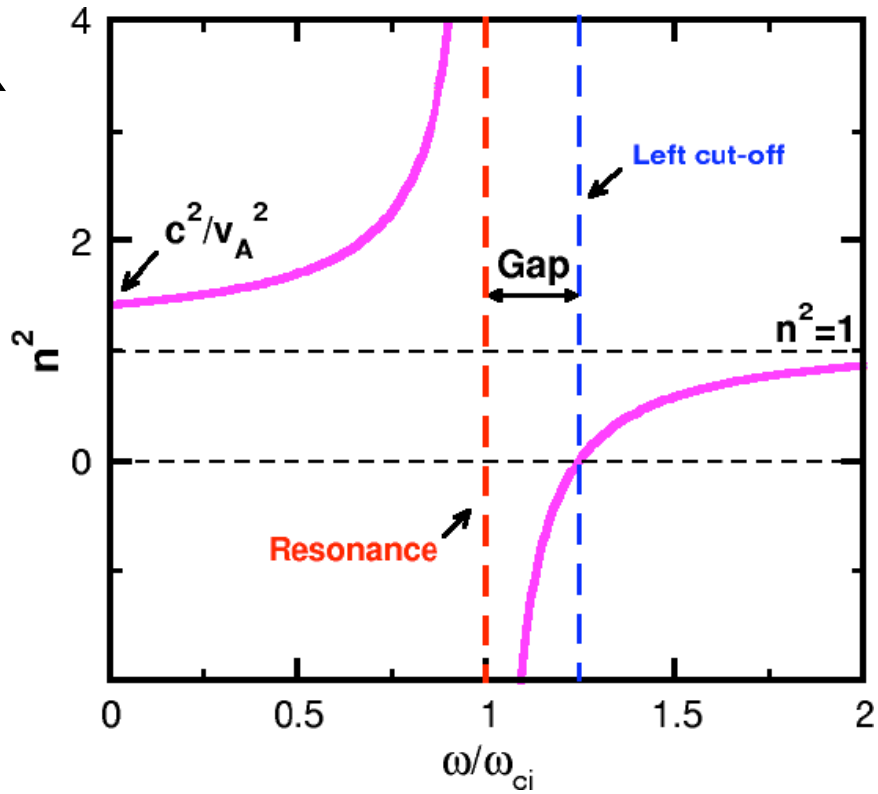
2. Whistlers

- Wave equation: $\begin{cases} (L - n^2)E^+ = 0 & \textbf{(1)} \\ (R - n^2)E^- = 0 & \textbf{(2)} \end{cases}$

- **1) Ion whistler** *(L-Wave)*:
  - Dispersion relation: $n^2 = L$

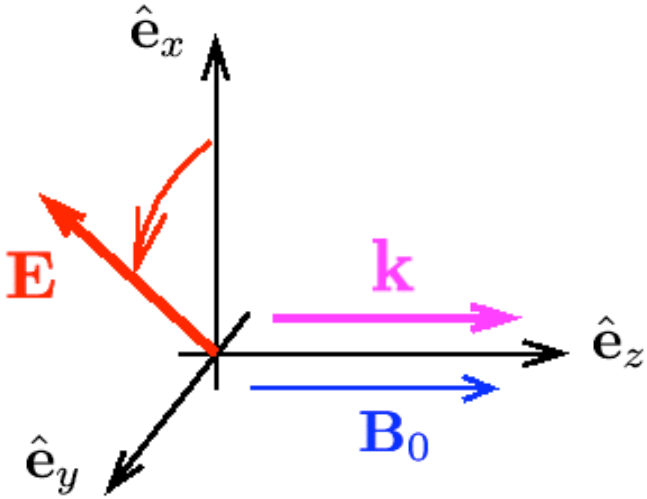


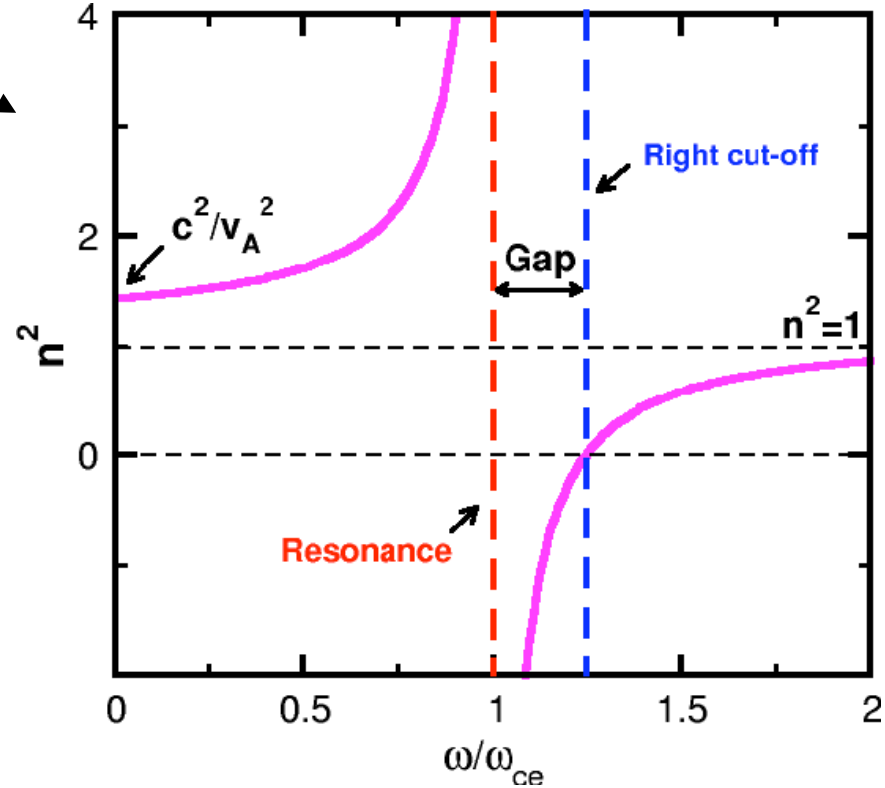
$E^- = 0 \qquad (E_x = iE_y)$



Electron whistler

- **2) Electron Whistler** *(R-Wave)*:
  - Dispersion relation: $n^2 = R$





$$E^+ = 0 \qquad (E_x = -iE_y)$$

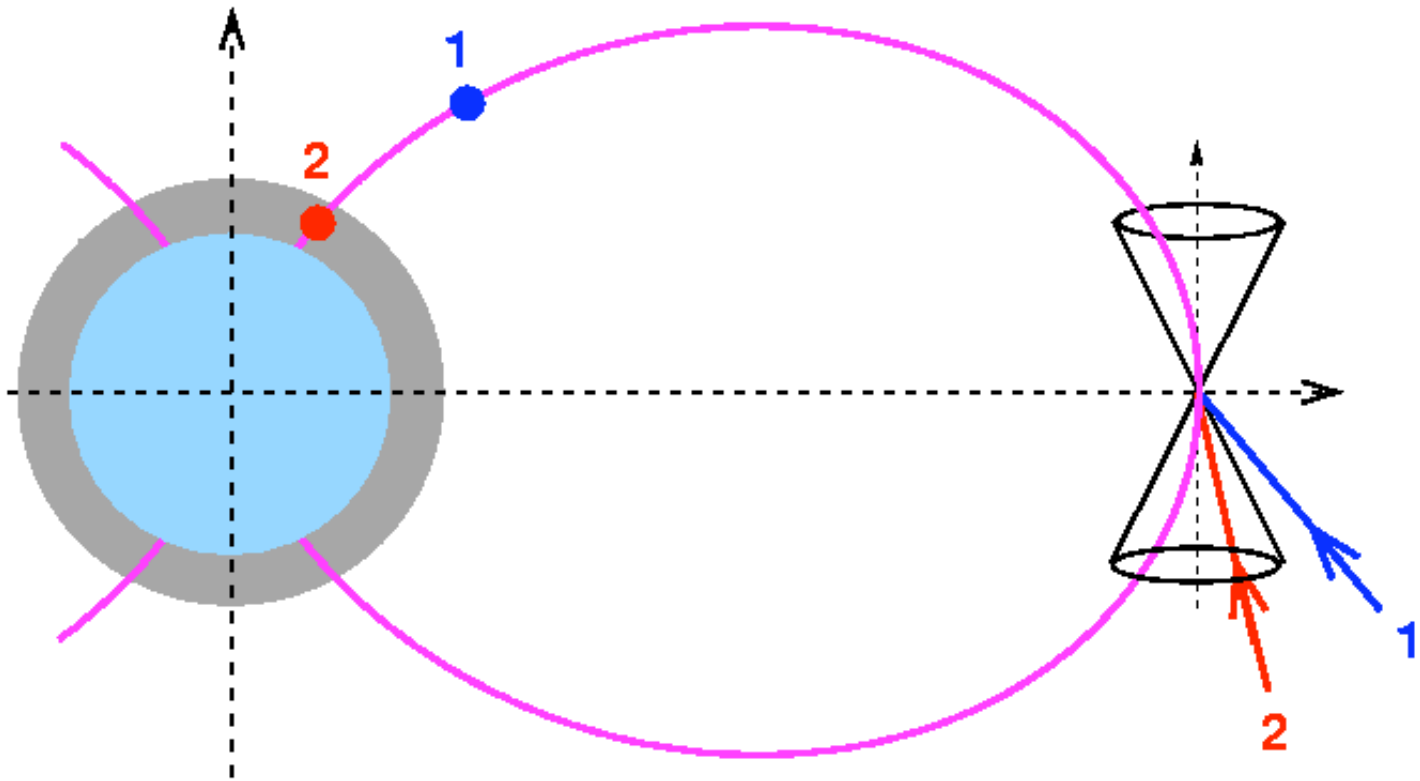
  - Phase velocity - group velocity:

$$v_\phi = \frac{c}{n} \approx c\sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}} \propto \sqrt{\omega} \qquad v_g = \frac{\partial\omega}{\partial k} \approx 2c\sqrt{\frac{\omega\omega_{ce}}{\omega_{pe}^2}} \propto \sqrt{\omega}$$

