41st Culham Plasma Physics Summer School

Wave-Particle Interaction in Plasmas

Rémi Dumont

Association Euratom-CEA
CEA/DSM/DRFC, Centre de Cadarache
13108 Saint-Paul lez Durance, France

Acknowledgement: A. Bécoulet, Y. Sarazin

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Introduction

The wave problem

- Maxwell's equations:

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_{ext}, \qquad \boldsymbol{\nabla} \cdot \mathbf{B} = 0$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}_{ext} + \frac{\partial \mathbf{D}}{\partial t}$$

  - $\mathbf{E}/\mathbf{D}$: Electric field / displacement
  - $\mathbf{H}/\mathbf{B}$: Magnetic intensity / induction
  - $\mathbf{j}_{ext}$: Source current
  - ρ_{ext}: External charge density

- Anisotropic medium:

$$\begin{cases} \mathbf{B} = \overline{\overline{\boldsymbol{\mu}}} \cdot \mathbf{H} \\ \mathbf{D} = \overline{\overline{\boldsymbol{\epsilon}}} \cdot \mathbf{E} \end{cases}$$

$\overline{\overline{\boldsymbol{\mu}}}$: magnetic permeability, $\overline{\overline{\boldsymbol{\epsilon}}}$: dielectric permittivity

- Plasma medium:

$$\begin{cases} \mathbf{B} = \mu_0 \mathbf{H} \\ \mathbf{D} = \overline{\overline{\boldsymbol{\epsilon}}} \cdot \mathbf{E} \end{cases}$$

- Link electric field - electric displacement:

$$\mathbf{j}_{tot} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j}_{ext} + \frac{\partial \mathbf{D}}{\partial t}$$

- Current coherence: $\mathbf{j}_{tot} \equiv \mathbf{j} + \mathbf{j}_{ext}$

$$\begin{cases} \mathbf{j}_{tot}\text{: total current} \\ \mathbf{j}\text{: “response current”} \end{cases}$$

- Harmonic fields: $(\mathbf{E}, \mathbf{B}) \propto \exp(i\omega t)$

$$\mathbf{D} = \epsilon_0 \left(\mathbf{E} + \frac{i}{\omega \epsilon_0} \mathbf{j} \right)$$

- Constitutive relation of the medium:

$$\mathbf{j} \stackrel{?}{=} \mathbf{j}(\mathbf{E})$$

- Linear response: $(\rho, \mathbf{j}) \propto \mathbf{E}$

$$\mathbf{j} = \bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{E} \quad \rightarrow \quad \mathbf{D} = \bar{\bar{\boldsymbol{\epsilon}}} \cdot \mathbf{E}$$

$\bar{\bar{\boldsymbol{\sigma}}}$: conductivity tensor, $\bar{\bar{\boldsymbol{\epsilon}}}$: dielectric tensor

- Statistical description of the plasma:

$$\mathbf{j} = \sum_s q_s \int d^3\mathbf{v}\, \mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \equiv \bar{\bar{\boldsymbol{\sigma}}} \cdot \mathbf{E}$$

f_s: Distribution function for species s

- Collisionless kinetic equation (Vlasov) for f_s:

$$\frac{df_s}{dt} \equiv \frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0$$

Over the wave-particle interaction timescale, collisions can be safely neglected.

- Linearisation:

$$\begin{cases} f_s \equiv f_{s,0} + \delta f_s(t) \\ \mathbf{E} \equiv \mathbf{E}_0 + \delta \mathbf{E}(t) \\ \mathbf{B} \equiv \mathbf{B}_0 + \delta \mathbf{B}(t) \end{cases}$$

  - $(\mathbf{B}_0, \mathbf{E}_0)$: equilibrium electromagnetic field
  - $f_{s,0}$: equilibrium distribution function
  - $(\delta\mathbf{B}, \delta\mathbf{E}) \propto \exp(i\omega t)$: wave electromagnetic field
  - $\delta f_s(t) \propto \exp(i\omega t)$: response distribution function

- Linearisation: zero-order

$$\mathbf{v} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = 0$$

- Linearisation: first-order

$$\begin{aligned} \frac{d\delta f_s}{dt} \equiv & \frac{\partial \delta f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial \delta f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \\ = & -\frac{q_s}{m_s}(\delta\mathbf{E} + \mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} \end{aligned}$$

- Solution (first order distribution function)

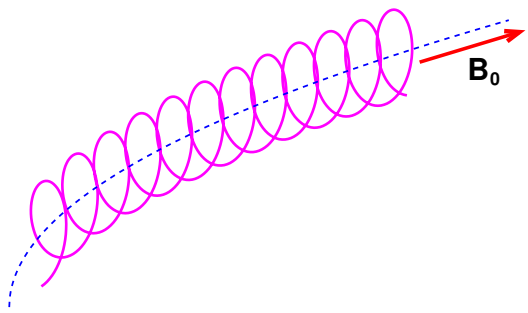
$$\delta f_s(\mathbf{r},\mathbf{v},t) = -\frac{q_s}{m_s}\int_{-\infty}^{t} dt'\big[\delta\mathbf{E} + \mathbf{v}' \times \delta\mathbf{B}\big] \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}'}$$

The integral is performed over *the unperturbed orbits* (characteristics of the Vlasov equation)

- Unperturbed orbits (characteristics):

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = \frac{q_s}{m_s}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0)$$

- In a tokamak: $\mathbf{E}_0 \approx \mathbf{0}$

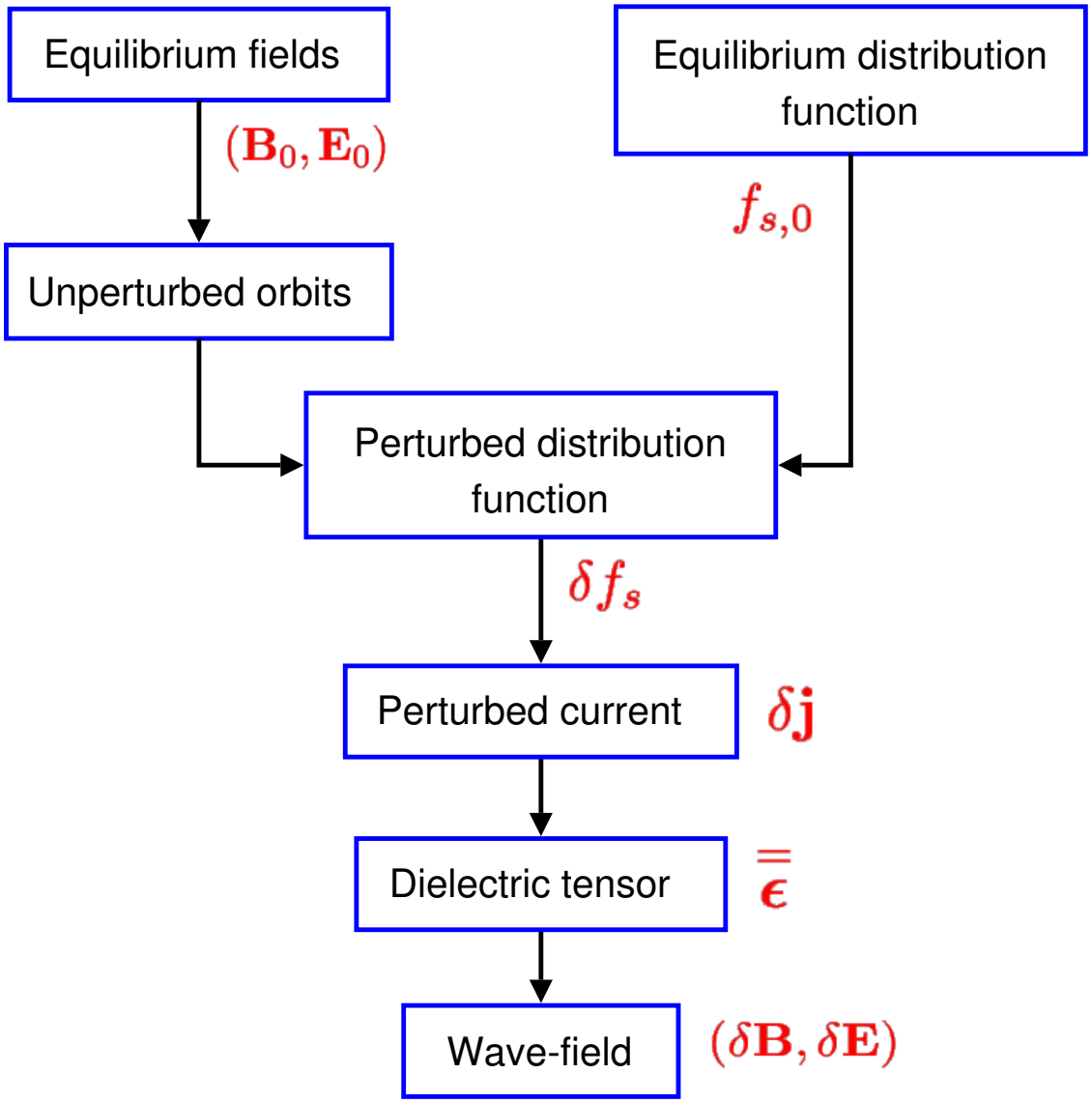


Various approximations are available to obtain the unperturbed orbits $\longrightarrow$ δf_s can be computed.

$$\delta\mathbf{j}(\mathbf{r},t) = \sum_s q_s \int d^3\mathbf{v}\, \mathbf{v}\, \delta f_s(\mathbf{r},\mathbf{v},t) = \bar{\bar{\boldsymbol{\sigma}}} \cdot \delta\mathbf{E}$$

$\longrightarrow$ $\bar{\bar{\boldsymbol{\sigma}}}, \bar{\bar{\boldsymbol{\epsilon}}}$ can be deduced.

- The wave problem: summary



The wave-field is obtained by solving the linearised Maxwell-Vlasov system

The kinetic problem

- Kinetic equation for f_s:

$$\frac{\partial f_s}{\partial t} + \mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = \left(\frac{\partial f_s}{\partial t}\right)_{coll}$$

- Linear treatment:
  - Linearisation:

$$f_s(\mathbf{r}, \mathbf{v}, t) \equiv f_{s,0}(\mathbf{r}, \mathbf{v}) + \delta f_s(\mathbf{r}, \mathbf{v}, t)$$

  - $\delta f_s \propto \exp(i\omega t)$ and $f_{s,0}$ is constant.
  - Collisions neglected (too infrequent to be relevant on the wave period timescale)

- Quasilinear treatment:
  - $f_{s,0}$ may vary, on a slow time scale compared to $2\pi/\omega$

$$\longrightarrow \begin{cases} \langle \delta f_s(t) \rangle = 0 \\ \langle f_s(t) \rangle = f_{s,0}(t) \end{cases}$$

    where $\langle \cdot \rangle$ denotes the time-average over the fast timescale.
  - Collisions have to be included.

- Time-averaged kinetic equation

$$\left\langle \frac{df_s}{dt} \right\rangle = \frac{df_{s,0}}{dt} \equiv \frac{\partial f_{s,0}}{\partial t} + \mathbf{v} \cdot \frac{\partial f_{s,0}}{\partial \mathbf{r}} + \ldots$$

$$\ldots + \frac{q_s}{m_s}(\mathbf{E}_0 + \mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{s,0}}{\partial \mathbf{v}} = \hat{C}(f_{s,0}) + \hat{Q}(f_{s,0})$$

  - $\hat{C}(f_{s,0})$: Collision (Fokker-Planck) operator
  - $\hat{Q}(f_{s,0})$: Quasilinear operator

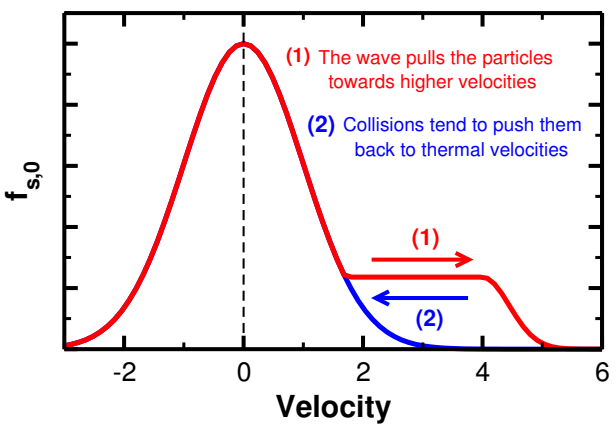
$$\hat{Q}(f_{s,0}) = -\frac{q_s}{m_s}\left\langle (\delta\mathbf{E} + \mathbf{v} \times \delta\mathbf{B}) \cdot \frac{\partial \delta f_s}{\partial \mathbf{v}} \right\rangle$$

