Phase Integral Methods

Roscoe White
[1]*Plasma Physics Laboratory, Princeton University, P.O.Box 451, Princeton, New Jersey 08543*

I. INTRODUCTION

The power and simplicity of Phase Integral Methods (Heading, 1962) for the approximate solution of differential equations make them a common tool in many branches of physics, and a particularly useful one in plasma physics, where the equations are often too cumbersome to solve by standard exact methods. Many of the differential equations of interest can be put in the form

$$\frac{d^2\psi}{dz^2} + Q(z,w)\psi = 0. \tag{1}$$

In these cases, the existence of solutions and the approximate complex eigenfrequencies ω can often be determined by phase integral methods. This technique has been used to examine unstable modes in plasmas in the areas of ballooning modes, drift waves, and other microinstabilities, as well as to investigate parametric instabilities associated with incident wave energy both in tokamak heating and laser pellet fusion problems. It is not possible to cover all aspects of the method of phase integrals here. We restrict ourselves to deriving the most essential results, which should permit the solution of most problems arising in plasma physics which can be addressed by these methods.

We take Eq. 1 as standard form for the differential equation to be examined. The complex frequency ω generally plays the role of an unknown eigenvalue, and $z = x+iy$ is a complex variable. The physical problem is initially defined on the real axis and the equation has been analytically continued into the complex plane. The physical problems include searching for the existence of an instability, in which case outgoing wave boundary conditions are imposed for a growing mode and ω becomes an eigenvalue to be determined, or finding the amplitudes and phases of reflected and transmitted waves given an incoming wave of frequency ω.

Briefly, the WKBJ approximate solutions of Eq. 1, so named after Wentzel (1926), Kramers (1926), Brillouin (1926), and Jeffreys (1923), take the form

$$\psi_\pm = Q^{-1/4} e^{\pm i \int^z Q^{1/2} dz}, \tag{2}$$

and provided that

$$\left| \frac{dQ}{dz} Q^{-3/2} \right| \ll 1 \tag{3}$$

a general solution of Eq. 1 can be approximated by

$$\psi = a_+\psi_+ + a_-\psi_-. \tag{4}$$

The solutions $\psi_\pm$ are local, not global solutions of Eq. 1. Clearly, inequality (3) is not valid in the vicinity of a zero of $Q(z,\omega)$, commonly called a turning point. Aside from this, however, $\psi_\pm$ are not approximations of a continuous solution of Eq. 1 in the whole z plane; *i.e.*, if ψ is to approximate a continuous solution of Eq. 1, then the coefficients $a_\pm$ are not fixed over the whole z plane. The Method of Phase Integrals consists in relating, for a given solution of Eq. 1, the WKBJ approximation in one region of the z plane to that in another.

These regions are separated by the so-called Stokes and anti-Stokes lines (Stokes, 1899) associated with $Q(z,\omega)$, and thus the qualitative properties of the solution are determined once these lines are known. The Stokes (anti-Stokes) lines associated with $Q(z,\omega)$ are paths in the z plane, emanating from zeros or singularities of $Q(z,\omega)$, along which $\int Q^{1/2}(z,\omega)dz$ is imaginary (real). We review first the characteristic properties of these lines and then the way in which they determine the global nature of a WKBJ solution.

Define a local anti-Stokes line to be, for any z_0 an infinitesimal path dz emanating from z_0 along which $Q^{l/2}dz$ is real. Along this path $|\psi_\pm|$ are essentially constant; *i.e.*, the solutions are oscillatory. If $Q(z_0,\omega)$ is finite and well behaved, the local anti-Stokes line is given by setting dz equal to a real number times $\pm Q(z_0)^{-1/2}$ *i.e.*, from z_0 there issue two oppositely directed lines. Points at which $Q(z_0)$ is zero or infinity must be analyzed with more care. When the zero is first order, consider an infinitesimal line emanating from z_0, with $dz = (z - z_0)$. Then write $Q(z) \simeq Q'(z_0)dz$, and require that dz be a local anti-Stokes line, *i.e.* that $Q(z)^{1/2}dz = Q'(z_0)^{1/2}dz^{3/2}$ be real. Since dz is then proportional to $Q'(z_0)^{-1/3}e^{i(2n\pi)/3}$ with n integer, we find that three anti-Stokes lines emanate from z_0. Similarly, one finds that from a double root there issue four anti-Stokes lines, from a simple pole a single line, *etc.* It is thus quite easy to read the locations of zeros, poles, *etc.*, of a function from a plot of the z plane upon which are displayed the local anti-Stokes lines, which we will refer to as a Stokes diagram. An example is shown in Fig. 1 for a function Q which possesses simple zeros in the first and third quadrants, a second order zero in the fourth quadrant, a pole in the second quadrant, and no other zeros or singularities. In refering to Stokes diagrams, we will refer to both zeros and singularities of Q(z) as singular points since it is the function $Q^{1/2}$ which is relevant in this diagram.