→ **Large dispersivity**

Whistlers in the magnetosphere

Electron population in the Van Allen belts

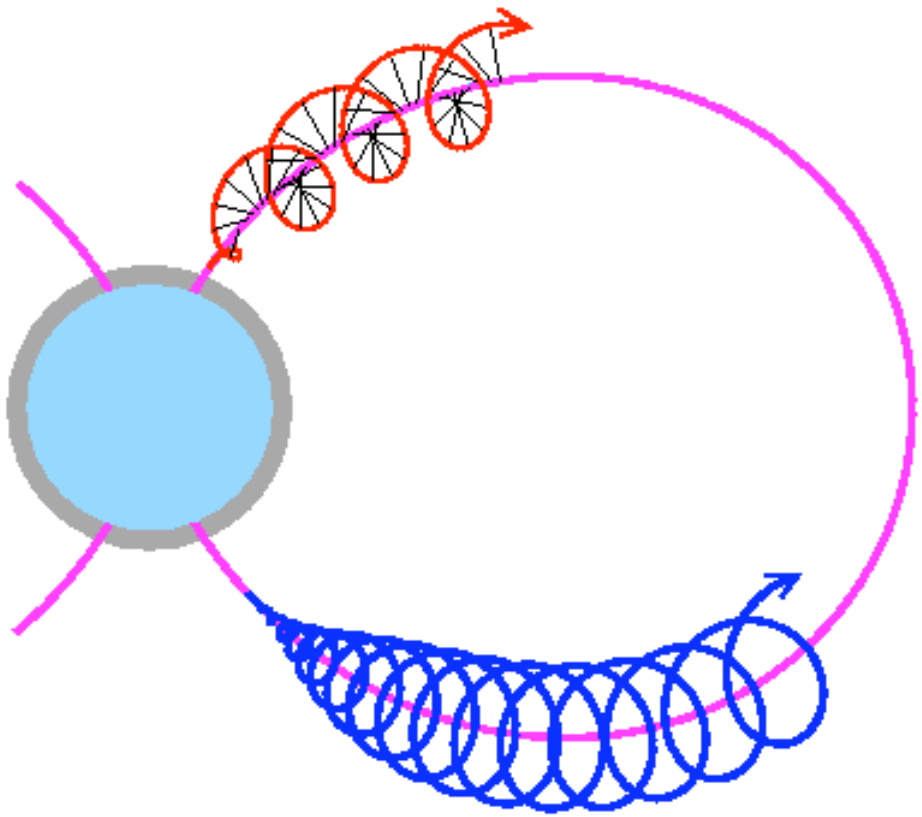


Collisions: pitch angle scattering: **(1)** □ **(2)**

→ **Weak losses**

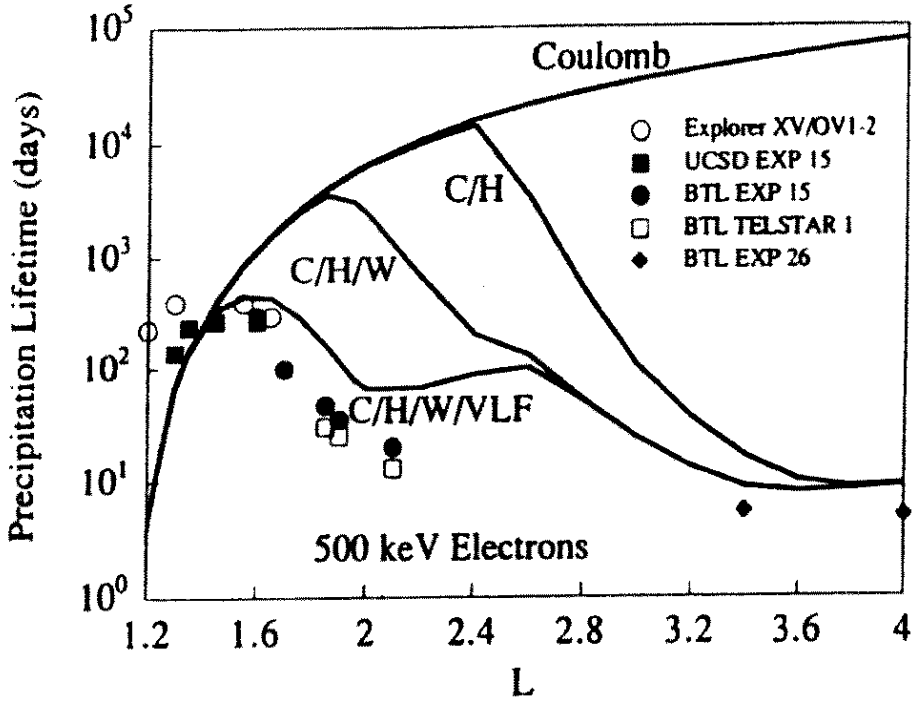
Whistlers in the magnetosphere (2)

Ducted whistlers



Interaction with the electrons trapped in the radiation belts

Electron lifetime



[*Source*: R.W. Abel and R.M. Thorne J. Geophys Res. 103 (1998) 2365]

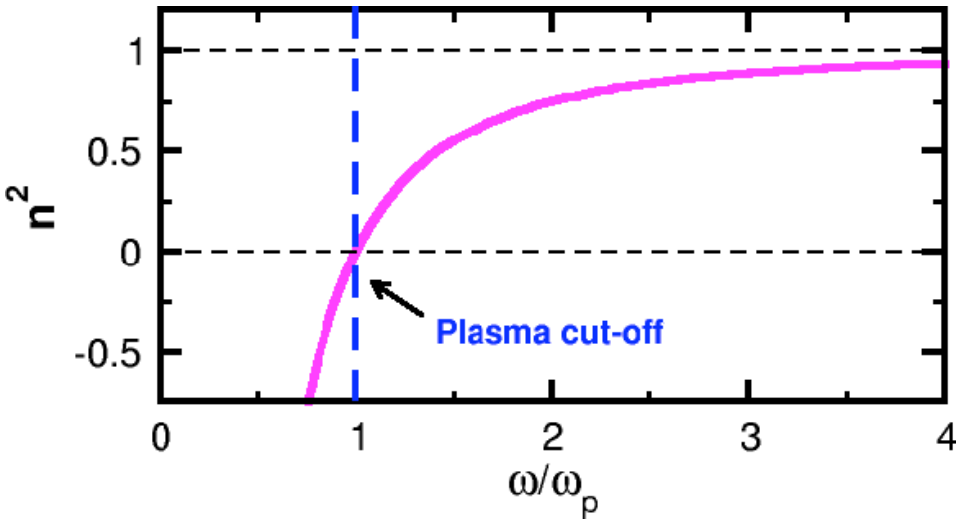
Perpendicular propagation

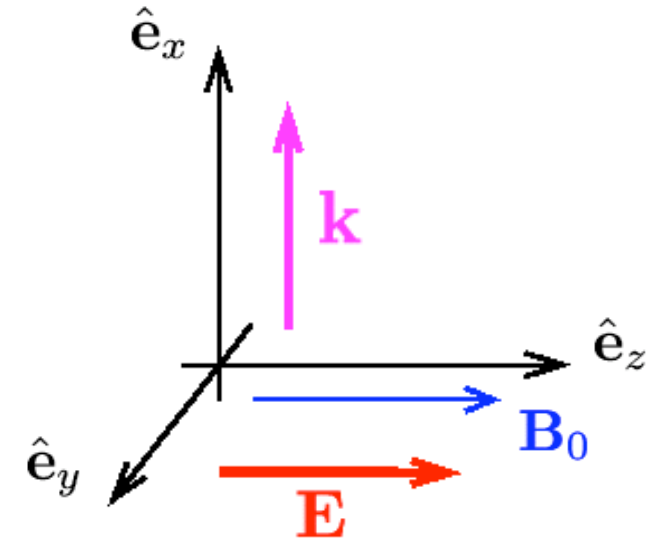
- Wave equation (θ=π/2):

$$\left\{\begin{array}{l} SE_x - iDE_y = 0 \\ iDE_x + (S-n^2)E_y = 0 \\ (P-n^2)E_z = 0 \end{array}\right\} \begin{array}{l} \\ \text{(2)} \\ \text{(1)} \end{array}$$