- Fokker-Planck form of the quasilinear equation

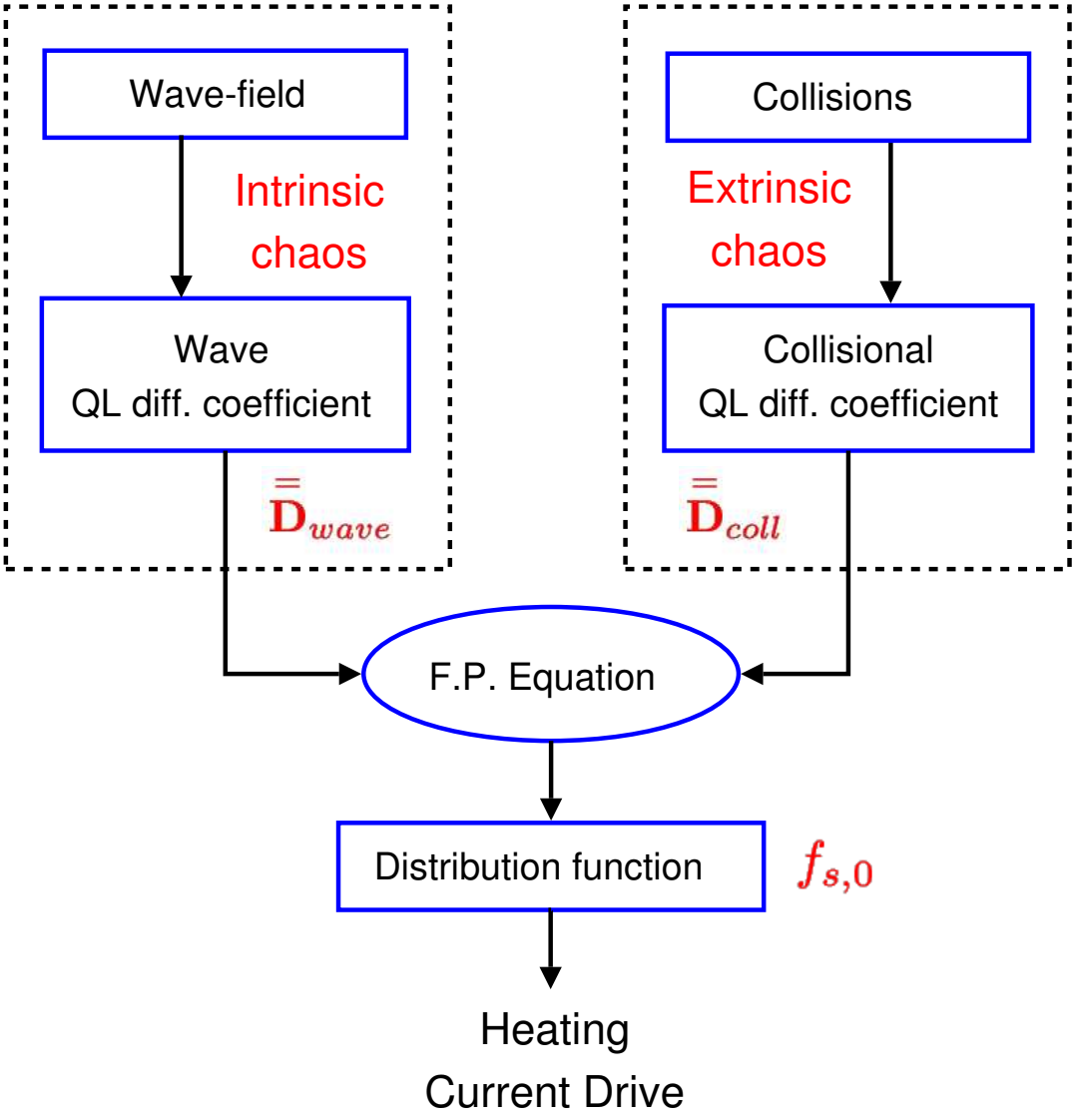
$$\frac{df_{s,0}}{dt} = \underbrace{\frac{\partial}{\partial v_i} D_{ij}^{coll} \frac{\partial f_{s,0}}{\partial v_j}}_{\text{Collisional diffusion}} + \underbrace{\frac{\partial}{\partial v_i} D_{ij}^{wave} \frac{\partial f_{s,0}}{\partial v_j}}_{\text{Wave-induced diffusion}}$$

$\longrightarrow$ Two competing mechanisms

$\longrightarrow$ New equilibrium distribution function $f_{s,0}$

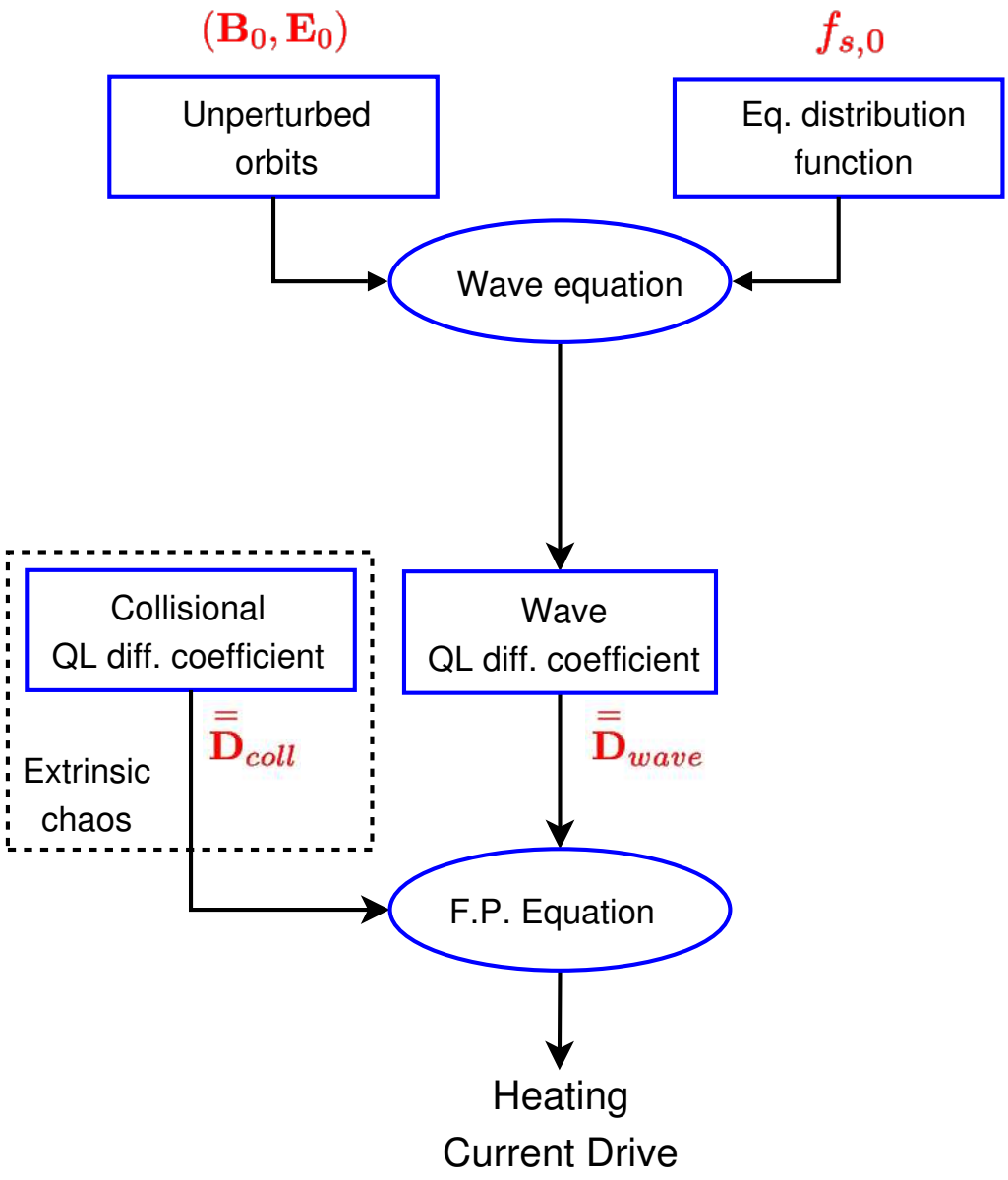


- The kinetic problem: summary



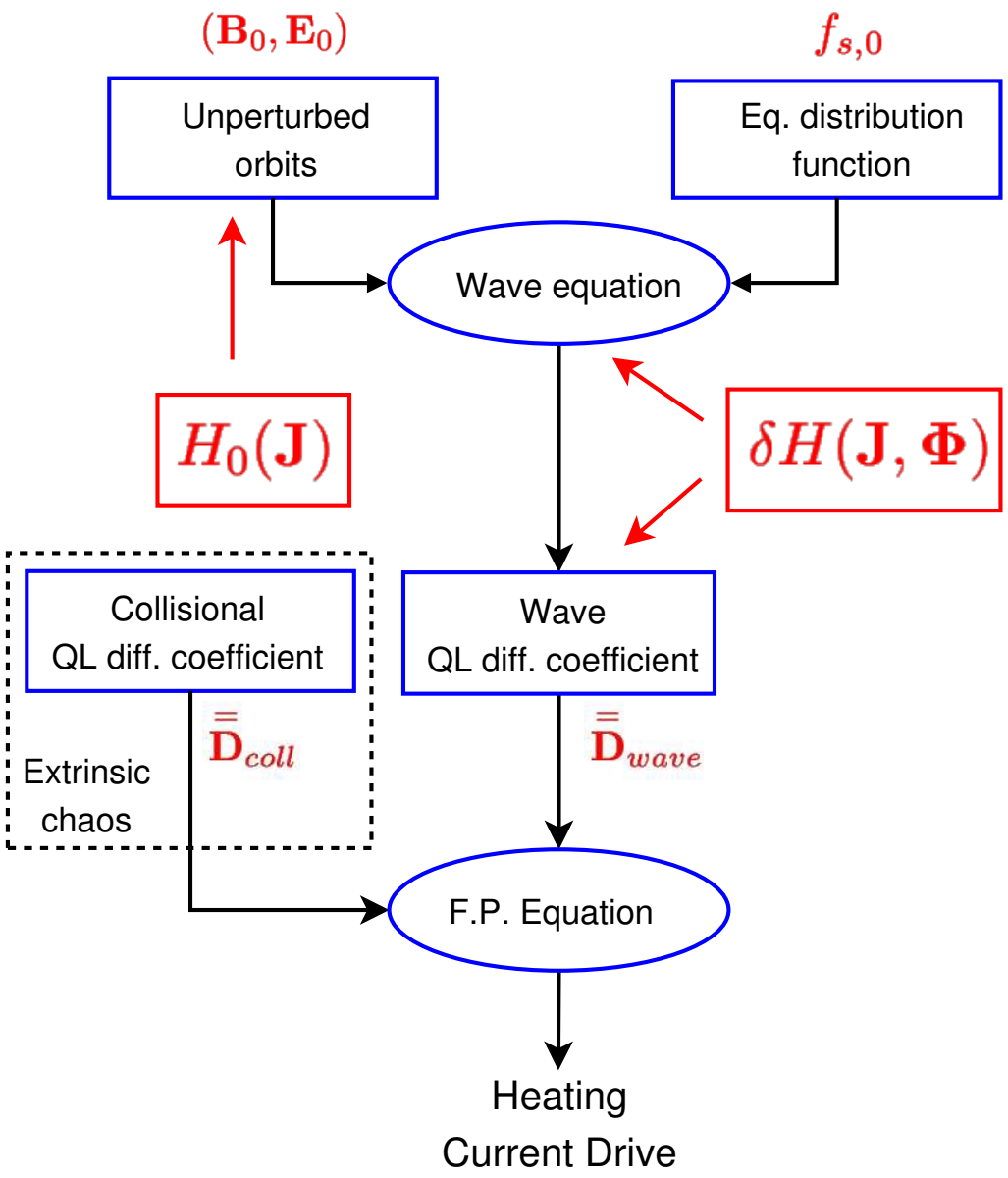
The distribution function taking into account the secular effect of the wave is computed by solving the Fokker-Planck equation

General picture



In a "classical" Newton description, the computation of the unperturbed orbits and of the quasilinear operators are two very complicated problems.

General picture



A Hamiltonian description is well adapted to tackle the wave + kinetic problems in a self-consistent way

The Hamiltonian approach

Unperturbed orbits

Mechanics fundamentals

Lagrangian approach: We define the *Lagrangian*

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) \equiv T(\mathbf{q}, \dot{\mathbf{q}}) - V(\mathbf{q}, t)$$

- T: Kinetic energy, V: Potential energy
- $\mathbf{q}, \dot{\mathbf{q}}$: generalised position, velocity

The equations of motion are $(1 \leq i \leq n)$:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

$\longrightarrow$ n 2nd-order equations (*Euler-Lagrange*).

N.B.: Noting that $T = 1/2m\dot{\mathbf{q}}^2$, the Euler-Lagrange equations yield

$$m\ddot{\mathbf{q}} + \boldsymbol{\nabla} V = 0$$

If we suppose that the force $\mathbf{f}$ derives from the potential V, i.e. $\mathbf{f} = -\boldsymbol{\nabla} V$, then we obtain

$$m\ddot{\mathbf{q}} = \mathbf{f}$$

which is the usual Newton equation of motion.

Hamilton approach: We define the *Hamiltonian*

$$H(\mathbf{p}, \mathbf{q}, t) \equiv \sum_i \dot{q}_i p_i - \mathcal{L}(\dot{\mathbf{q}}, \mathbf{q}, t)$$

and the *generalised momentum*

$$p_i \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

The *Hamilton's equations* are

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad -\dot{p}_i = \frac{\partial H}{\partial q_i}$$

$\longrightarrow$ Set of $2n$ first-order equations

$\longrightarrow$ $(\mathbf{q}, \mathbf{p})$ are said to be *conjugate*.

N.B.: In many cases, when the system is conservative, we have $H = T + V = E$, with E the total energy. Beware, however, this is not always true !

The power of Hamilton's approach lies in the freedom of choice of the generalised coordinates. We can use a *canonical transformation* to

transform the set $(\mathbf{p}, \mathbf{q})$ into the set $(\mathbf{p}', \mathbf{q}')$, with corresponding Hamiltonian H'

$$H'(\mathbf{p}', \mathbf{q}', t) = H(\mathbf{p}, \mathbf{q}, t) + \frac{\partial}{\partial t} F_1(\mathbf{q}, \mathbf{q}', t)$$

$F_1(\mathbf{q}, \mathbf{q}', t)$ is the *generating function* and is determined by the relations

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad p_i' = -\frac{\partial F_1}{\partial q_i'}$$

Cyclic coordinates: if q_j does not appear in $\mathcal{L}$, then

$$\frac{d}{dt} p_j = 0 \quad \rightarrow \quad p_j = \text{const.}$$

$\longrightarrow$ q_j is called a *cyclic* coordinate.