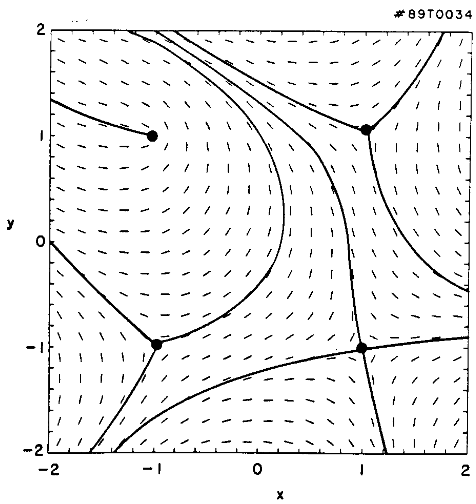

FIG. 1. Stokes diagram for $Q = (z - z_l)(z - z_2)(z - z_3)^2/(z - z4)$ with $z_1 = l + i$, $z_2 = -1 - i$, $z_3 = 1 - i$, and $z_4 = -1 + i$.

A display of this nature allows a qualitative survey of the analytic structure of a function, without the numerical complication of an actual search for roots (White, 1979, 2000).

Using the local anti-Stokes lines as guides, we can form global, continuous anti-Stokes lines for those particular lines which emerge from the singular points of the Stokes plot, and these lines have been added to Fig. 1. Along the global anti-Stokes lines the functions $\psi_\pm$ are, within the validity of the WKBJ approximation, of constant amplitude, *i.e.*, oscillatory. We similarly define local and global Stokes lines to be lines emerging from the singular points for which the integral $\int Q^{1/2}dz$ is imaginary. Along the Stokes lines the WKBJ solutions are exponentially increasing or decreasing with fixed phase. Except at singular points the Stokes and anti-Stokes lines are orthogonal. The global anti-Stokes and Stokes lines which are attached to the singular points of the Stokes diagram, along with the Riemann cut lines, determine the global properties of the WKBJ solutions.

In the notation of Heading, including the slow $Q^{-1/4}$ dependence, a WKBJ solution is denoted by

$$(a, z)_s = Q^{-1/4}e^{i\int_a^z Q^{1/2}dz} \tag{5}$$

where the subscript s(d) indicates that the solution is subdominant (dominant); *i.e.*, exponentially decreasing (increasing) for increasing $|z - a|$ in a particular region of the z plane, bounded by Stokes and anti-Stokes lines. The point a is taken to be a nearby singular point to which the dominancy or subdominancy refers. The two independent local WKBJ approximate solutions of Eq. 1 in this notation are given by (z,a) and (a,z). Clearly if (z,a) is subdominant, then (a,z) is dominant. It is readily verified that upon crossing an anti-Stokes line these two solutions reverse character. Thus we find that upon crossing an anti-Stokes line we must make the change $(a, z)_d \rightarrow (a, z)_s$ and $(z, a)_s \rightarrow (z, a)_d$. This is the first of the connection formula, a collection of rules for continuing a solution through the z plane in the presence of cuts, Stokes lines, and anti-Stokes lines.

II. CONNECTION FORMULAE

We first consider the continuation of a solution about an isolated turning point, located at z = 0. Later we show how one can pass from one turning point to another, allowing continuation through the entire z plane. The connection formulae depend on the nature of the turning point, and for simplicity we first consider a first order turning point, $Q \sim -z$, the associated Stokes diagram shown in Fig. 2.

First consider crossing a cut. Analytically continuing the solution, Eq. 5, counter-clockwise around the turning point a we find that $\psi_\pm \to -i\psi_\mp$. Thus in crossing the cut in a clockwise sense, in order to insure continuity of our continued solution, we must make the changes

$$\begin{aligned}(0,z) &\to i(z,0)\\ (z,0) &\to i(0,z).\end{aligned} \tag{6}$$

Dominancy (or subdominancy) is not changed.

Now consider the process of crossing an anti-Stokes line, where dominant and subdominant solutions exchange character. Begin in the vicinity of a nearby Stokes line, and suppose the solution to Eq. 1 is approximated by a dominant expression $(z,0)_d$ given by Eq. 5. A small subdominant part could also be present, so to speak lost in the noise of the WKBJ approximation.