1. Ordinary wave (O):

- Dispersion relation: $n^2 = P$
- Propagative for $\omega^2 > \omega_{pe}^2 + \omega_{pi}^2$



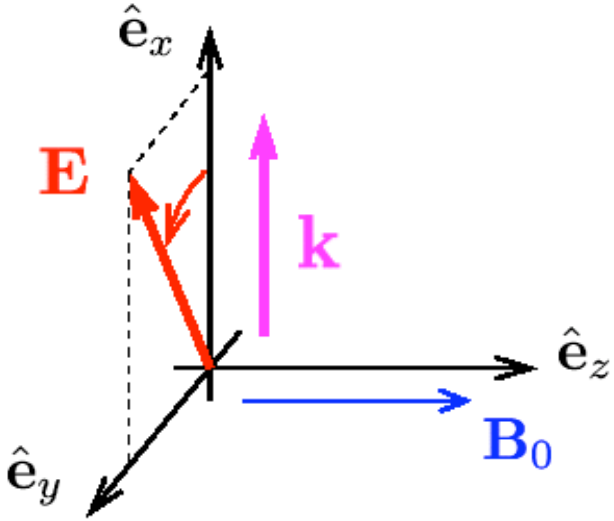


$E_x = E_y = 0 \qquad (\mathbf{E} \parallel \mathbf{B}_0)$

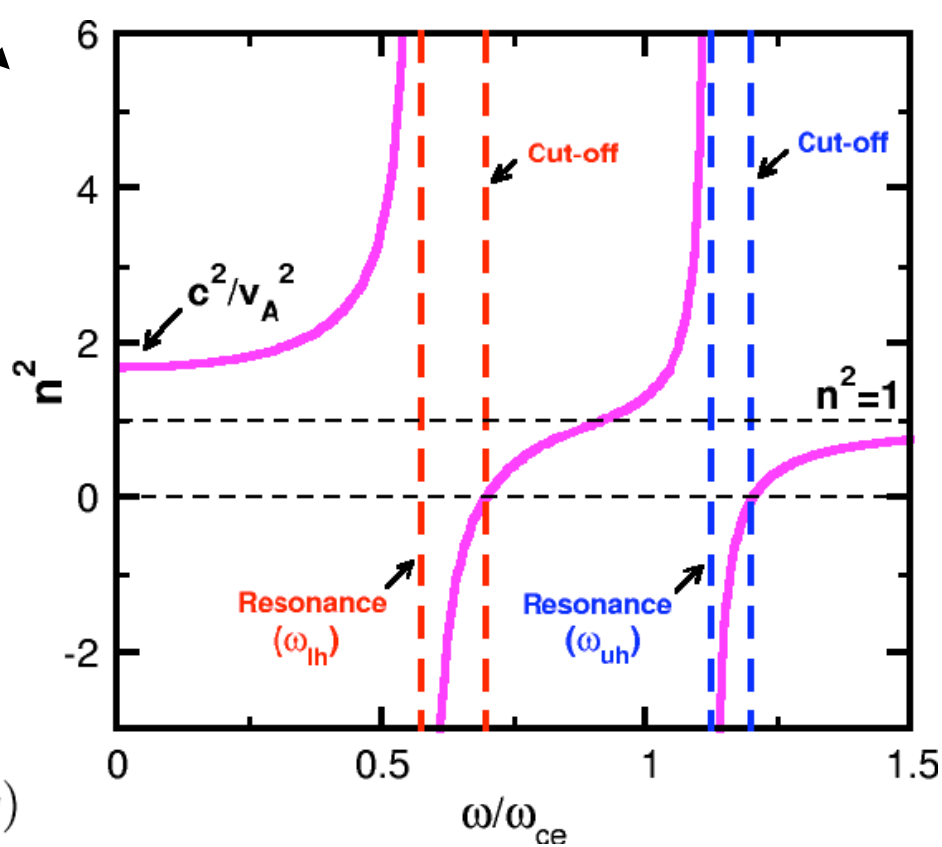
Perpendicular propagation (2)

2. Extraordinary wave (X):

- Dispersion relation: $n^2 = RL/S$



$$\frac{E_x}{E_y} = i\frac{R-L}{R+L}$$



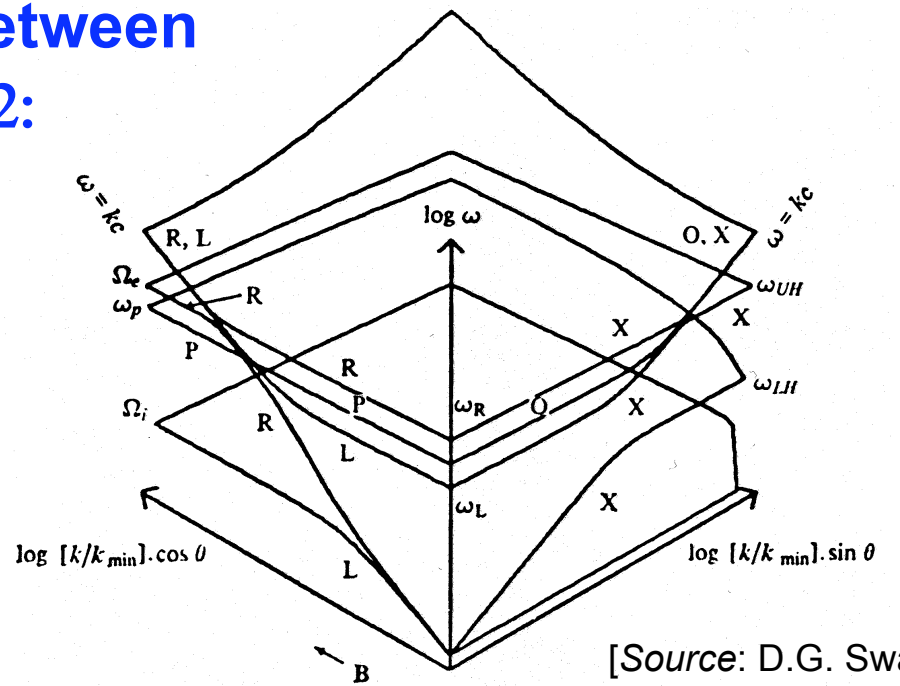
- Screening phenomenon:

$$\omega \to \omega_{ci}: \quad E_+ \to 0 \quad (E_x \to -iEy)$$

$$\omega \to \omega_{ce}: \quad E_- \to 0 \quad (E_x \to iEy)$$

Arbitrary angle propagation

- **Continuity between $\theta=0$ et $\theta=\pi/2$:**



[*Source*: D.G. Swanson, Plasma Waves (A.P., San Diego, 1989)]

- **Example: MHD waves: ($\omega \ll \Omega_{ci}$)**

$$\begin{cases} \left(-n^2\cos(\theta)+1+\dfrac{c^2}{v_a^2}\right)\cdot E_x = 0 & \textbf{(Torsional Alfvén wave)} \\ \left(-n^2+1+\dfrac{c^2}{v_a^2}\right)\cdot E_y = 0 & \textbf{(Compressional Alfvén wave)} \\ P\cdot E_z = 0 \end{cases}$$

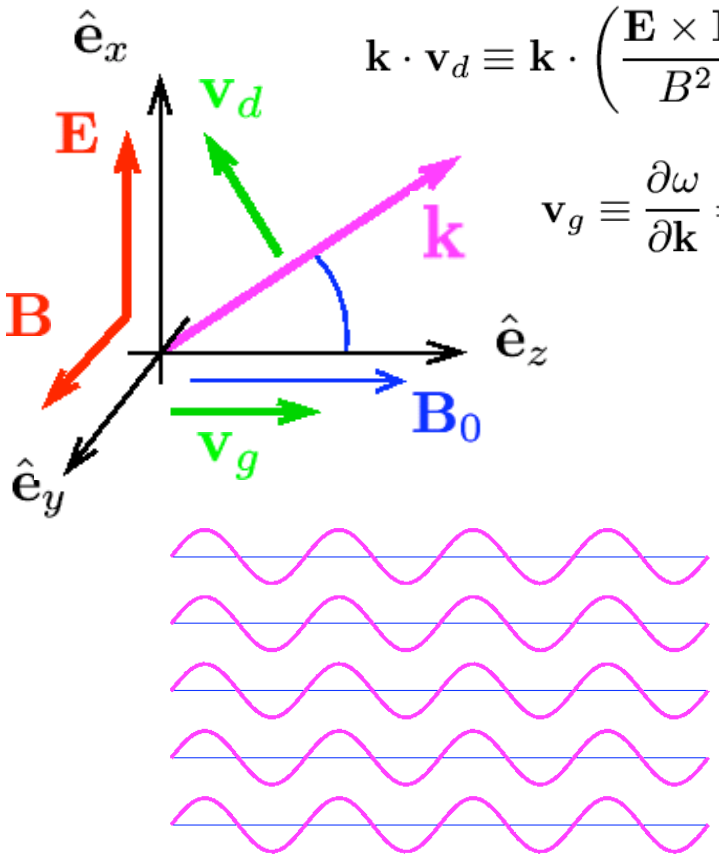
Low frequency: Alfvén waves

Torsional Alfvén wave

$$n^2\cos^2(\theta) = 1 + \frac{c^2}{v_a^2}$$

$$\mathbf{k}\cdot\mathbf{v}_d \equiv \mathbf{k}\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{B^2}\right) = 0$$

$$\mathbf{v}_g \equiv \frac{\partial\omega}{\partial\mathbf{k}} = v_g\hat{\mathbf{e}}_z$$