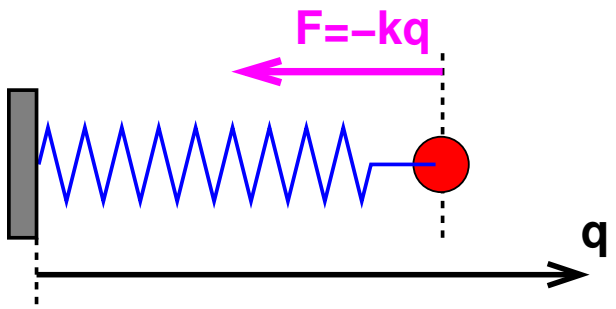
Suppose that all q_i's are cyclic, then $H = H(p_1, \ldots, p_n)$ and

$$p_i \equiv \alpha_i = \text{const.}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \equiv \omega_i \quad \rightarrow \quad q_i = \omega_i t + \beta_i$$

If we can find the generating function F_1 to transform our original set of coordinates into this one, the integration of the equations of motion will be trivial.

The harmonic oscillator: a classical example



The Hamiltonian for this system is

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} \equiv \frac{1}{2m}(p^2 + m^2\omega^2 q^2)$$

By using Hamilton equations or standard Newton technique, it is readily shown that the equations of motion write (E is the system energy)

$$q = \frac{2E}{m\omega^2}\sin(\omega t + \alpha), \quad p = \sqrt{2mE}\cos(\omega t + \alpha)$$

where α is an integration constant.

Instead, we can use the generating function

$$F_1(q, q') = \frac{m\omega q^2}{2} \cot(q')$$

to obtain a new set of coordinates (p', q') related to (p, q) through

$$q = \sqrt{\frac{2p'}{m\omega}} \sin(q'), \quad p = \sqrt{2p'm\omega} \cos(q')$$

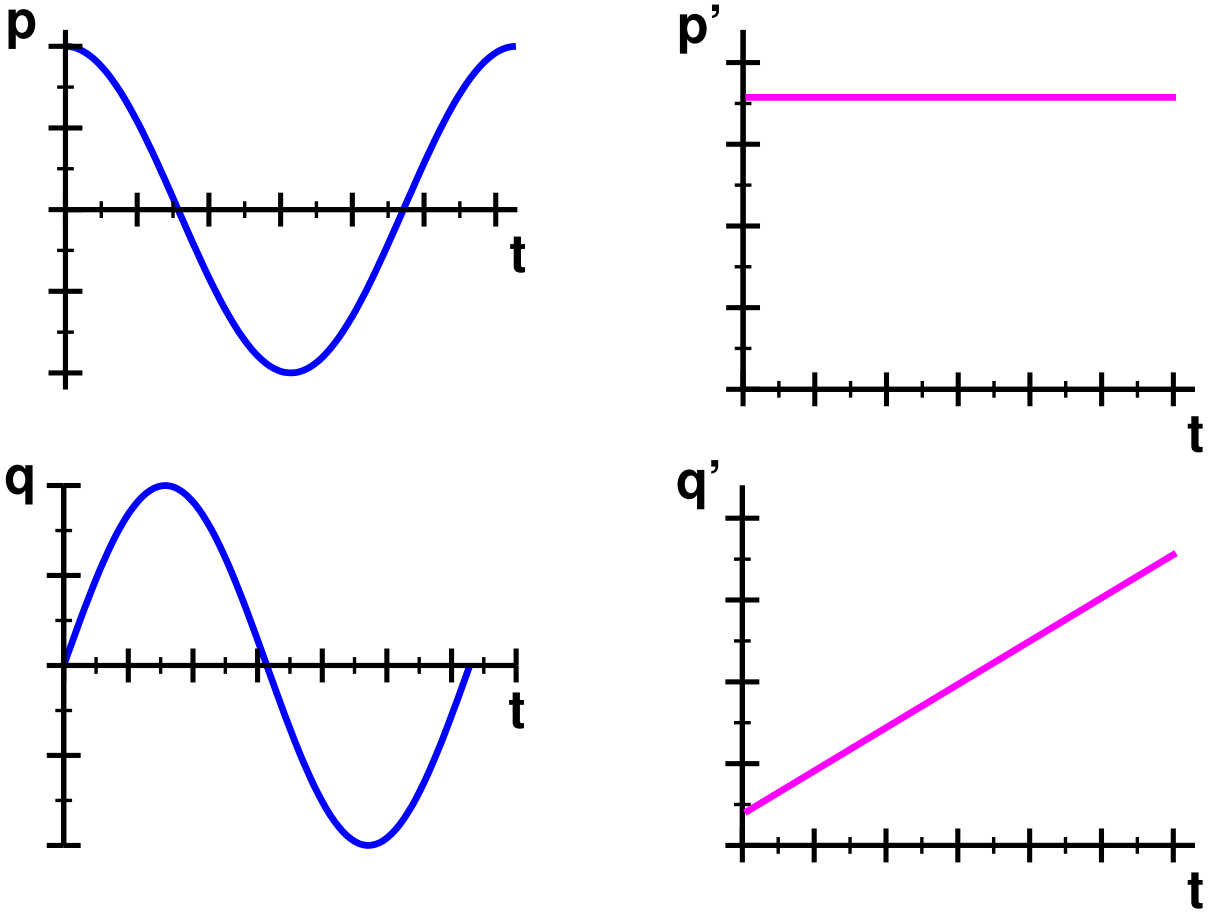
The new Hamiltonian is $H = \omega p'$ and q' is a cyclic variable, so that

$$p' = \text{const.} = E/\omega$$

Moreover

$$\dot{q}' = \frac{\partial H}{\partial p'} = \omega \quad \rightarrow \quad q' = \omega t + \alpha$$

We can compare the motion in the old *(left)* and new *(right)* coordinate systems

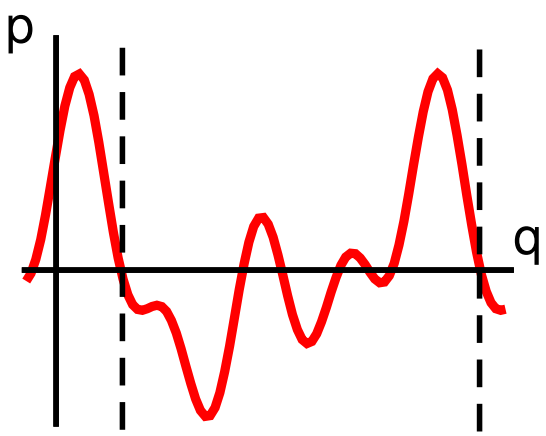


Question: how does one find the appropriate generating function ?

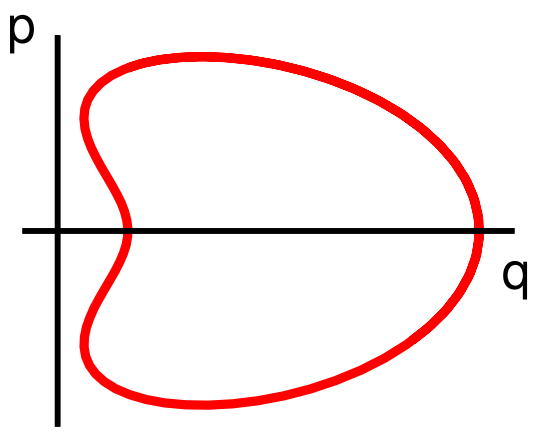
Answer: The *Hamilton-Jacobi theory* provides a framework to obtain the canonical transform in a more systematic fashion.

Periodic motion:

Types of periodic motion:



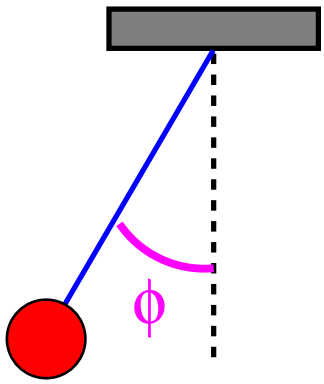
Rotation



Libration

- 1) Rotation: p_i is a periodic function of q_i.
  e.g. *simple pendulum*
- 2) Libration: p_i and q_i are periodic
  e.g *harmonic oscillator, simple pendulum*

Illustration: the simple pendulum

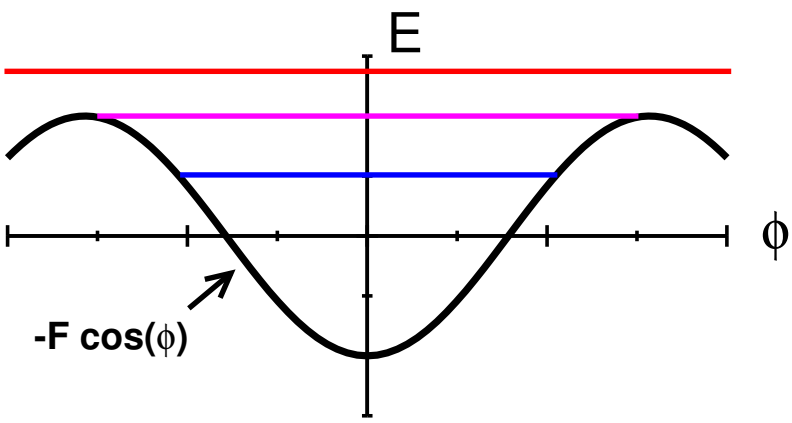


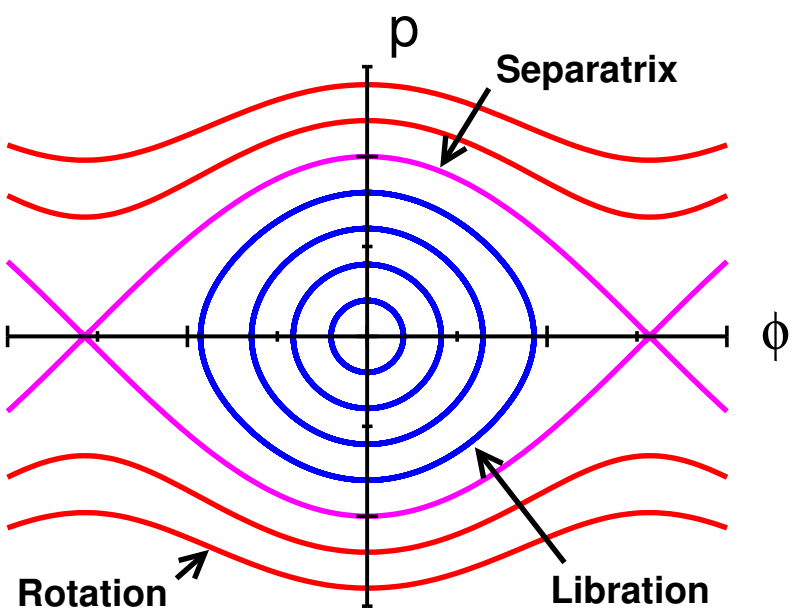
The Hamiltonian can be written as

$$H = \frac{p^2}{2m} - F\cos(\phi)$$

with $F \equiv mgl$, l being the pendulum length.

Both the libration and the rotation are possible motions $\longrightarrow$ the *island structure*





Action-Angle variables

We define the *action*:

$$J \equiv \frac{1}{2\pi} \oint p dq$$

The integral is to be performed over a period of libration or rotation of the system.

- For a time-invariant Hamiltonian $H(p,q) = \alpha$, we can show that :
  - J is a constant of motion
  - $H = H(J)$ and Φ, the conjugate variable to J is obtained using Hamilton's equation

$$\dot{\Phi} = \frac{\partial H}{\partial J} = \omega(J) \quad \rightarrow \quad \Phi = \omega t + \beta$$

  with ω the frequency of motion.
- This means that once $H(J)$ is known, ω, **the frequency, is directly available** without needing to solve the equations of motion. (J, Φ) are the *action-angle variables*.
- This formalism may be extended to the case of a slowly varying Hamiltonian $H(p,q,t)$ (adiabatic system). J is then called an *adiabatic invariant*.

Application to the harmonic oscillator:

The Hamiltonian for this system

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} = E$$

is used to compute the action

$$J = \frac{1}{2\pi} \oint p dq = \frac{1}{2\pi} \oint \sqrt{2mE - mkq^2} dq$$

which yields

$$H = J\sqrt{\frac{k}{m}}$$

We obtain immediately

$$\omega = \frac{\partial H}{\partial J} = \sqrt{\frac{k}{m}}$$

Note that we did not have to actually solve the equations of motion !

Unperturbed particle trajectories

Generalisation to a three-dimensional (3-D) system:

The confinement hypothesis implies that the system is quasi-periodical

This has the strong consequence that the unperturbed orbits can be described using:

- **Three action variables**: J_1, J_2 and J_3
- **Three angle variables**: Φ_1, Φ_2 and Φ_3
- **Three frequencies**: ω_1, ω_2 and ω_3

In a tokamak, the transformation from $(\mathbf{q}, \mathbf{p})$ to $(\mathbf{J}, \mathbf{\Phi})$ is approximately known, so that the unperturbed particles orbits can be described using the unperturbed Hamiltonian $H_0 = H_0(\mathbf{J})$ and the associated relations

$$\frac{dJ_k}{dt} = -\frac{\partial H_0}{\partial \Phi_k} = 0$$

and

$$\frac{d\Phi_k}{dt} = \frac{\partial H_0}{\partial J_k} = \omega_k(\mathbf{J})$$

The Hamiltonian approach

Plasma-Wave interaction

Hamiltonian perturbation theory

Resonant perturbation:

The wave is considered as a perturbation to the system, and its effects are described by δH

$$H = H_0(\mathbf{J}) + \delta H(\mathbf{J}, \mathbf{\Phi}, t)$$

H_0 is the unperturbed Hamiltonian, used to obtain the particle orbits. A Fourier expansion of δH over the angle variables can be performed

$$\delta H(\mathbf{J}, \mathbf{\Phi}, t) = \sum_{\mathbf{N}} h_{\mathbf{N}}(\mathbf{J}) \exp\left(i(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)\right)$$

where $\mathbf{N} \equiv (N_1, N_2, N_3)$ is a triplet of integers.