Trying to continue the solution past an anti-Stokes line creates a problem, because the previously small subdominant part, with an unknown coefficient, becomes dominant, making our solution totally inaccurate. To correct this one must, in the vicinity of the Stokes line, choose the coefficient of the subdominant solution so that the continuation to a nearby anti-Stokes line will give the correct solution. The necessary coefficient of the subdominant solution is called the Stokes constant.

For a first order turning point the Stokes constant can be derived simply by requiring that the solution be single valued upon continuation about the turning point. We know this is true because for $Q = -z$ the point $z = 0$ is a regular point of the differential equation and the solution is representable by a Taylor series with infinite radius of convergence. This is not the case for all forms of Q, in some cases the solution itself may possess cuts originating at zeros or singularities of Q. Begin with a subdominant solution, $(0,z)_s$ along the positive real axis of Fig. 2, a Stokes line. We deliberately begin with a subdominant solution so that the solution is small and cannot contain any dominant part due to the approximate nature of the WKBJ solution. Now continue this solution in both directions about the turning point.

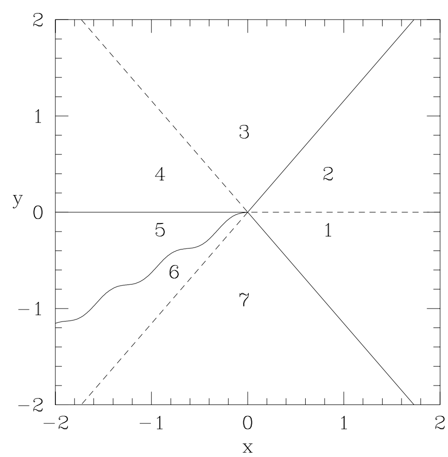

FIG. 2. Stokes diagram for a first order turning point.

Passing upward into domain 3, or downward into domain 7, an anti-Stokes line is passed and the solution becomes dominant, reaching maximal dominancy along the Stokes lines separating domains 3,4 and domains 6,7. On crossing these Stokes lines we must add subdominant parts to make up for any lost by the WKBJ approximation. Thus on passing into domain 4 the solution becomes $(0,z)_d + T_1(z,0)_s$ and in domain 6 it becomes $(0,z)_d - T_2(z,0)_s$ where T_1 and T_2 are Stokes constants, and the signs are chosen to reflect the fact that the Stokes line is crossed clockwise in one case and counterclockwise in the other. Now continue both solutions into domain 5. Coming from domain 6 we cross the cut clockwise, giving $i(z,0)_d - iT_2(0,z)_s$. From domain 4 we cross an anti-Stokes line, giving $(0,z)_s + T_1(z,0)_d$. Equating these determines the Stokes constants to be $T_1 = T_2 = i$.

We can now use this result to relate the asymptotic behavior at $+\infty$ to that at $-\infty$. For large positive x the solution has the form $(0,z)_s \sim e^{-\frac{2}{3}x^{3/2}}/x^{1/4}$. Note that the branch of the square root is determined by the fact that this solution must be subdominant, and that we have multiplied by a constant phase to choose the solution to be real. Writing $z = re^{i\pi}$, a phase choice determined by our placement of the cut in Fig. 2, we find $(0,z) = e^{i(\frac{2}{3}r^{3/2}-\pi/4)}/r^{1/4}$ and $(z,0) = e^{-i(\frac{2}{3}r^{3/2}+\pi/4)}/r^{1/4}$. Placing the cut in another position will modify some expressions, but not final results. The solution in domain 4 is $(0,z)_d + i(z,0)_s = 2sin(2r^{3/2}/3+\pi/4)$. Choose overall normalization for positive x to be $Ai(x) \simeq e^{-\frac{2}{3}x^{3/2}}/(2\sqrt{\pi}x^{1/4})$, giving for large negative x $Ai(x) \simeq sin(2r^{3/2}/3+\pi/4)/(\sqrt{\pi}r^{1/4})$, with $r=|x|$, which is of course the correct asymptotic expression for the Airy function. The continuation of the second function, $Bi(x)$, is slightly more subtle, since the solution involves dominant and subdominant solutions on a Stokes line.