Field lines torsion

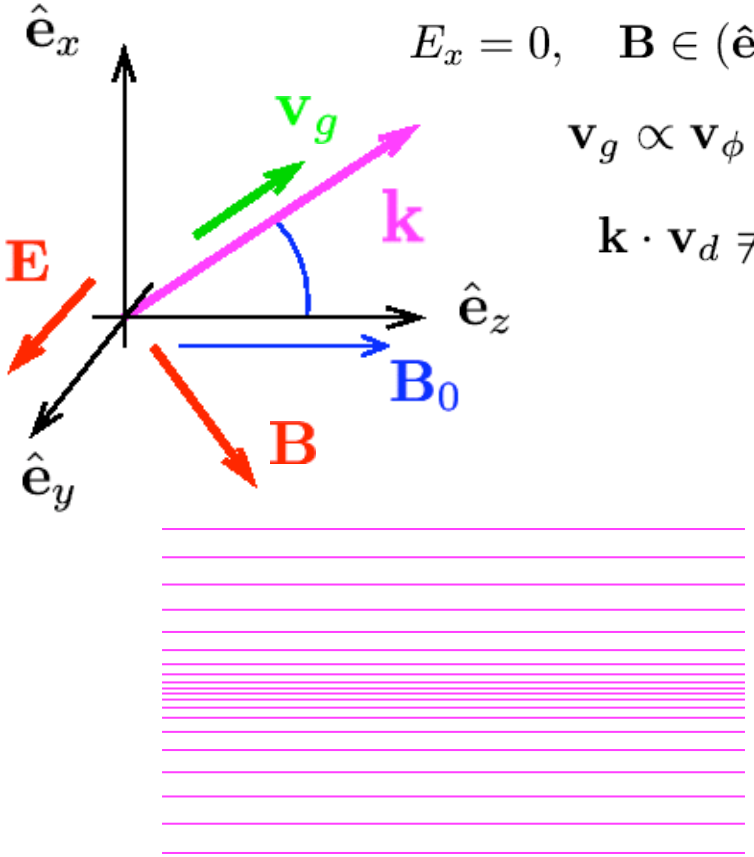
Compressional Alvén wave

$$n^2 = 1 + \frac{c^2}{v_a^2}$$

$$E_x = 0, \quad \mathbf{B} \in (\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_z)$$

$$\mathbf{v}_g \propto \mathbf{v}_\phi \propto \mathbf{k}$$

$$\mathbf{k}\cdot\mathbf{v}_d \neq 0$$



Field lines compression

Flight map: the CMA diagram

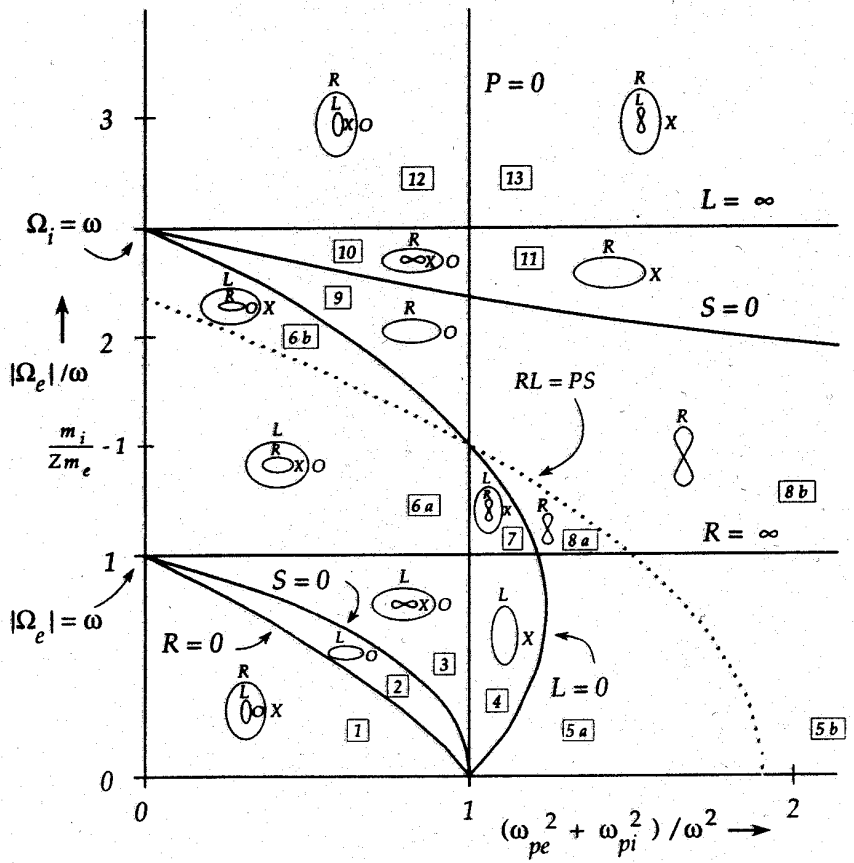
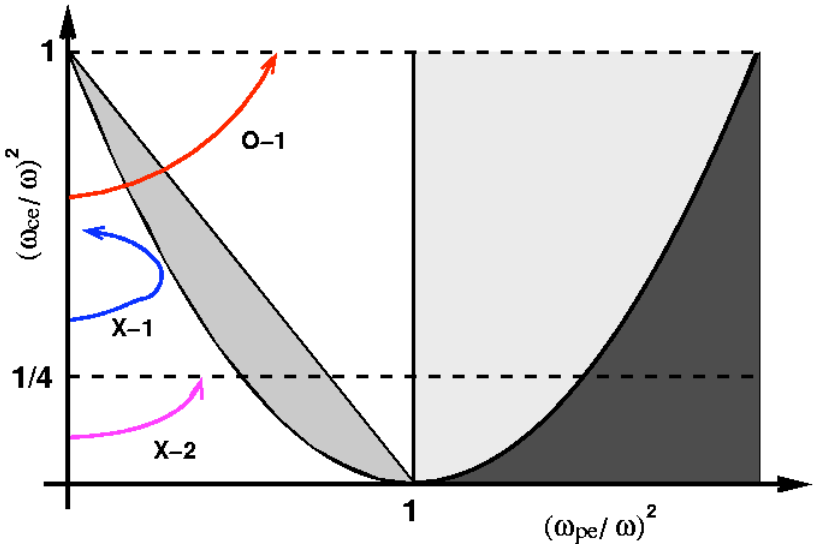


Fig. 2-1. CMA diagram for a two-component plasma. The ion-to-electron mass ratio is chosen to be 2.5. Bounding surfaces appear as lines in this two-dimensional parameter space. Cross sections of wave-normal surfaces are sketched and labeled for each region. For these sketches the direction of the magnetic field is vertical. The small mass ratio can be misleading here: the $L=0$ line intersects $P=0$ at $\Omega_i/\omega_i=1-(Zm_e/m_i)$.

Reference: Stix (1992)

Example: EC Waves in a tokamak



Ray-tracing

The homogeneity hypothesis

- Plane waves: $(\mathbf{E},\mathbf{B}) \equiv (\mathbf{E}_0,\mathbf{B}_0)\cdot\exp\left(i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right)$

  → Valid in a **stationary** and **homogeneous** plasma

  → $\Delta t \gg \dfrac{2\pi}{\omega}$ **[x]**

  → $\Delta r \gg \lambda$ **[o]**

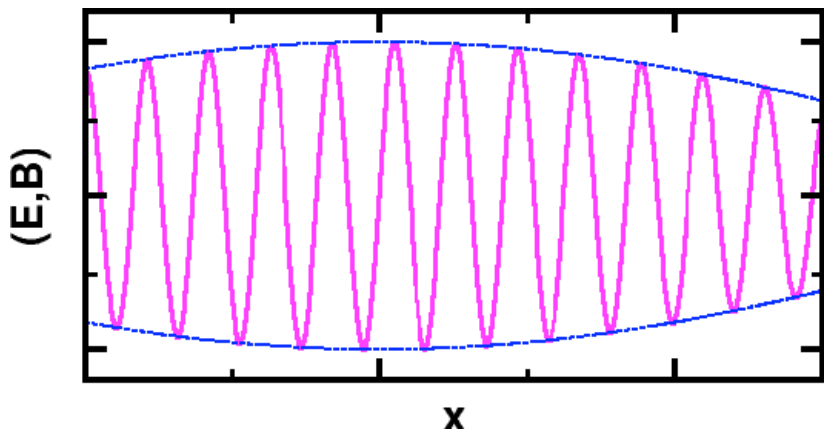
- Wave absorption: $\omega \to \omega_r - i\gamma, \quad \mathbf{k} \to \mathbf{k}_r + i\mathbf{k}_i$