The resonance between the mode $\mathbf{N}$ of the wave and the particle occurs when (stationary phase)

$$\Omega_{\mathbf{N}}(\mathbf{J}) = \frac{d(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)}{dt} = \sum_{k=1}^{3} N_k \omega_k(\mathbf{J}) + \omega = 0$$

$\rightarrow$ Surface $\mathbf{J} = \mathbf{J}_R$ in action space $(\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3)$

Single resonant mode: idealised case

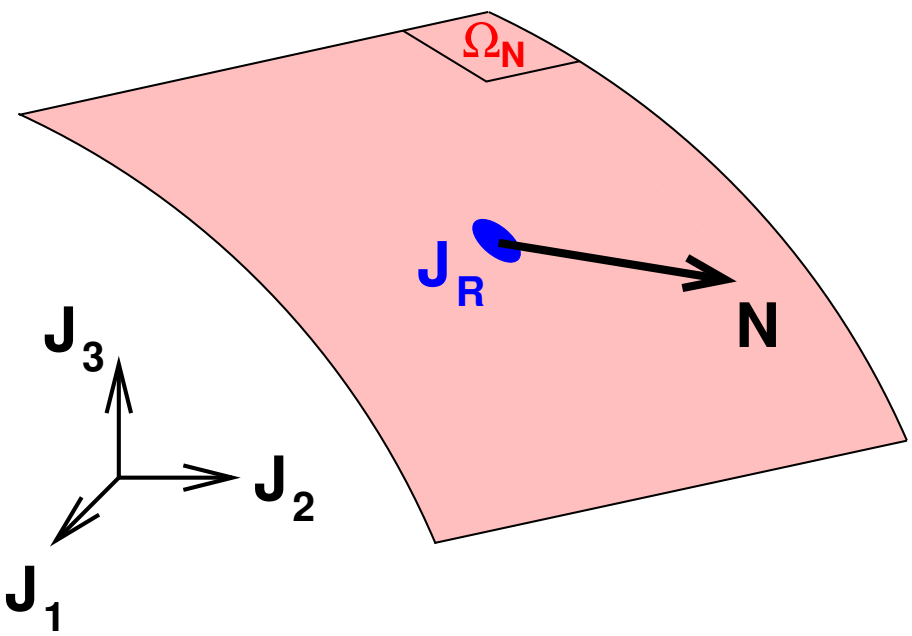
Let us consider the excitation of a mode $(\mathbf{N})$

$$\delta H(\mathbf{J}, \mathbf{\Phi}, t) = s(\mathbf{J}) \cos(\mathbf{N} \cdot \mathbf{\Phi} - \omega t)$$

The particle motion in the wave field is obtained from Hamilton's equation

$$\frac{d\mathbf{J}}{dt} = -\frac{\partial(H_0 + \delta H)}{\partial \mathbf{\Phi}} = -\frac{\partial \delta H}{\partial \mathbf{\Phi}} = \ldots$$
$$\ldots = \mathbf{N} s(\mathbf{J}) \sin(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)$$

$\rightarrow$ The particle oscillates in the direction determined by $\mathbf{N} \equiv (N_1, N_2, N_3)$.



Letting $\mathbf{J} \equiv \mathbf{J}_R + \lambda \mathbf{N}$ and $\varphi \equiv \mathbf{N} \cdot \mathbf{\Phi} + \omega t$, we obtain

$$\frac{d\lambda}{dt} = s(\mathbf{J}) \sin(\varphi)$$

We deduce from Hamilton's equations

$$\frac{d\mathbf{\Phi}}{dt} = \frac{\partial(H_0 + \delta H)}{\partial \mathbf{J}} = \boldsymbol{\omega}(\mathbf{J}) + \frac{ds}{d\mathbf{J}} \cos(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)$$

with $\boldsymbol{\omega} \equiv (\omega_1, \omega_2, \omega_3)$, which can be rewritten as

$$\frac{d\varphi}{dt} = \big(\mathbf{N} \cdot \boldsymbol{\omega}(\mathbf{J}) + \omega\big) + \mathbf{N} \cdot \frac{ds}{d\mathbf{J}} \cos(\varphi)$$

In the vicinity of the resonance, we can perform the expansion

$$\mathbf{N} \cdot \boldsymbol{\omega}(\mathbf{J}) + \omega \approx \underbrace{\mathbf{N} \cdot \boldsymbol{\omega}(\mathbf{J}_R) + \omega}_{0} + \lambda \sum_{k,l=1}^{3} N_k N_l \frac{\partial \omega_k}{\partial J_l}\bigg|_{\mathbf{J}_R}$$

$$= \lambda \sum_{k,l=1}^{3} N_k N_l \frac{\partial^2 H_0}{\partial J_k \partial J_l}\bigg|_{\mathbf{J}=\mathbf{J}_R}$$

We define the *local (unperturbed) Hamiltonian curvature*

$$\alpha \equiv \sum_{k,l=1}^{3} N_k N_l \frac{\partial^2 H_0}{\partial J_k \partial J_l}\bigg|_{\mathbf{J}=\mathbf{J}_R}$$

so that we obtain the equations of motion

$$\frac{d\lambda}{dt} = s(\mathbf{J})\sin(\varphi)$$

and

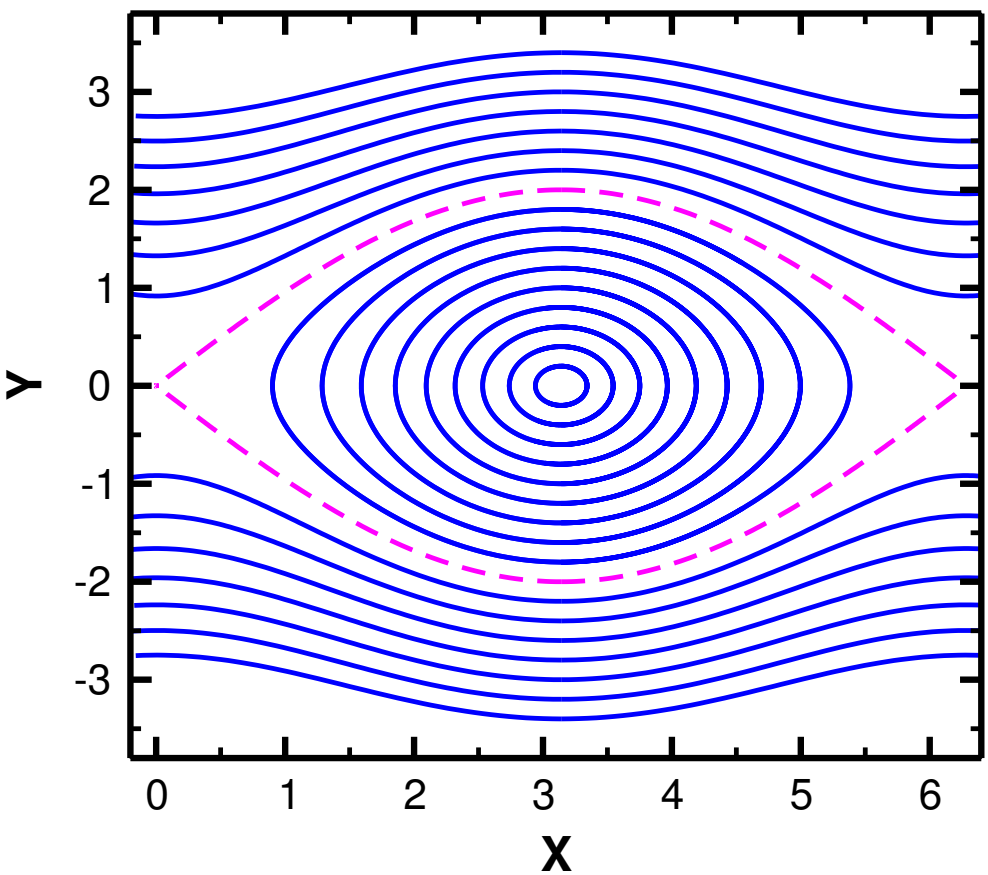
$$\frac{d\varphi}{dt} = \alpha\lambda + \mathbf{N}\cdot\frac{ds}{d\mathbf{J}}\cos(\varphi)$$

Solution for $s = \mathbf{const.}$:

We renormalise the variables, letting $X \equiv \varphi$, $Y \equiv \lambda(\alpha/s)^{1/2}$ and $\tau \equiv (\alpha s)^{1/2} t$ to obtain the renormalised equations

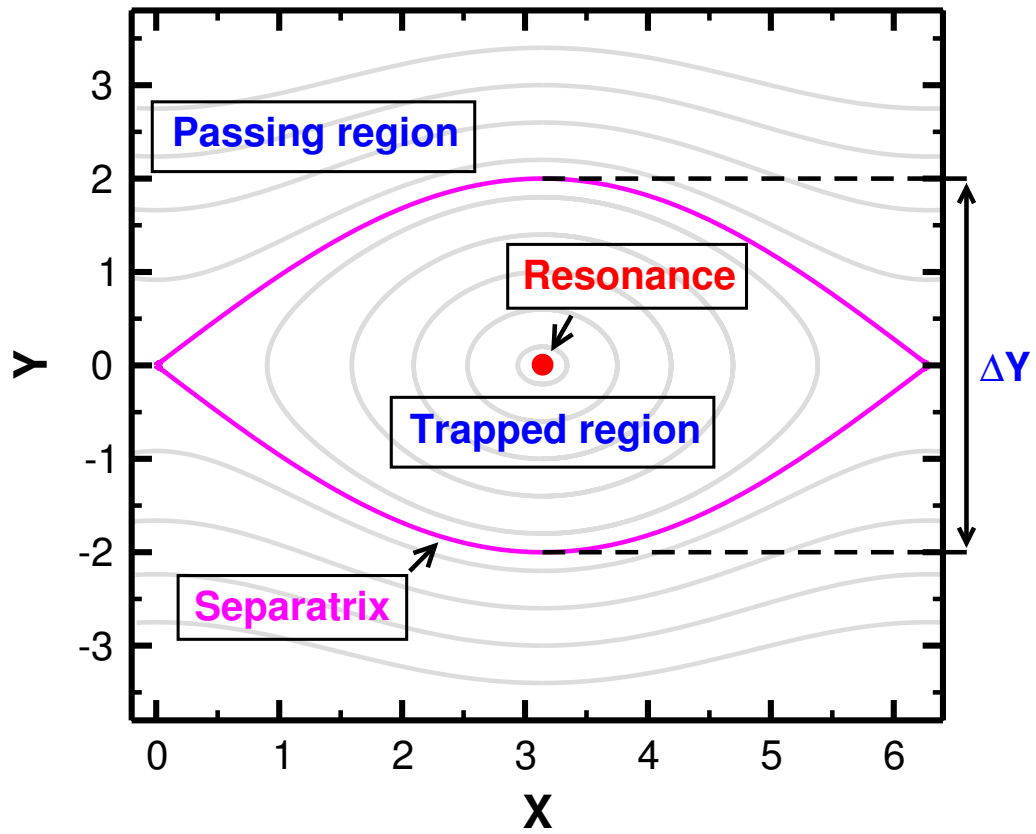
$$\frac{dX}{d\tau} = Y, \quad \frac{dY}{d\tau} = \sin(X)$$

Phase portrait:



A familiar shape. . .

The island structure: Topological properties



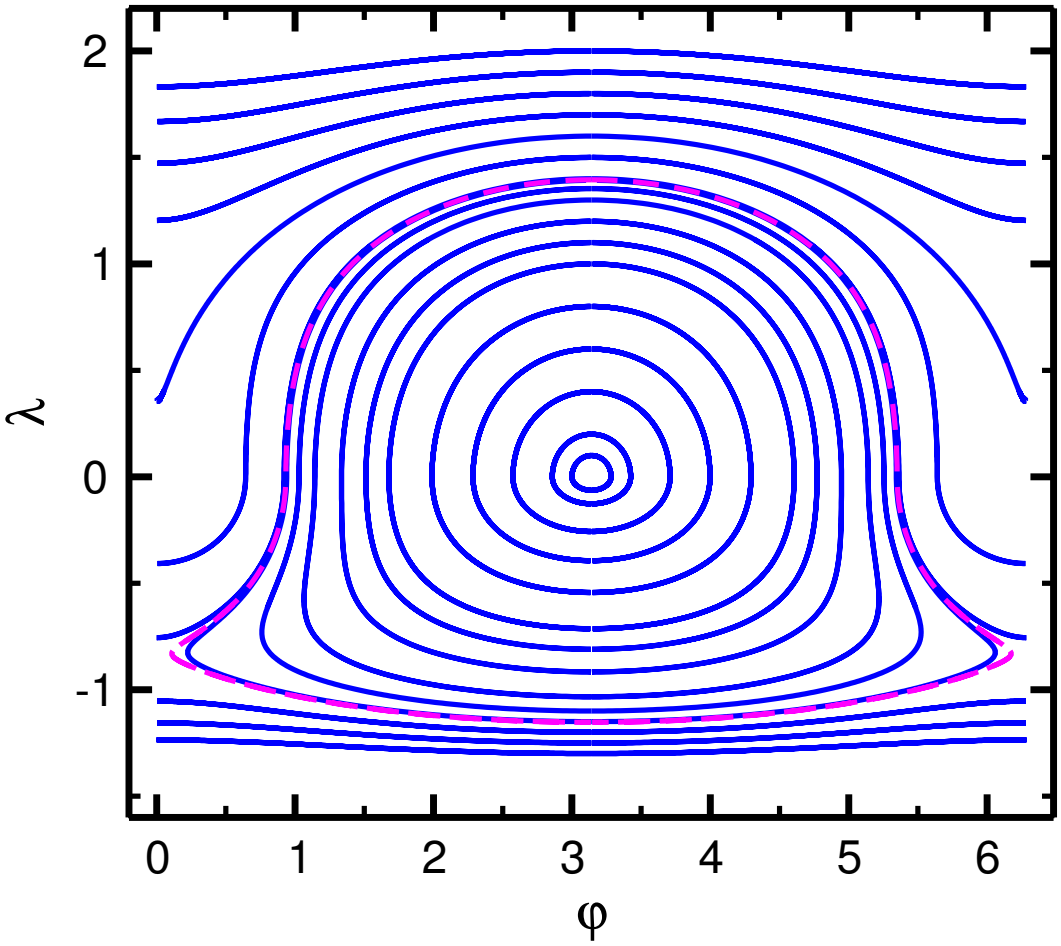
Trapped region width:

$$\Delta\lambda = \Delta Y (s/\alpha)^{1/2} = 4(s/\alpha)^{1/2}$$

In both region, the perturbed motion is quasi-periodical, integrable, thus leading to a possible redefinition of the *action-angle variables for the perturbed system*. However, two different sets are needed for both regions (topological separation).