To obtain the correct expression for a solution on a Stokes line, we must consider flux conservation. Multiply the differential equation $d^2\psi/dz^2 + Q\psi = 0$ by ψ^* and its complex conjugate by ψ and subtract, giving, for Q real,

$$\frac{d}{dz}[\psi^*\psi' - \psi'^*\psi] = 0. \tag{7}$$

Thus $S = Im\psi^*\psi'$, referred to as the flux, is conserved if Q is real. This is not an approximate WKB result, it is exact. Now consider the solution $(0,z) = e^{i\frac{2}{3}z^{3/2}}/z^{1/4}$ for z negative and real in Fig. 2. We have $(0,z)' = ie^{i\frac{2}{3}z^{3/2}}z^{1/4}$ plus a higher order term in $1/z$, giving for the flux $S=1$. Now continue to large positive x. In domain 4 this solution is exponentially decreasing so we have $(0,z)_s$, and the same in domain 3. Finally in domain 2 we have $(0,z)_d$. Now continue in the lower half plane. In 5 we have $(0,z)_d$, and in 6 $-i(z,0)_d$, giving in 7 $-i(z,0)_d + (0,z)_s$, and in 1 $-i(z,0)_s + (0,z)_d$. Now we recognize a problem, since the solutions in domains 1 and 2 disagree by the presence of the subdominant term. The correct expression can be obtained by considering the flux, since the flux for the correct solution must be 1, and this is an exact result. Write the solution as $(0,z)_d - T(z,0)_s$. The flux is then $T - T^*$ and thus the correct value for T is $i/2$, not i. Thus the correct value on the real line, a Stokes line, is just the average of the values obtained in domains 1 and 2, above and below the line. We thus obtain a special rule regarding Stokes lines lying on the real axis. Use half the Stokes constant to step on to the Stokes line, and again half to step off. The value exactly on the line is given by the mean of the values above and below the line.

Return to the Airy function $Bi(x)$. Choose for large negative x the asymptotic form $Bi(x) \simeq cos(\frac{2}{3}r^{3/2} + \pi/4)/(\sqrt{\pi}r^{1/4})$, express this in terms of $(0,z)$ and $(z,0)$ and continue to the right, giving for the solution exactly on the real line, $Bi(x) \simeq e^{\frac{2}{3}x^{3/2}}/(\sqrt{\pi}x^{1/4})$. It is often stated that the choice of an asymptotically dominant form does not uniquely determine the solution because of possible subdominant parts, but if the "half Stokes constant" rule is used, the solution is unique.

It is possible to find the Stokes constant for a turning point of arbitrary order by analytically solving the exact differential equation in the vicinity of the turning point. Consider the vicinity of a turning point of order n, where the differential equation has the form

$$\frac{d^2\phi}{dw^2} + w^n\phi = 0. \tag{8}$$

Substitute $\phi = w^{1/2}u$, $z = 2w^{(n+2)/2}/(n+2)$, giving Bessel's equation of order 1/(n+2),

$$u'' + \frac{1}{z}u' + \left(1 - \frac{\nu^2}{z^2}\right)u = 0, \tag{9}$$

with $\nu = 1/(n+2)$. To find the Stokes constants, we analytically continue the solution to the Bessel equation. The canonical form, Eq. 1 can be obtained in the variable z by letting $u = \psi/z^{1/2}$, giving $\psi'' + Q(z)\psi = 0$, with $Q(z) = 1 + (1/4 - \nu^2)/z^2$. For large $|z|$ the solutions have the form

$$u \simeq \frac{1}{\sqrt{z}}e^{\pm iz}. \tag{10}$$

The Stokes diagram is shown in Fig. 3. The origin is a second order pole of Q, the positive and negative real axes are anti-Stokes lines, and the positive and negative imaginary axes are Stokes lines.

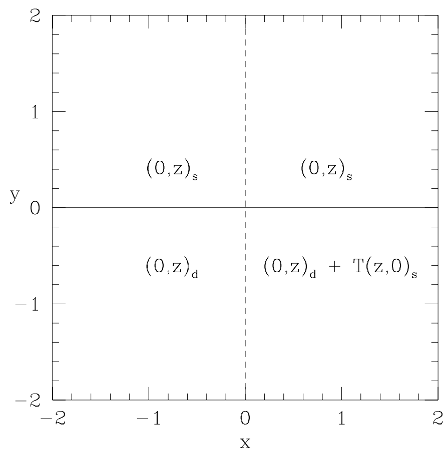

FIG. 3. Stokes diagram for the Bessel function.

Also shown are the WKBJ approximations to the solution, where we have chosen the form along the positive real axis to be $u \sim \frac{1}{\sqrt{x}} e^{ix}$, subdominant in the upper half plane. This solution has been continued counterclockwise around the origin to obtain the expressions shown in each quadrant. Note that the origin is a regular singular point of the differential equation, so the expression in quadrant 4 cannot be continued across the real axis and equated to the expression in quadrant 1, because the solution itself possesses a cut.