  → $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = e^{i(\mathbf{k}_r\cdot\mathbf{r}-\omega_r t)}\cdot\exp(-\mathbf{k}_i\cdot\mathbf{r})\cdot\exp(-\gamma t)$

- WKB hypothesis: $(\mathbf{E},\mathbf{B}) \equiv \left(\mathbf{E}_0(\mathbf{r},t),\mathbf{B}_0(\mathbf{r},t)\right)\cdot\exp(i\psi)$

  Eikonal function

$$\begin{cases} \nabla\psi \equiv \mathbf{k}(\mathbf{r},t) \\ \dfrac{\partial\psi}{\partial t} \equiv -\omega(\mathbf{r},t) \end{cases}$$



→ Time-scale splitting:
- Fast: □
- Slow: $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{k}$, □

→ Applicable outside: cut-offs, resonances, □#□r

Energy balance

- **Poynting theorem**: $$\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \left[\mathbf{B} \cdot \frac{\partial \mathbf{H}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \right] = -\mathbf{j}_s \cdot \mathbf{E}$$

  (1) **(2)**

- **"Classical" medium** (solid…):

  (1) Atoms, nuclei motion…: polarization current (reversible)

  (2) Free charges motion: conduction current (irreversible)

  Generally: $\square_{mfp} << \square$ ⟶ Quasi-instantaneous equilibrium (e.g.: Joule effect)

- **Plasma**:
  - No clear separation between conduction current / polarization current
  - All charges are free, but contribute to the polarization ($\square_{mfp} >> \square$)

  ⟶ No equilibrium **over one wave period**

  ⟶ No equivalent to the electromagnetic energy density, to the instantaneous Poynting vector **over one wave period**

  ⟶ Use of **averaged quantities** is necessary

Energy balance (2)

- **Two conditions:**
  - Weakly inhomogeneous plasma: $\mathbf{Q} \equiv \mathbf{Q}_0 \times \exp(i\psi)$

    $\psi \equiv \psi_r + i\psi_i \longrightarrow \begin{cases} \mathbf{k}_i \equiv \boldsymbol{\nabla}\psi_i \\ \omega_i \equiv -\dfrac{\partial \psi_i}{\partial t} \end{cases}$
  - Use of averaged quantities (over many wave periods)

- **Poynting theorem:**

$$W \equiv \frac{1}{2}\left[\frac{B_0^2}{\mu_0} + \mathbf{E}_0^* \cdot \frac{\partial}{\partial \omega}(\omega \overline{\overline{\epsilon}}^h) \cdot \mathbf{E}_0\right]$$

[Electromagnetic energy]

$$\mathbf{T} \equiv -\frac{\omega}{2}\mathbf{E}_0^* \cdot \frac{\partial \overline{\overline{\epsilon}}^h}{\partial \mathbf{k}} \cdot \mathbf{E}_0$$

[Kinetic flux]

$$\boxed{\frac{\partial W}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{S}_0 + \mathbf{T}) = -P_{abs}}$$

$$\mathbf{S}_0 \equiv \frac{1}{\mu_0}\Re(\mathbf{E}_0^* \times \mathbf{B}_0)$$

[Poynting flux]

$$P_{abs} \equiv \omega \mathbf{E}_0^* \cdot \overline{\overline{\epsilon}}^a \cdot \mathbf{E}_0$$

[Absorbed power]

- **Dielectric tensor:** $\overline{\overline{\epsilon}} \equiv \overline{\overline{\epsilon}}^h + i\overline{\overline{\epsilon}}^a$

Collisional absorption

- **Cold plasma:**

$$\left.\begin{aligned} \overline{\overline{\epsilon}}^{a} &= 0 \quad \rightarrow \quad P_{abs} = 0 \\ \frac{\partial \overline{\overline{\epsilon}}^{h}}{\partial \mathbf{k}} = \frac{\partial \overline{\overline{\epsilon}}}{\partial \mathbf{k}} &= 0 \quad \rightarrow \quad \mathbf{T} = 0 \end{aligned}\right\}$$

No absorption,
No energy exchange

- **Friction term (Crook):** $\frac{d\mathbf{v}_s}{dt} = \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \nu_s \mathbf{v}_s$

- **Dielectric tensor:**

$$\left.\begin{aligned} L &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})} \quad \rightarrow \quad 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs} + i\nu_s)} \\ R &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} \quad \rightarrow \quad 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs} + i\nu_s)} \\ P &\equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \quad \rightarrow \quad 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + i\nu_s)} \end{aligned}\right\} \quad \overline{\overline{\epsilon}}^{a} \neq \overline{\overline{0}}$$

- **Absorbed power:**

$$P_{abs} = \sum_s \nu_s \frac{\omega_{ps}^2}{\omega^2} \left[A_1 |E^+|^2 + A_2 |E^-|^2 + A_3 |E_z|^2\right] \neq 0$$

$\longrightarrow$ **Absorption directly proportional to friction coefficient**

Thermal corrections

- **Momentum conservation:**

$$n_s m_s \frac{d\mathbf{v}_s}{dt} = n_s m_s \left(\frac{\partial \mathbf{v}_s}{\partial t} + (\mathbf{v}_s \cdot \boldsymbol{\nabla})\mathbf{v}_s \right) = n_s q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}) - \boldsymbol{\nabla} \cdot \overline{\overline{\boldsymbol{\Phi}}}_s$$

- **Cold plasma:** $\overline{\overline{\boldsymbol{\Phi}}}_s = \overline{\overline{\mathbf{0}}}$

$$\longrightarrow \quad -i\omega m_s \mathbf{v}_s = q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0)$$

- **Thermal effects:** $\overline{\overline{\boldsymbol{\Phi}}}_s = p_s \times \overline{\overline{\mathbf{1}}}$

$$\longrightarrow \quad -i\omega m_s \mathbf{v}_s = q_s (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0) - \frac{1}{n_s} \boldsymbol{\nabla} p_s$$

- **Closure:** $p_s = c_s^2 n_s m_s \,, \quad c_s^2 \equiv \gamma_s \frac{k_b T_s}{m_s}$

zero for transverse perturbations

$$\longrightarrow \quad \mathbf{v}_s = i \frac{q_s}{m_s \omega} (\mathbf{E} + \mathbf{v}_s \times \mathbf{B}_0) + \frac{c_s^2}{\omega^2} (\mathbf{k} \cdot \mathbf{v}_s) \cdot \mathbf{k}$$

- **New dielectric tensor:** $\overline{\overline{\boldsymbol{\epsilon}}} = \overline{\overline{\boldsymbol{\epsilon}}}(\mathbf{k}) = \overline{\overline{\boldsymbol{\epsilon}}}_{froid} + i \left(\frac{kc}{\omega} \right) \cdot \overline{\overline{\boldsymbol{\epsilon}}}^{(1)} - \left(\frac{kc}{\omega} \right)^2 \cdot \overline{\overline{\boldsymbol{\epsilon}}}^{(2)}$

Space-dispersive

Thermal correction: HF waves

1. High frequency: extraordinary wave

- Cold plasma (electrons only):

$$n_{X,c}^2 = \frac{(\omega^2 - \omega_{pe}^2)\cdot(\omega^2 - \omega_e^2) - \omega_{ce}^2\omega_{pe}^2}{\omega^2\cdot(\omega^2 - \omega_e^2)}$$