Solution for $s = s(\mathbf{J})$:

The source is now supposed to be a function of J centred around the resonance.

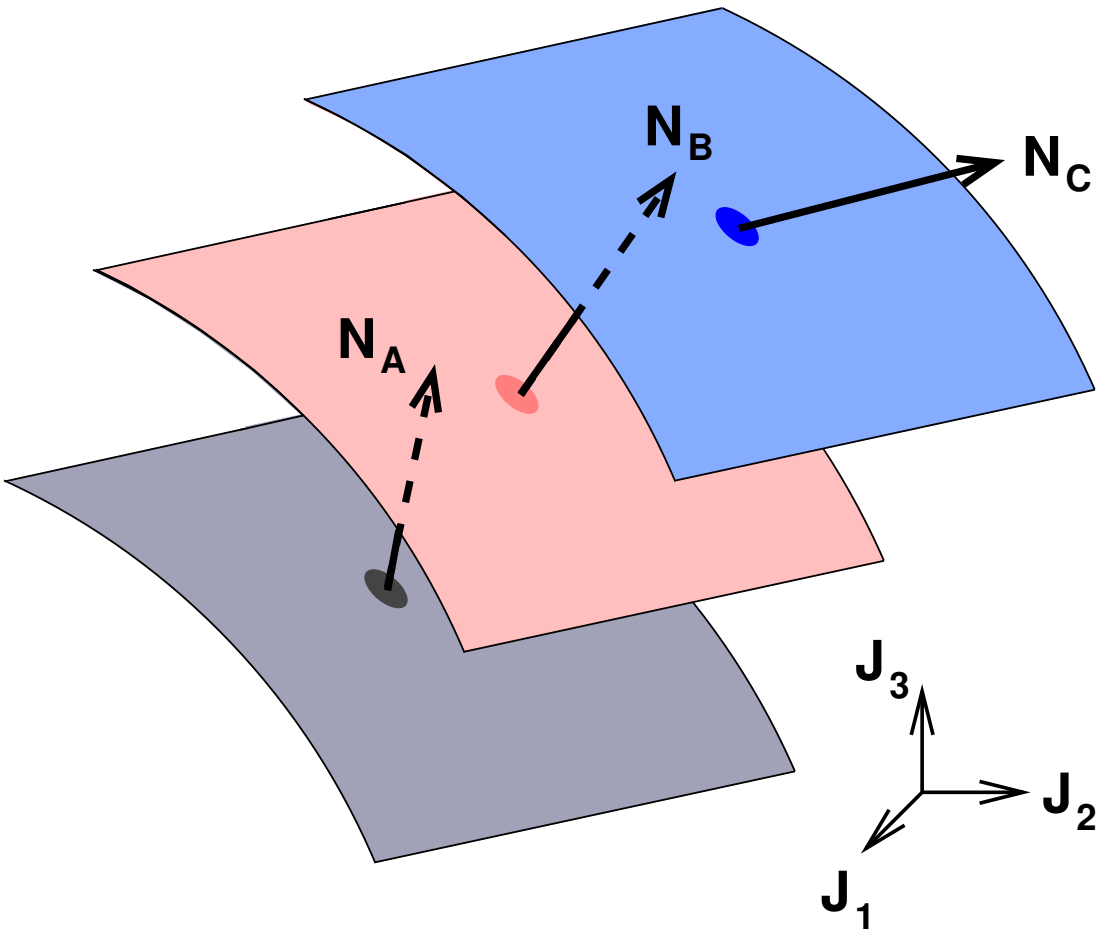


The structure has a more complicated shape, but the same topological properties are observed **(trapped region - separatrix - passing region)**

Multiple resonances:

In real systems, the wave-particle interaction usually generates a set of resonant surfaces

$$\Omega_{\mathbf{N}} = \mathbf{N} \cdot \boldsymbol{\omega}(\mathbf{J}) + \omega = 0$$



Particle under the effects of two waves:

The Hamiltonian writes (1-D system)

$$H(J,\Phi,t) = H_0(J) + \delta H(J,\Phi,t)$$

with

$$\delta H = A_1 \cos(N_1\Phi + \omega_1 t) + A_2 \cos(N_2\Phi + \omega_2 t)$$

with A_1 and A_2 the respective intensities of both waves, ω_1 and ω_2 their frequencies. We assume

$$\begin{cases} N_1 = N_2 \equiv N \\ \omega_1 \equiv \omega_0, \quad \omega_2 \equiv \omega_0 - \Delta\omega \end{cases}$$

The resonance condition for the first wave writes $N\omega(J_R) + \omega_0 = 0$. We write $J \equiv J_R + \lambda N$ and define $\varphi \equiv N\Phi + \omega_0 t$ to obtain the coupled equations

$$\frac{d\varphi}{dt} = \alpha\lambda$$

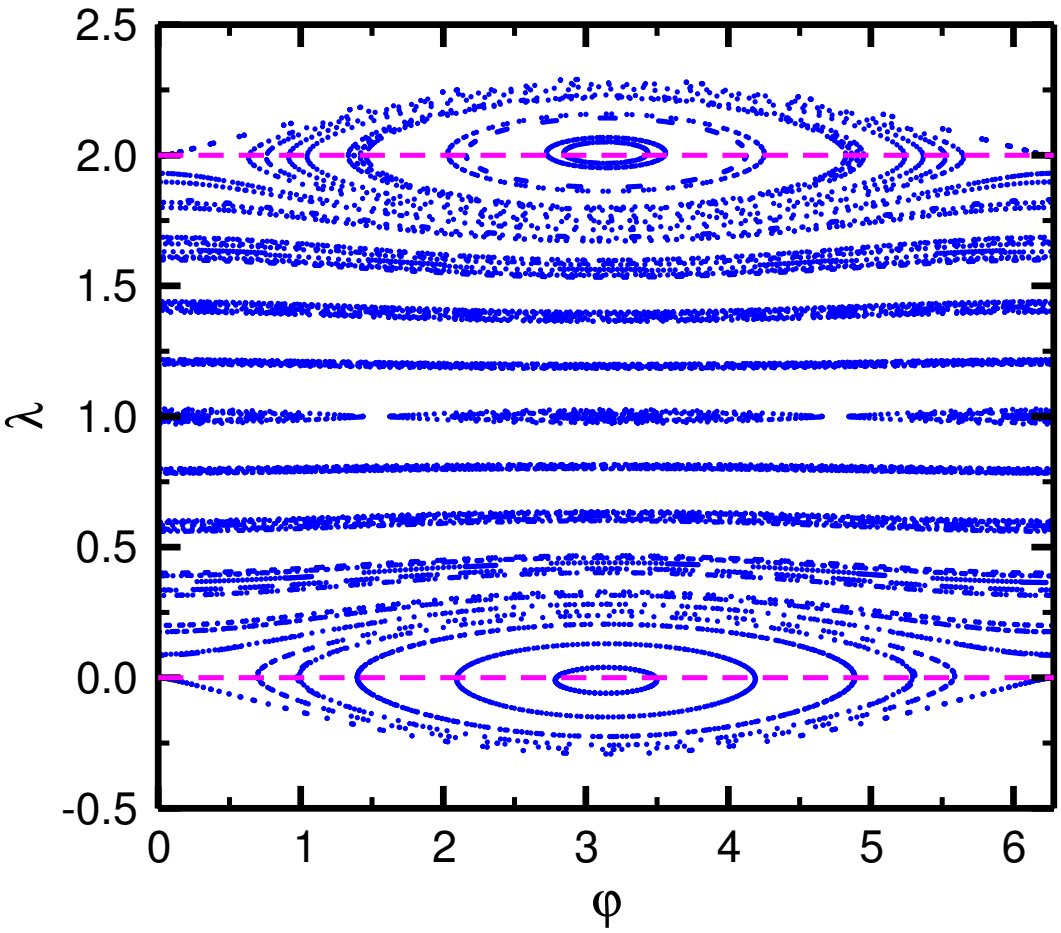
and

$$\frac{d\lambda}{dt} = A_1 \sin(\varphi) + A_2 \sin(\varphi - \Delta\omega t)$$

We obtain two islands, of widths $\Delta\lambda_i \approx 4\sqrt{A_i}$, distant from $\Delta\lambda_{res.} = \Delta\omega$. Their overlaping is characterized by the *Chirikov parameter*:

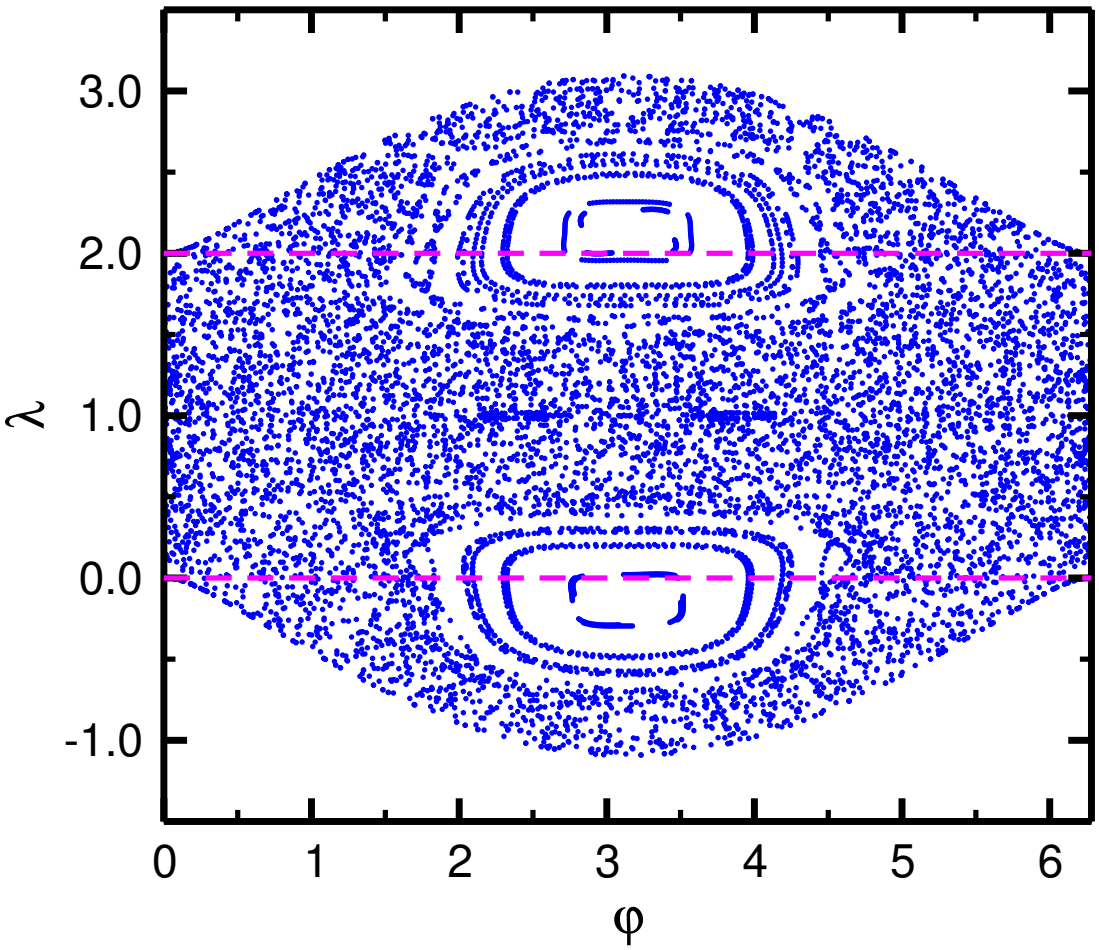
$$\sigma \equiv \frac{\Delta\lambda_{isl.}}{\Delta\lambda_{res.}} \approx \frac{\Delta\lambda_1 + \Delta\lambda_2}{2\Delta\lambda_{res.}} = \frac{2(\sqrt{A_1} + \sqrt{A_2})}{\Delta\omega}$$

Weak perturbation: $\sigma \approx 0.3$



$\longrightarrow$ **The particle motion is still integrable**

Strong perturbation: $\sigma \approx 1.0$



The overlapping of Hamiltonian resonances leads to a stochastic motion of the particle in phase space

$\rightarrow$ **Velocity-space diffusion sets in**

Quasilinear theory

Quasilinear equation

The time evolution of the particle distribution function in phase space is governed by the *kinetic equation* (see part I)

$$\frac{df}{dt}(\mathbf{r}, \mathbf{p}, t) = \left(\frac{\partial f}{\partial t}\right)_{coll.}$$

Neglecting the collisions, the latter can be written in action-angle space as

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{J}} \cdot \frac{d\mathbf{J}}{dt} + \frac{\partial f}{\partial \mathbf{\Phi}} \cdot \frac{d\mathbf{\Phi}}{dt} = 0$$

or, using Hamilton's equations

$$\frac{\partial f}{\partial t} - \frac{\partial f}{\partial \mathbf{J}} \cdot \frac{\partial H}{\partial \mathbf{\Phi}} + \frac{\partial f}{\partial \mathbf{\Phi}} \cdot \frac{\partial H}{\partial \mathbf{J}} = 0$$

The unperturbed motion is characterised by $H_0 = H_0(\mathbf{J})$, so that the unperturbed (equilibrium) distribution function is obtained from

$$\frac{\partial f_0}{\partial \mathbf{\Phi}} \cdot \frac{\partial H_0}{\partial \mathbf{J}} = 0 \quad \rightarrow \quad f_0 = f_0(\mathbf{J})$$

Perturbed motion:

$$\begin{cases} H(\mathbf{J}, \mathbf{\Phi}, t) = H_0(\mathbf{J}) + \delta H(\mathbf{J}, \mathbf{\Phi}, t) \\ f(\mathbf{J}, \mathbf{\Phi}, t) = f_0(\mathbf{J}) + \delta f(\mathbf{J}, \mathbf{\Phi}, t) \end{cases}$$

We can write the *linearised* Vlasov equation as

$$\frac{\partial \delta f}{\partial t} - \frac{\partial f}{\partial \mathbf{J}} \cdot \frac{\partial \delta H}{\partial \mathbf{\Phi}} + \frac{\partial \delta f}{\partial \mathbf{\Phi}} \cdot \frac{\partial H}{\partial \mathbf{J}} = 0$$

where all second order contributions have been neglected. As before, the Hamiltonian perturbation describing the wave is decomposed as

$$\delta H(\mathbf{J}, \mathbf{\Phi}, t) = \sum_{\mathbf{N}} h_{\mathbf{N}}(\mathbf{J}) e^{i(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)} + \text{c.c.}$$

The harmonic response of the particles to the wave is contained in δf, which it is therefore natural to expand as

$$\delta f(\mathbf{J}, \mathbf{\Phi}, t) = \sum_{\mathbf{N}} f_{1,\mathbf{N}}(\mathbf{J}) e^{i(\mathbf{N} \cdot \mathbf{\Phi} + \omega t)} + \text{c.c.}$$

The linearised Vlasov equation then leads to

$$i\omega f_{1,\mathbf{N}} - ih_{\mathbf{N}}\mathbf{N}\cdot\frac{\partial f_0}{\partial \mathbf{J}} + if_{1,\mathbf{N}}\mathbf{N}\cdot\frac{\partial H_0}{\partial \mathbf{J}} = 0$$

Recalling that $\partial H_0/\partial \mathbf{J} = \boldsymbol{\omega}$, we obtain

$$f_{1,\mathbf{N}}(\mathbf{J}) = h_{\mathbf{N}}\frac{\mathbf{N}}{\omega + \mathbf{N}\cdot\boldsymbol{\omega}}\frac{\partial f_0}{\partial \mathbf{J}}$$

The Landau prescription (principle of causality) allows to rewrite this expression as

$$f_{1,\mathbf{N}}(\mathbf{J}) = \lim_{\varepsilon\to 0} h_{\mathbf{N}}\frac{\mathbf{N}}{\omega + \mathbf{N}\cdot\boldsymbol{\omega} + i\varepsilon}\frac{\partial f_0}{\partial \mathbf{J}}$$

To derive δf, the *linear response* of the system to the wave, we have considered that f_0 was constant. The *quasilinear approach* consists in letting it vary on a timescale long compared to the wave period:

$$f_0(\mathbf{J}) \to f_0(\mathbf{J}, t)$$

The average-operation over $\mathbf{\Phi}$ eliminates all fast-varying quantities and we are left with the secular variation of f:

$$\frac{1}{(2\pi)^3}\int d^3\mathbf{\Phi} f(\mathbf{J},\mathbf{\Phi},t) = f_0(\mathbf{J},t)$$

The Vlasov equation is written, *with second-order terms kept back*, and is averaged

$$\frac{1}{(2\pi)^3}\int d^3\mathbf{\Phi}\left\{\frac{\partial f}{\partial t}-\frac{\partial \delta f}{\partial \mathbf{J}}\cdot\frac{\partial \delta H}{\partial \mathbf{\Phi}}+\frac{\partial \delta f}{\partial \Phi}\cdot\frac{\partial \delta H}{\partial \mathbf{J}}\right\} = 0$$

to give

$$\frac{\partial f_0}{\partial t} + i\sum_{\mathbf{N}}\mathbf{N}\cdot\frac{\partial}{\partial \mathbf{J}}\left[f^*_{1,\mathbf{N}}h_{\mathbf{N}} - f_{1,\mathbf{N}}h^*_{\mathbf{N}}\right] = 0$$

where $*$ denotes the complex conjugate.

Using the expression for $f_{1,\mathbf{N}}$ derived above yields

$$\frac{\partial f_0}{\partial t} = \lim_{\varepsilon\to 0}\mathbf{N}\cdot\frac{\partial}{\partial \mathbf{J}}\left[|h_{\mathbf{N}}|^2\frac{2\varepsilon\mathbf{N}}{(\omega+\mathbf{N}\cdot\boldsymbol{\omega})^2+\varepsilon^2}\right]\cdot\frac{\partial f_0}{\partial \mathbf{J}}$$

which can be written under the compact form

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \overline{\overline{\mathbf{D}}}_{ql} \cdot \frac{\partial f_0}{\partial \mathbf{J}}$$

This expression is a diffusion equation in phase space with the terms of the *quasilinear tensor* $\overline{\overline{\mathbf{D}}}_{ql}$ given by

$$\left(\overline{\overline{\mathbf{D}}}_{ql}\right)_{kl} = 2\pi \sum_{\mathbf{N}} N_k N_l |h_{\mathbf{N}}|^2 \delta(\omega + \mathbf{N} \cdot \boldsymbol{\omega})$$

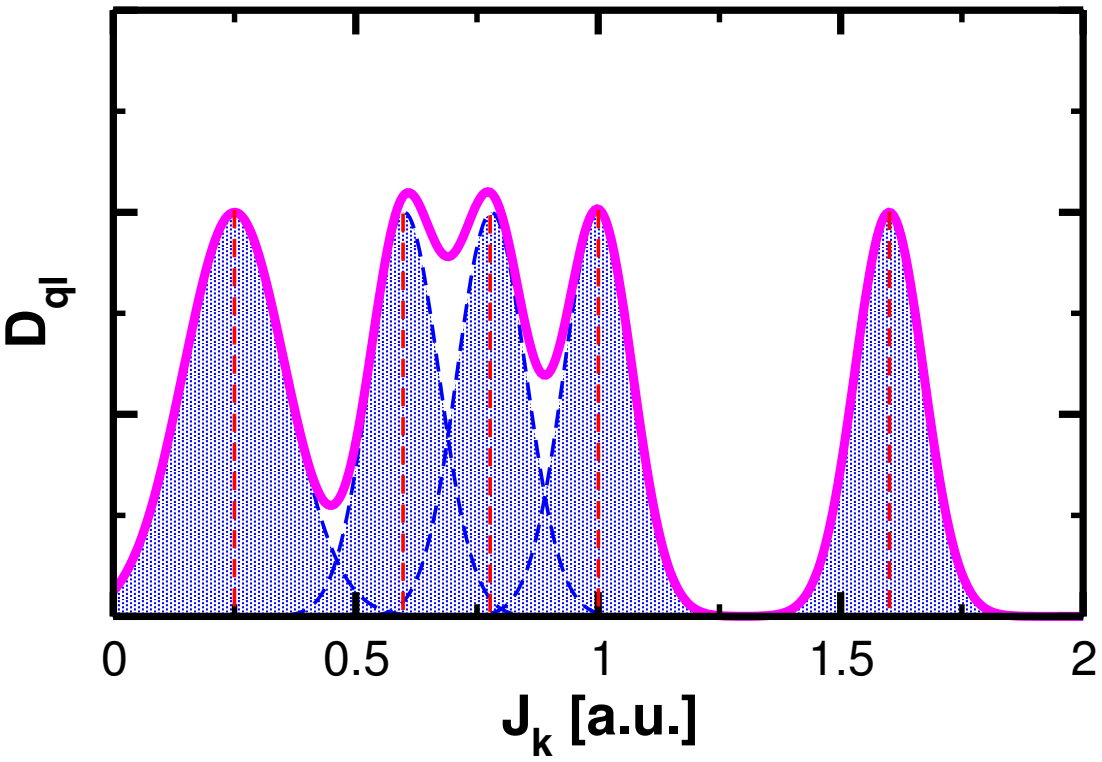
N.B.: The Vlasov equation becomes a diffusion equation only if a *phase decorrelation process* takes place. Mathematically, this was provided by the $i\varepsilon$ term added to the denominator of the linear solution δf. Physically, it is caused by:

1. The interaction itself (intrinsic chaos)
2. The collisions (extrinsic chaos)

Quasilinear diffusion

Under the effect of the wave, the particles diffuse in phase space. Each resonance $\Omega_{\mathbf{N}} = 0$ contributes to the quasilinear diffusion coefficient, the intensity being determined by $|h_{\mathbf{N}}|$.

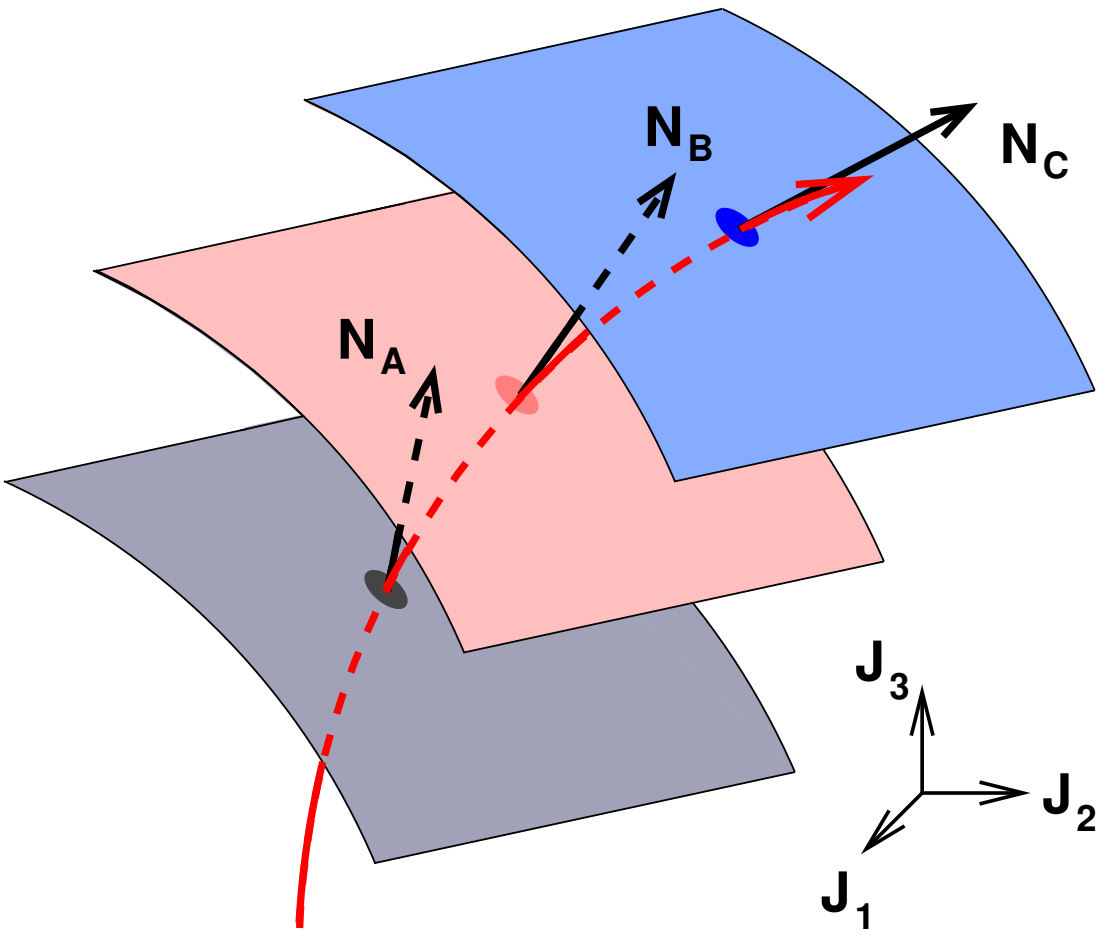
$$\left(\bar{\bar{\mathbf{D}}}_{ql}\right)_{kl} = 2\pi \sum_{\mathbf{N}} N_k N_l |h_{\mathbf{N}}|^2 \delta(\omega + \mathbf{N} \cdot \boldsymbol{\omega})$$