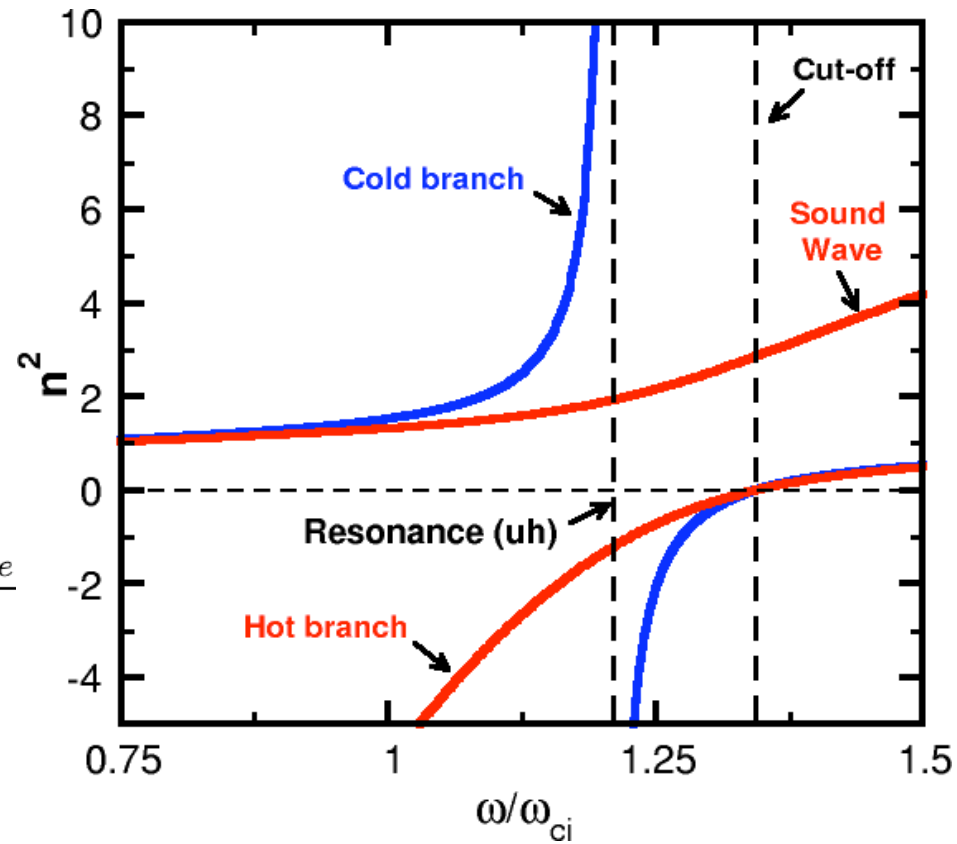
$$\omega_e^2 \equiv \omega_{pe}^2 + \omega_{ce}^2$$

- Including thermal corrections:

$$n_{X,h}^2 = \frac{(\omega^2 - \omega_{pe}^2)\cdot(\omega^2 - \omega_e^2 - k^2c_e^2) - \omega_{ce}^2\omega_{pe}^2}{\omega^2\cdot(\omega^2 - \omega_e^2 - k^2c_e^2)}$$

Thermal velocity

Dispersion relation



→ Resonance disappears

→ A new branch appears

Thermal corrections: MHD waves

2. Low frequency: MHD waves

- **Cold plasma: 2 modes**
  - Torsional Alfvén wave:

$$\frac{\omega}{\omega_{ci}} = \left(\frac{kc_s}{\omega_{ci}}\right) \cdot \left(\frac{v_a}{c_s}\right) \cdot \cos(\theta)$$

  - Compressional Alfvén wave:

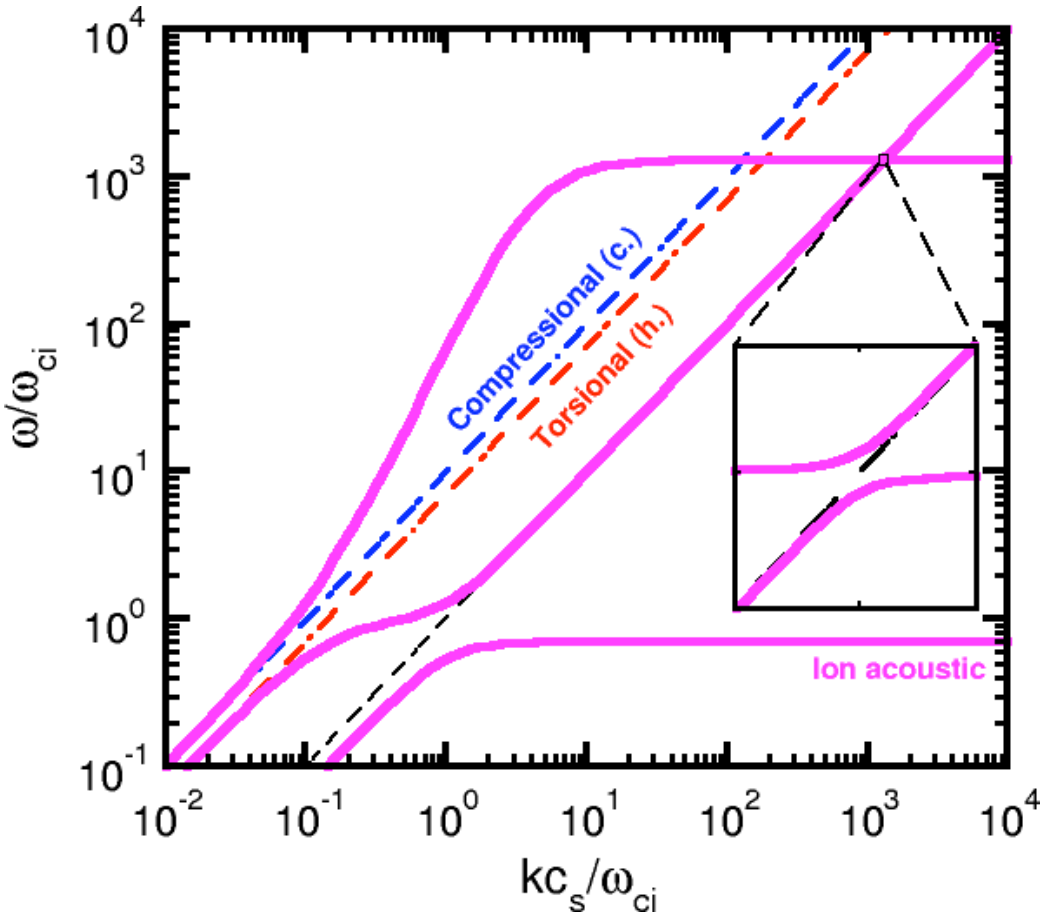
$$\frac{\omega}{\omega_{ci}} = \left(\frac{kc_s}{\omega_{ci}}\right) \cdot \left(\frac{v_a}{c_s}\right)$$

- **Hot plasma:**

  Dispersion relation of 3rd order in n^2

  → **New branch.**

Stringer diagram



Waves in a kinetic plasma

Kinetic effects

- **Microscopic effects not considered in a fluid model**
  - Non-collisional absorption phenomena
  - Finite Larmor radius effect
  - Non-Maxwellian distributions

$\longrightarrow$ **The particles must be described individually**

- **General procedure**
  - Vlasov equation:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0 \quad \longrightarrow \quad \left.\frac{df_s}{dt}\right|_{\mathbf{R}} = 0$$

Unperturbed trajectories

  - Unperturbed trajectories: $\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{q_s}{m_s}\mathbf{v} \times \mathbf{B}_0$

Equilibrium distribution

  - Perturbation (1st order) of the distribution function

$$\tilde{f}_s(\mathbf{r}, \mathbf{v}, t) = -\frac{q_s}{m_s} \int_{-\infty}^{t} dt' \left[\mathbf{E}(\mathbf{r}', t') + \mathbf{v}' \times \mathbf{B}(\mathbf{r}', t')\right] \cdot \frac{\partial F_s}{\partial \mathbf{v}'}$$

$\longrightarrow$ **New dielectric tensor**

Non-collisional absorption

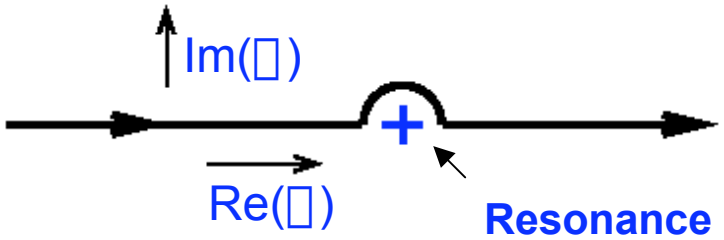
- **New dielectric tensor:** $\bar{\bar{\epsilon}} \equiv \bar{\bar{1}} + \sum_s \bar{\bar{\chi}}_s$

- **Susceptibility:**

$$\bar{\bar{\chi}}_s \equiv \frac{\omega_{ps}^2}{\omega\omega_{cs}} \sum_{n=-\infty}^{\infty} \int_0^{\infty} dv_\perp 2\pi v_\perp \int_{-\infty}^{\infty} dv_\parallel \left(\frac{\omega_{cs}}{\omega - n\omega_{cs} - k_\parallel v_\parallel} \cdot \bar{\bar{\mathbf{S}}}_n \right)$$

$$\bar{\bar{\mathbf{S}}}_n \equiv \bar{\bar{\mathbf{S}}}_n \left(\frac{\partial F_s}{\partial v_\perp}, \frac{\partial F_s}{\partial v_\parallel}, k_\perp, k_\parallel \right)$$

- **Singular denominator:**



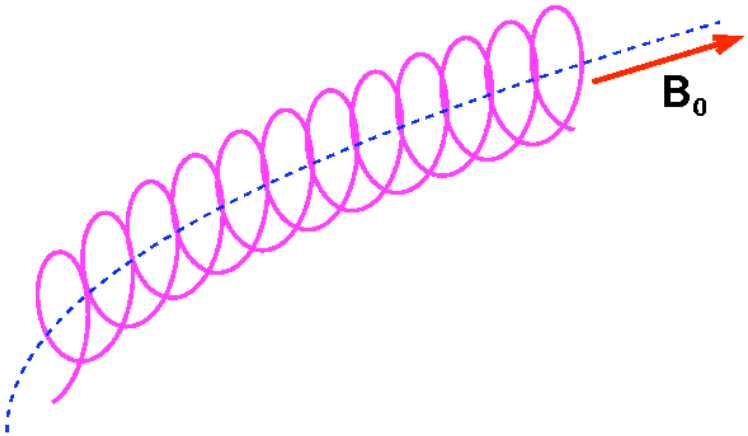
$$\lim_{\nu \to 0} \frac{1}{\omega - n\omega_{cs} - k_\parallel v_\parallel + i\nu} = \mathcal{P}\left(\frac{1}{\omega - n\omega_{cs} - k_\parallel v_\parallel} \right) - i\pi\delta(\omega - n\omega_{cs} - k_\parallel v_\parallel)$$

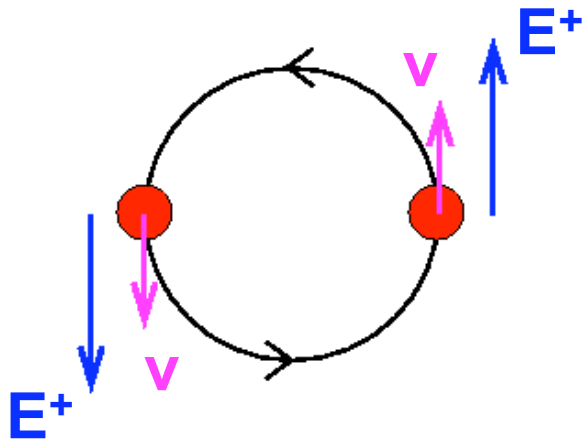
- Doppler effect ($k_{//}v_{//}$)
- Cyclotron absorption **(n ≠ 0)**
- Landau damping, TTMP **(n = 0)**

Non-collisional absorption processes

Cyclotron absorption

- **Interaction with particle gyration:** $\omega = n\Omega_{cs} + k_{//}v_{//}$

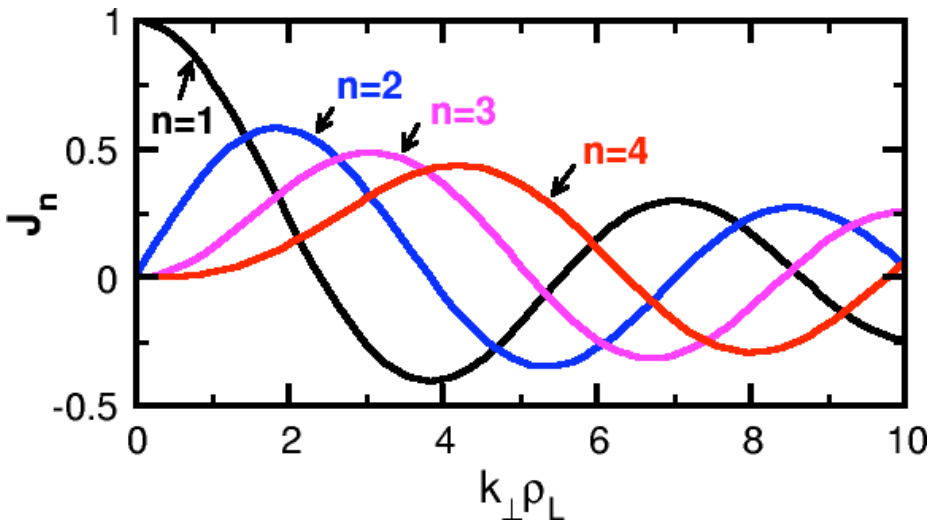




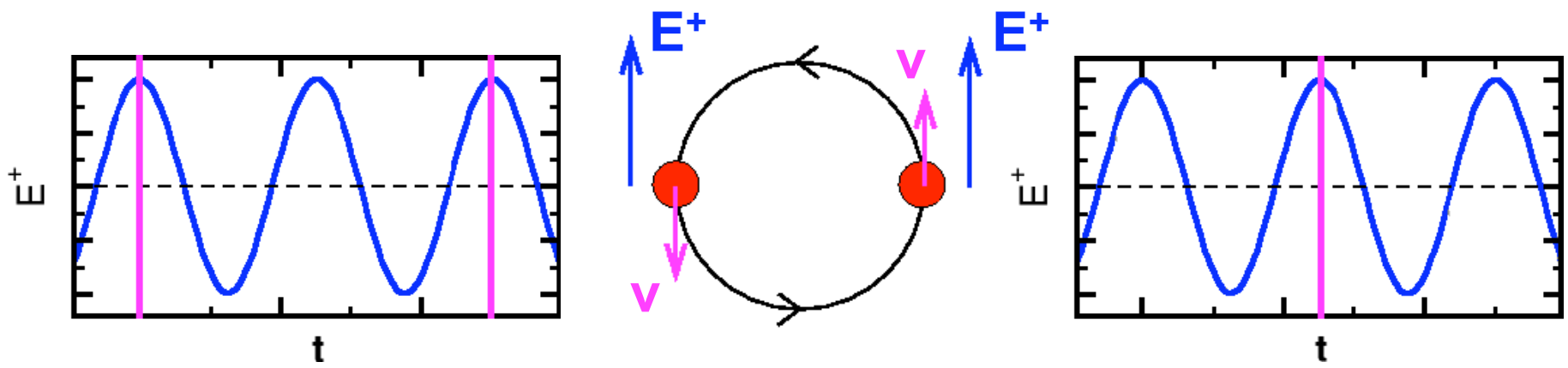
- **Particle acceleration:**

$$\frac{dW_\perp}{dt} \equiv \frac{d}{dt}\left(\frac{m_s v_\perp^2}{2}\right) = q_s \mathbf{v}_\perp \cdot \mathbf{E}$$

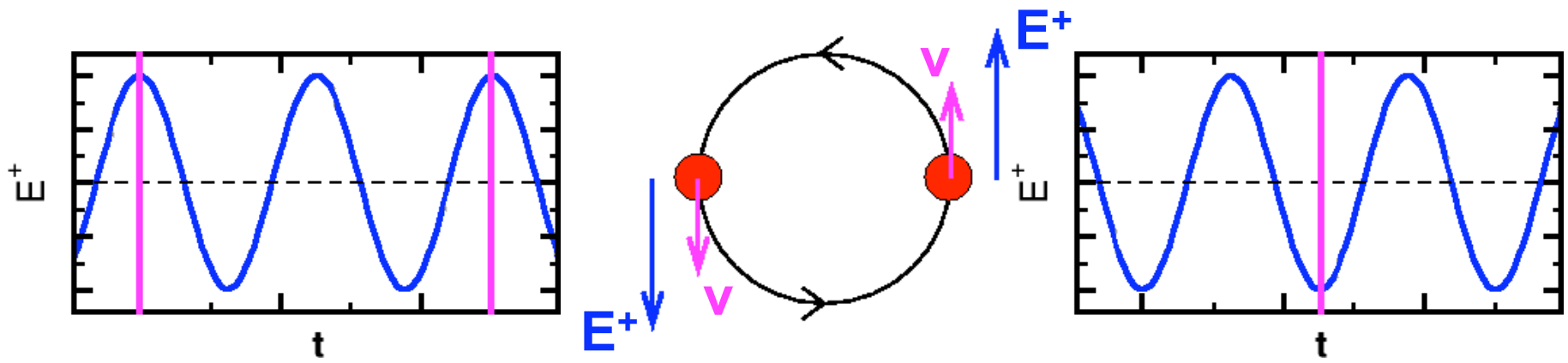
$$\longrightarrow \quad \frac{dW_\perp}{dt} \propto J_n(k_\perp \rho_L)$$



First harmonic absorption



$E=E_0\exp(i\omega t)$ ⟶ **No energy is transferred**



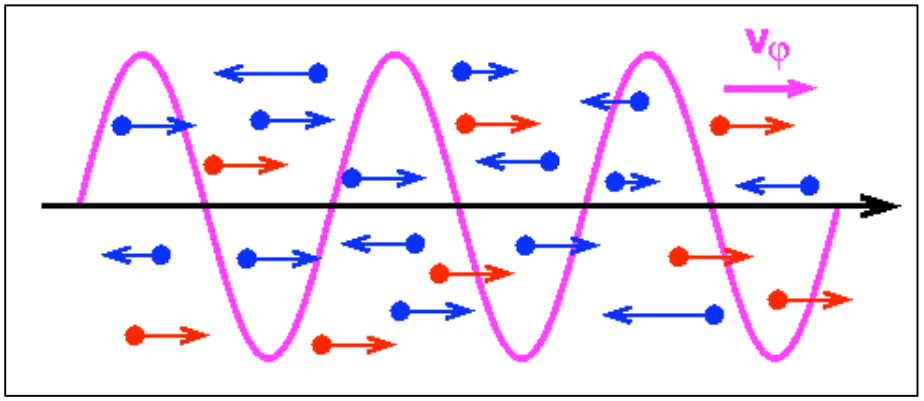
$E=E_0\exp(i(k_x x-\omega t))$, $k_x=\omega/2\beta_L$ ⟶ **Particle gets accelerated**