Quasilinear diffusion lines

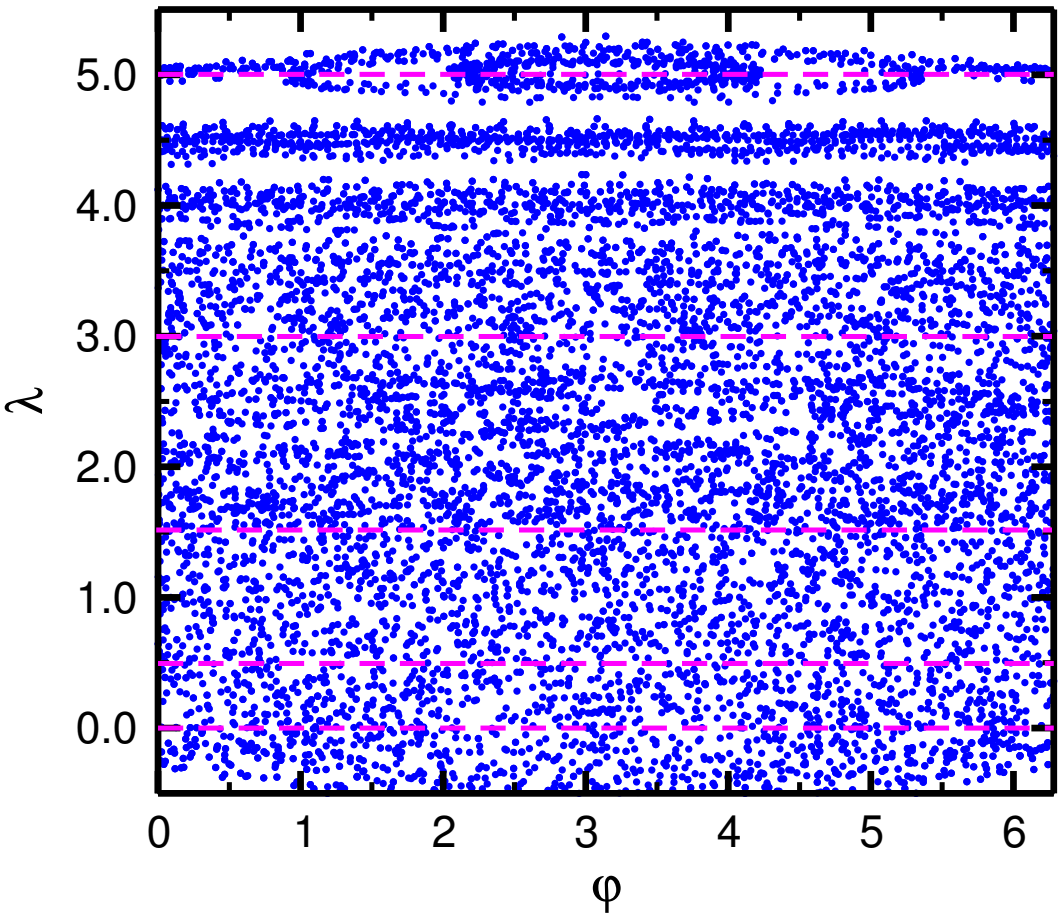
The diffusion curves follow the directions determined by the successive $\mathbf{N}$-vectors

$$\left(\bar{\bar{\mathbf{D}}}_{ql}\right)_{kl} = 2\pi \sum_{\mathbf{N}} N_k N_l |h_{\mathbf{N}}|^2 \delta(\omega + \mathbf{N} \cdot \boldsymbol{\omega})$$



Super-adiabaticity:

It is possible to show that the wave-particle interaction becomes less efficient when the particle energy increases. This leads to the appearance of one (or several) adiabatic barrier(s)



A particle originating from the stochastic region below the adiabatic barrier (here, $\lambda_{ab} \approx 4.25$) can not reach higher values of λ, where the trajectories become integrable.

Summary

Unperturbed system:

$$H_0(\mathbf{r},\mathbf{p}) \quad \rightarrow \quad (\mathbf{J},\mathbf{\Phi}), \boldsymbol{\omega} \equiv (\omega_1,\omega_2,\omega_3)(\mathbf{J})$$

Wave perturbation:

$$\delta H(\mathbf{J},\mathbf{\Phi},t) \equiv \sum_{\mathbf{N}} h_{\mathbf{N}}(\mathbf{J}) e^{i(\mathbf{N}\cdot\mathbf{\Phi}+\omega t)} + \text{c.c.}$$

Resonances identification: $\Omega_{\mathbf{N}} = \mathbf{N}\cdot\boldsymbol{\omega} + \omega = 0$

Chaos considerations: Chirikov parameter, σ

Quasilinear diffusion operator:

$$\left(\bar{\bar{\mathbf{D}}}_{ql}\right)_{kl} = 2\pi \sum_{\mathbf{N}} N_k N_l |h_{\mathbf{N}}|^2 \delta(\omega + \mathbf{N}\cdot\boldsymbol{\omega})$$

Kinetic equation:

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \bar{\bar{\mathbf{D}}}_{ql} \cdot \frac{\partial f_0}{\partial \mathbf{J}}$$

ICRF heating in a tokamak

Hamiltonian

We consider the case of a charged particle in a confinement magnetic field

1. The particle has charge q and mass m
2. The confinement field is written in terms of the potential vector $\mathbf{A}_0$ and scalar U_0
3. The wave is a perturbation of the electromagnetic field: $\delta\mathbf{A}$, δU

The Hamiltonian for this system writes

$$H(\mathbf{r},\mathbf{p},t) = \frac{\big(\mathbf{p} - q\mathbf{A}(\mathbf{r},t)\big)^2}{2m} + qU(\mathbf{r},\mathbf{t})$$

and can be decomposed as

$$H(\mathbf{r},\mathbf{p},t) \equiv H_0(\mathbf{r},\mathbf{p}) + \delta H(\mathbf{r},\mathbf{p},t)$$

with

$$H_0(\mathbf{r},\mathbf{p}) = \frac{\big(\mathbf{p} - q\mathbf{A}_0(\mathbf{r})\big)^2}{2m} + qU_0(\mathbf{r})$$

$\longrightarrow$ Action-angle coordinates system

$\longrightarrow$ Unperturbed orbits

The perturbation writes

$$\delta H(\mathbf{r},\mathbf{p},t) = q\delta U(\mathbf{r},t) + \ldots$$
$$\ldots - \frac{q}{m}\big(\mathbf{p} - q\mathbf{A}_0(\mathbf{r},\mathbf{t})\big)\cdot\delta\mathbf{A}(\mathbf{r},\mathbf{t}) + \ldots$$
$$\ldots + \frac{q^2}{2m}\big(\delta\mathbf{A}(\mathbf{r},t)\big)^2$$

The last term is a non-linear term, corresponding to the ponderomotive force, which is ignored here.

$\delta\mathbf{A}$ and δU are expanded over the angle-coordinate

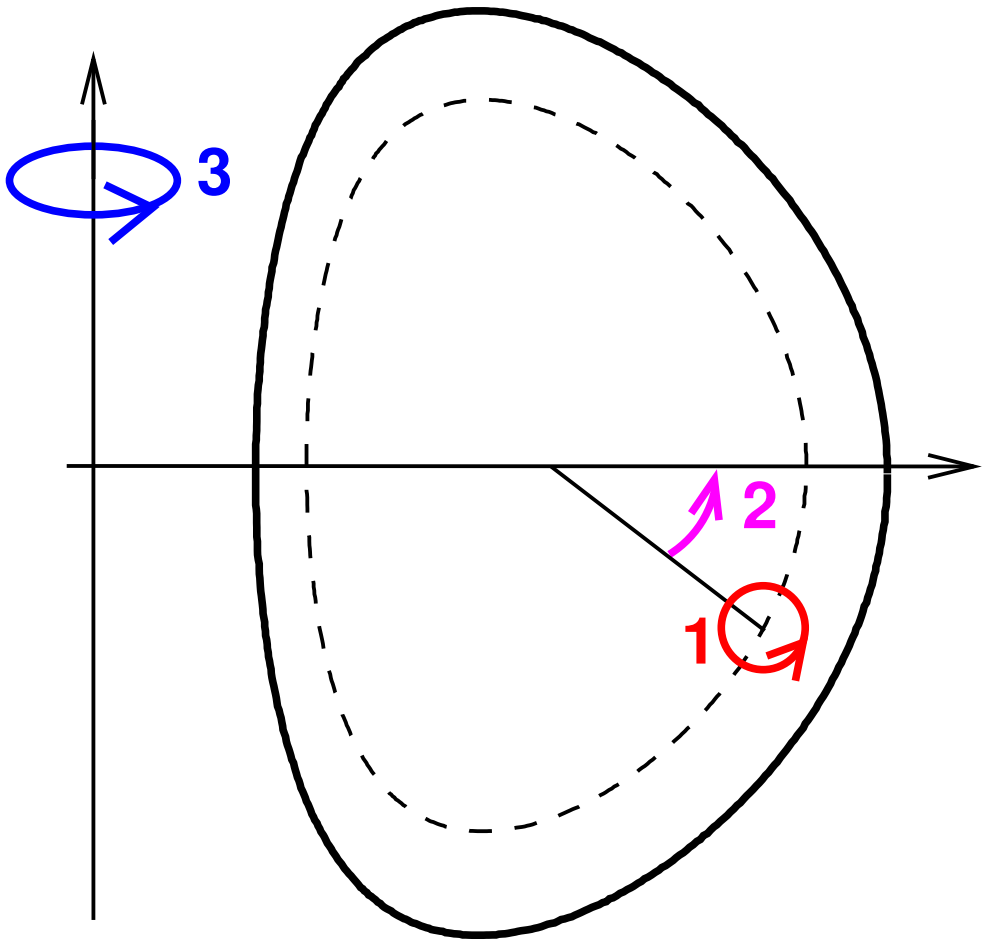
$$\begin{pmatrix} \delta\mathbf{A}(\mathbf{J},\mathbf{\Phi},t) \\ \delta U(\mathbf{J},\mathbf{\Phi},t) \end{pmatrix} = \sum_{\mathbf{N}} \begin{pmatrix} \mathbf{a}_{\mathbf{N}}(\mathbf{J}) \\ \mathbf{u}_{\mathbf{N}}(\mathbf{J}) \end{pmatrix} e^{i(\mathbf{N}\cdot\mathbf{\Phi}+\omega t)}$$

to obtain the Hamiltonian perturbation

$$\delta H(\mathbf{J},\mathbf{\Phi},t) = \sum_{\mathbf{N}} h_{\mathbf{N}}(\mathbf{J}) e^{i(\mathbf{N}\cdot\mathbf{\Phi}+\omega t)} + \text{c.c.}$$

Unperturbed dynamics

Decomposition of a particle orbit in the tokamak geometry:

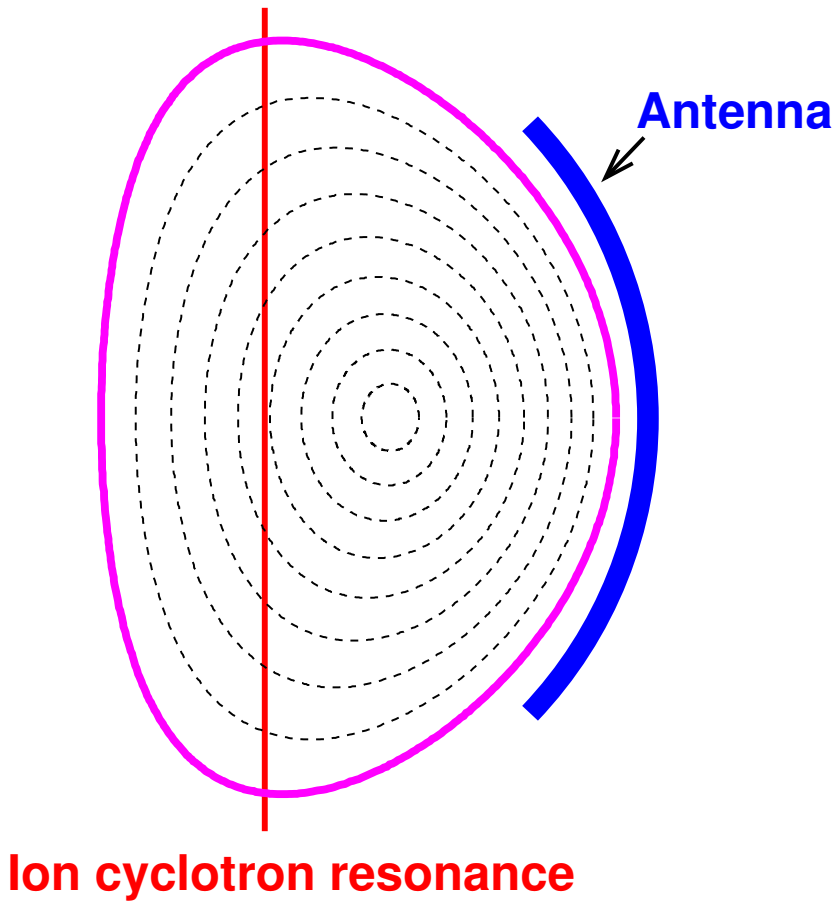


Three components of the unperturbed motion:

1.	Cyclotron motion	$\longrightarrow$	ω_1
2.	Poloidal motion	$\longrightarrow$	ω_2
3.	Toroidal motion	$\longrightarrow$	ω_3

ICRF (= **I**on **C**yclotron **R**esonance **H**eating))

General principle:

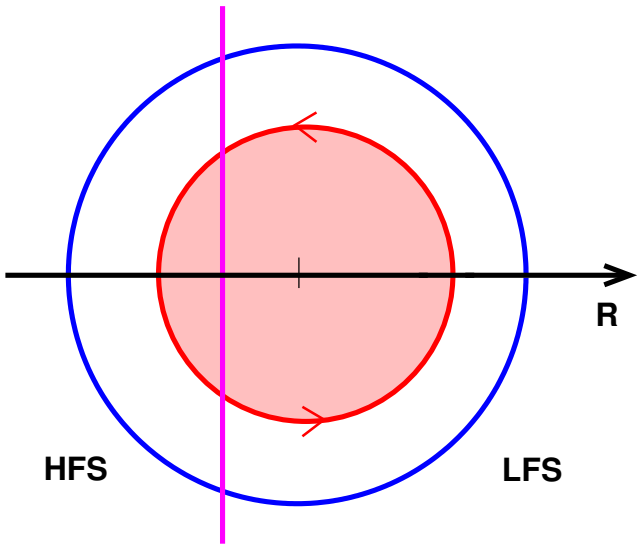


- The wave is injected from the Low Field Side of the tokamak
- Its frequency lies in the ion cyclotron frequency range ($f \approx 50\mathrm{MHz}$)
- In most scenarios, it is aimed at interacting with the ion gyro-motion and transfering its energy to the resonant population

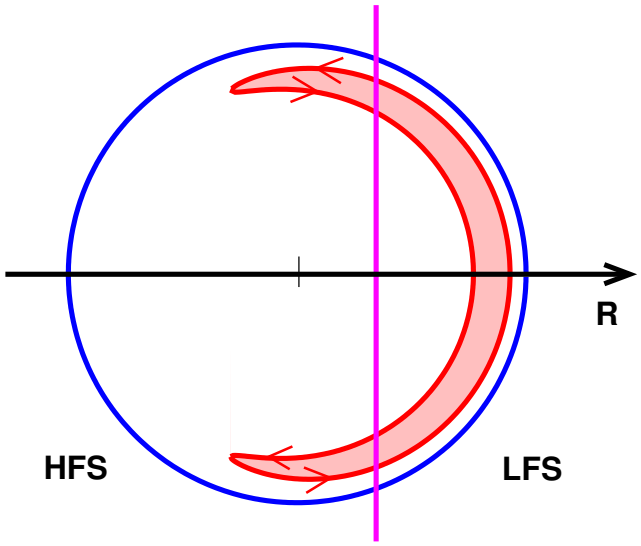
ICRF wave-ion interaction:

In a tokamak, two main classes of particles can be distinguished:

- *Passing particles* perform complete toroidal revolutions



- *Trapped particles* bounce back and forth in the toroidal direction $\longrightarrow$ *banana orbits*



ICRF heating in a tokamak: Main features

- The magnetic field $\mathbf{B}_0$ varies radially and poloidally

- The ions have large Larmor radii

- The antenna has a wide toroidal spectrum

  $\longrightarrow$ **huge number of resonances**

- At low energy, the collisions are efficient and provide an extrinsic phase decorrelation process.