Landau damping, TTMP

- **Landau damping:** $\omega = k_{//} v_{//}$
  - **Oscillating field:** $\mathbf{E = E_0 \cos(\omega t - k_{//} z)}$

    ⟶ Interaction with the particles having $v_{//} = \omega / k_{//}$ ($\pm \delta v_{//}$)

    ⟶ Emission / Absorption

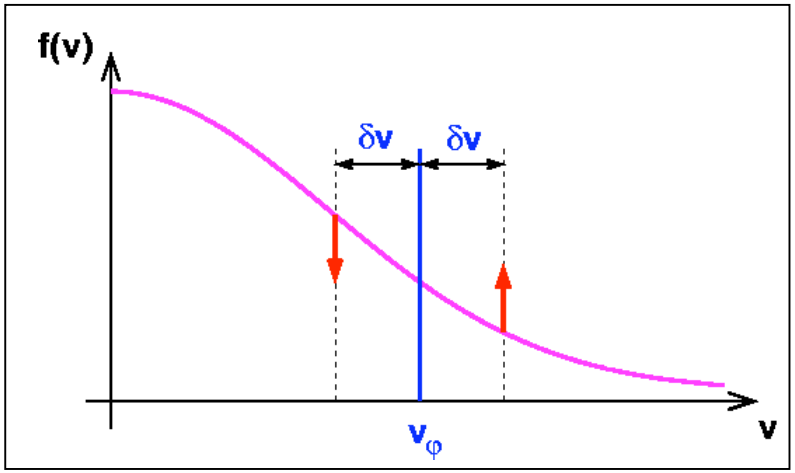


  - **Parallel energy modification ($W_{//}$):**

$$\frac{dW_{\|}}{dt} = W_{\|} \times \pi \cdot \frac{\omega \omega_p^2}{k_{\|} |k_{\|}|} \cdot \left. \frac{\partial F_s}{\partial v_{\|}} \right|_{v_{\|} = \omega / k_{\|}}$$

    ⟶ Sensitivity to the detailed shape of F_s

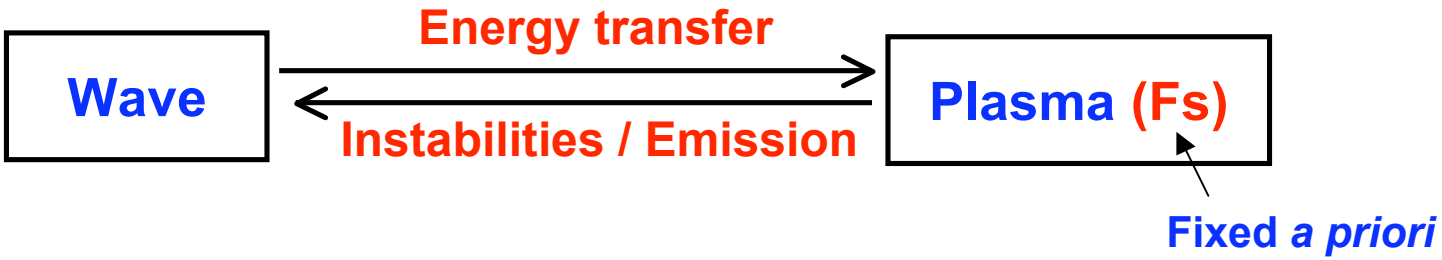
    ⟶ Linear theory framework becomes inadequate



- **Transit Time Magnetic Pumping:** $\omega = k_{//} v_{//}$

  ⟶ Similar to Landau damping, but with $\mathbf{q_s E \propto -\nabla B}$

Limitations of a linear description



Ex: unstable distribution function

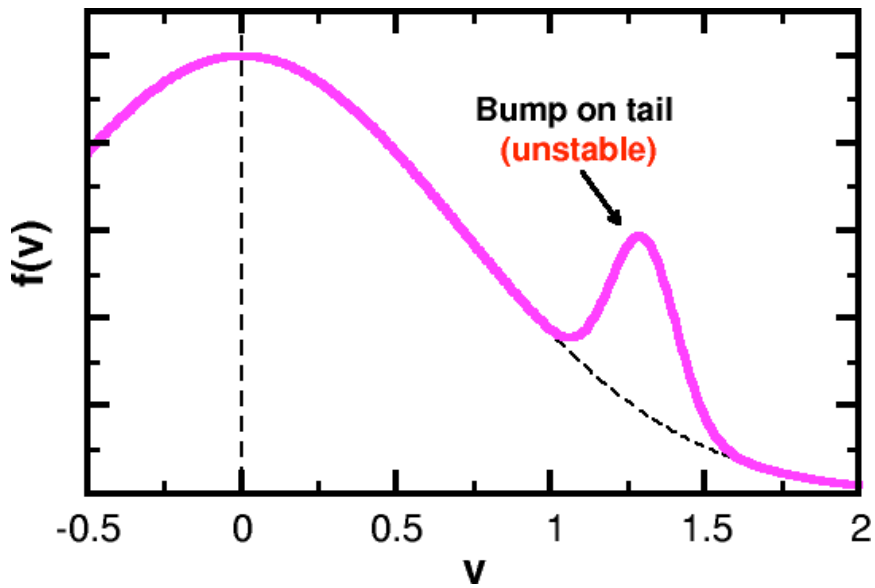
- Linear growing rate

$$\gamma = \frac{\pi}{2} \cdot \frac{\omega \omega_p^2}{k|k|} \cdot \left. \frac{\partial F_s}{\partial v} \right|_{v=\omega/k}$$

- Instability amplitude:

$$A \propto |E|^2 \propto \exp(2\gamma t)$$

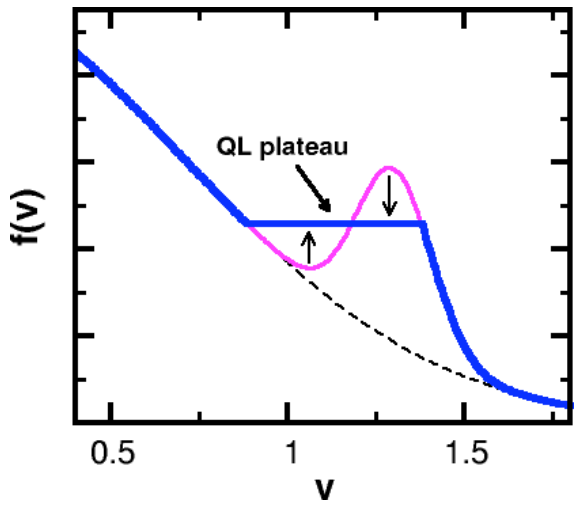
→ Infinite growth !?



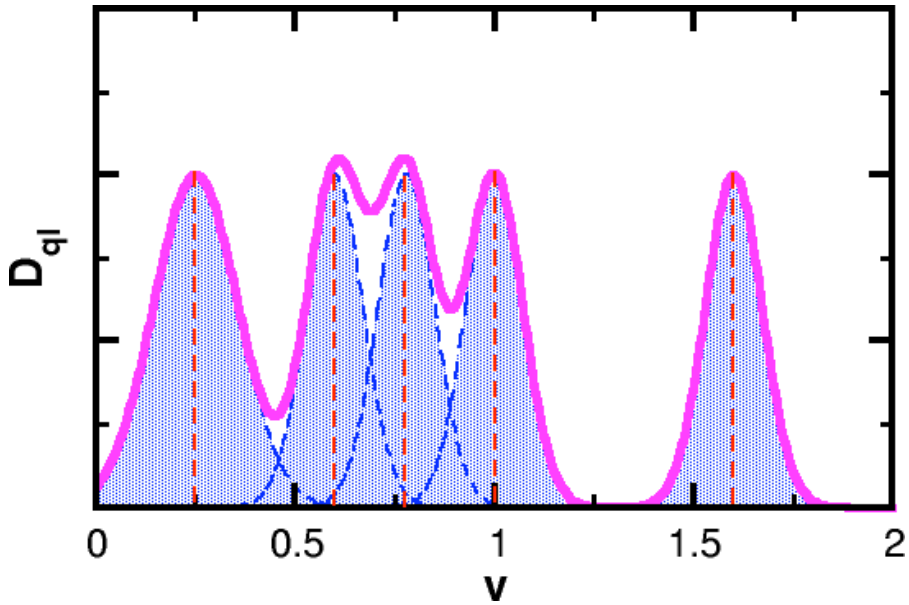
- **Up to what level does the instability really grow ?**
- **What is the saturation mechanism ?**
- **How is the equilibrium distribution function modified ?**

Quasilinear theory

- **A few features:**
  - **Feedback of the wave on F_s**
    - → F_s diffuses in velocity space
    - → Flattening in the vicinity of resonances
  - **Diffusion coefficient: $D_{ql} \propto |E_{lin}|^2$**



- **Fundamental hypotheses:**
  - **Moderate amplitudes**
    - → zero-th order orbits still defined
  - **Dense wave spectrum**
    - → No coherent phenomenon



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