- For usual parameters, the Chirikov parameter σ is larger than one over a wide range of phase space, even in regions where the collisions are inefficient.

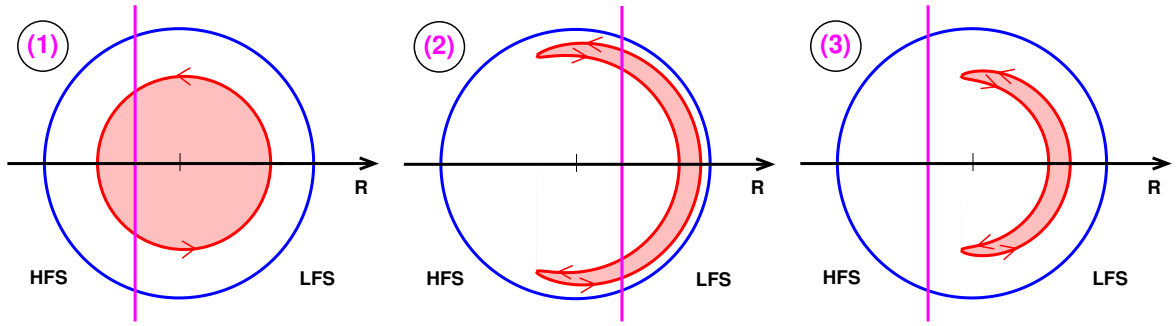
  $\longrightarrow$ **wave-induced diffusion**

- The Chrikov parameter decreases with the ion energy, which provide an upper energy boundary to the intrinsic chaos, typically in the MeV range.

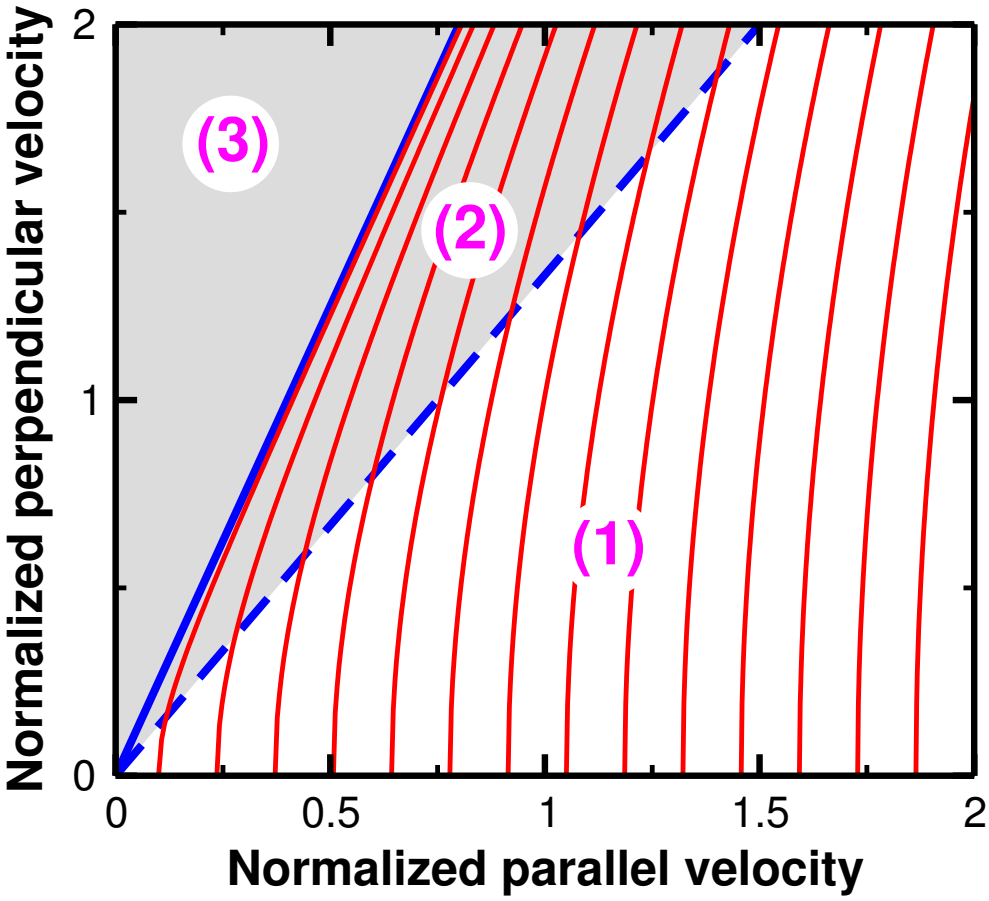
  $\longrightarrow$ **adiabatic barrier**

Diffusion lines in velocity space

Classes of particles:

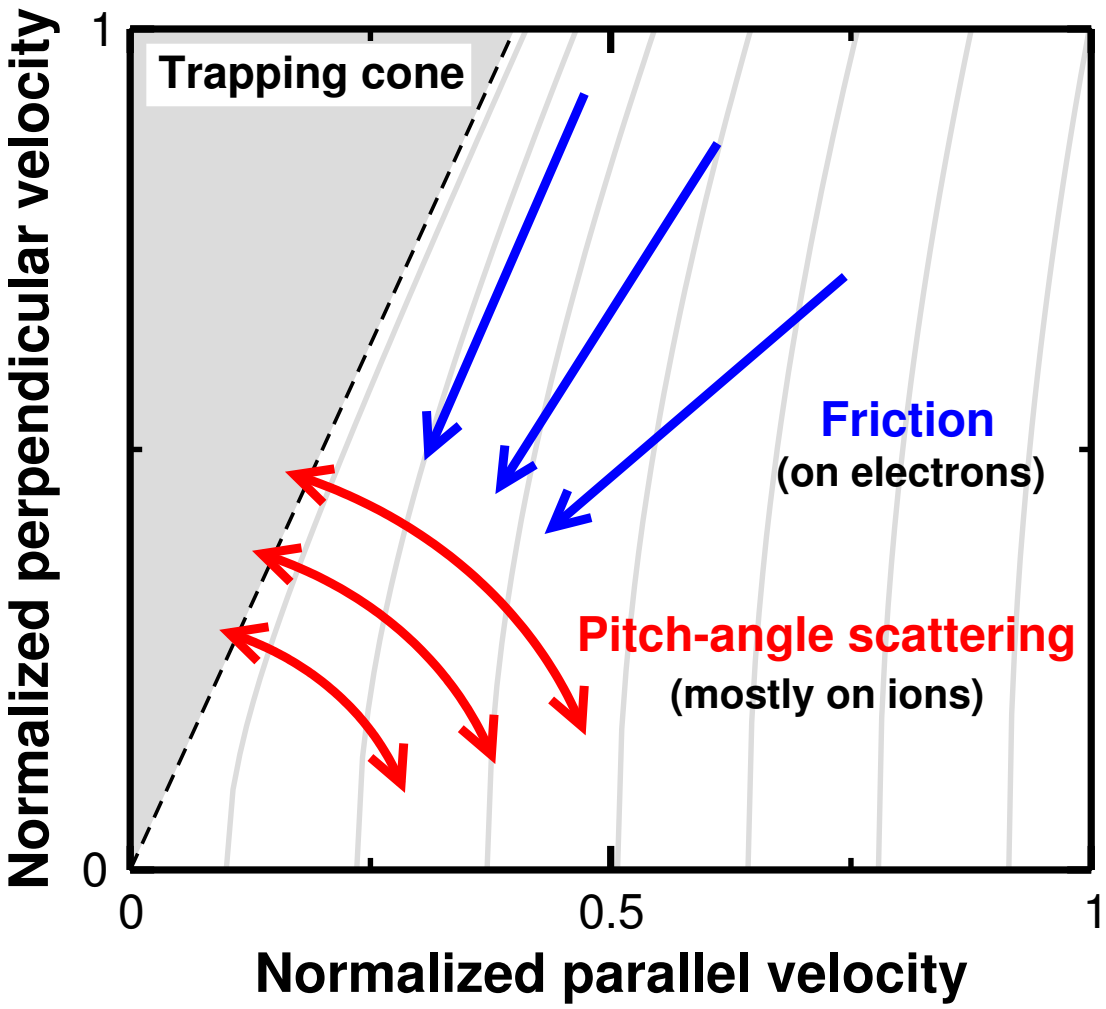


Wave-induced diffusion:



Collisions

Extrinsic decorrelation process:

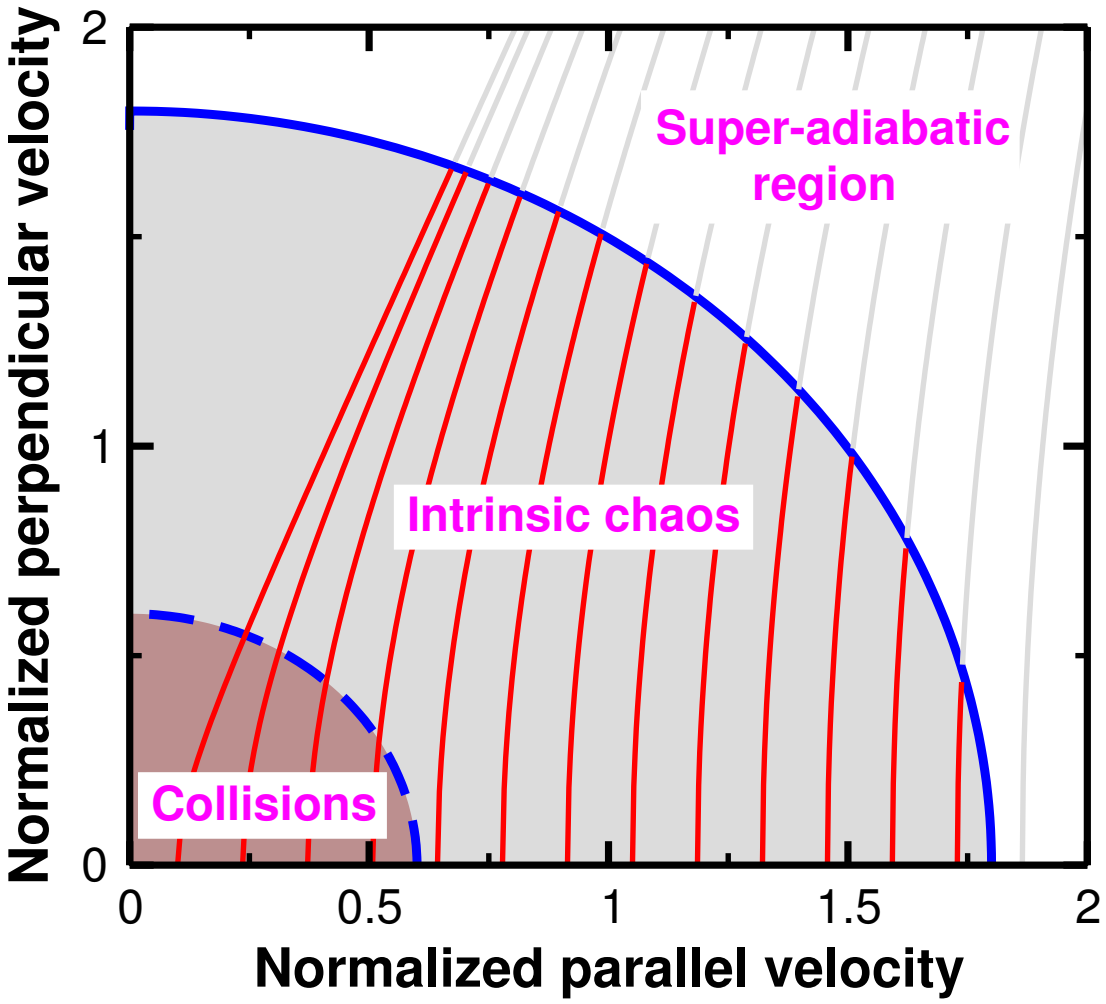


For fast ions, the main collisional processes are:

- Pitch-angle scattering (with bulk ions)
- Collisional friction (on electrons)

Super-adiabaticity

Presence of an adiabatic barrier:



The particles energy can increase up to the adiabatic barrier, which constitute an upper boundary. In typical current fusion experiments, this limit lies in the MeV range.

Conclusions

- Radiofrequency waves launched from outside the machine are routinely used for plasma heating, current drive and diagnostics.

- A deep understanding of the wave-particle interaction physics is a key point for fusion experiments design and interpretation.

- The Hamiltonian theory provides a powerful framework to describe the particle-wave interaction physics in plasmas.

- In a tokamak, it allows a complete separation between the complicated unperturbed dynamics and the perturbation (wave).

- The same approach provides the quasilinear diffusion operators and their domain of validity (stochastic - adiabatic).

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