A general solution of the Bessel equation can be written

$$u = Az^{\nu} P_1(z^2) + Bz^{-\nu} P_2(z^2) \tag{11}$$

where P_1 and P_2 are convergent series in z^2. See for example Whittaker and Watson (1962), Sec. 17.2. We choose A and B so that the asymptotic form of u matches the expressions in Fig. 3. Along the real axis in the upper half plane we have for large x

$$Ax^{\nu} P_1(x^2) + Bx^{-\nu} P_2(x^2) = \frac{1}{\sqrt{x}} e^{ix}. \tag{12}$$

Continuing counter clockwise by taking $z = e^{i\pi} x$ and matching to the subdominant form we find for large x

$$Ae^{i\nu\pi} x^{\nu} P_1(x^2) + Be^{-i\nu\pi} x^{-\nu} P_2(x^2) = \frac{e^{-i\pi/2}}{\sqrt{x}} e^{-ix}. \tag{13}$$

Similarly letting $z = xe^{i2\pi}$ and matching to the form in the lower right quadrant we find

$$Ae^{i2m\pi} x^{\nu} P_1(x^2) + Be^{-i2\nu\pi} x^{-\nu} P_2(x^2) = \frac{e^{-i\pi}}{\sqrt{x}} e^{ix} + T\frac{e^{-i\pi}}{\sqrt{x}} e^{-ix}. \tag{14}$$

These three equations can be written as a matrix equation for the vector $(Ax^{\nu} P_1(x^2), Bx^{\nu} P_2(x^2), x^{-1/2} e^{ix})$. There is a solution only if

$$det \begin{pmatrix} 1 & 1 & -1 \\ e^{i\nu\pi} & e^{-i\nu\pi} & e^{-i2x} \\ e^{i2\nu\pi} & e^{-i2\nu\pi} & 1 + Te^{-i2x} \end{pmatrix} = 0 \tag{15}$$

which gives $T = 2icos\nu\pi$ for the Stokes constant. Returning to Eq. 8 we find the Stokes constant for a turning point of order n to be given by

$$T = 2icos\left(\frac{\pi}{n+2}\right). \tag{16}$$

The form of the WKBJ solutions in the w plane is $\phi \sim e^{\pm i \int w^{n/2}} dw$, and in the z plane $u \sim e^{\pm i \int dz}$. Since $dz \sim w^{n/2} dw$ the Stokes and anti-Stokes lines in the two planes correspond as they must. The Stokes constant for crossing Stokes lines in the complex w plane of Eq. 8 is thus given by Eq. 16.

These results can be simply summarized. They prescribe a set of rules, first given by Heading, for obtaining a globally defined WKBJ solution which corresponds to the approximation of a single solution of the differential equation.

Begin with a particular solution in one region of the z plane, choosing that combination of subdominant and dominant solutions which gives the desired boundary conditions in this region. The global solution is obtained by continuing this solution through the whole z plane effecting the following changes:

1. If a_d and a_s are respectively the coefficients of the dominant and subdominant terms of a solution, then upon crossing a Stokes line in a counterclockwise sense a_s must be replaced by $a_s + Ta_d$ where T is called the Stokes constant. When the Stokes line originates at an isolated zero of order n, $T = 2icos(\pi/(n+2))$.

2. Upon crossing a cut in a counter clockwise sense, the cut originating from a first order zero of Q at the point a, we have

$$\begin{aligned} (a, z) &\rightarrow -i(z, a) \\ (z, a) &\rightarrow -i(a, z) \end{aligned} \tag{17}$$

The property of dominancy or subdominancy is preserved in this process.

3. Upon crossing an anti-Stokes line, subdominant solutions become dominant and vice verse.

4. Reconnect from singularity a to singularity b using $(z, a) = (z, b)[b, a]$ with $[b, a] = e^{i\int_b^a Q^{1/2}dz}$. If a,b are joined by a Stokes line, reconnect while on the line, using 1/2 the usual Stokes constant to step on the line, and again 1/2 to step off.

Using these rules we can pass from region to region across the cuts, Stokes and anti-Stokes lines emanating from a turning point. Beginning with any combination of dominant and subdominant solutions in one region, this process leads to a globally defined single valued approximate solution of Eq. 1. Although it would appear that the first rule gives rise to a discontinuous solution, this is not the case. At the Stokes line, in the presence of a dominant solution, the discontinuity produced is small compared to the error due to the WKBJ approximation itself. As one continues further away from the Stokes line, however, the subdominant term will begin to be important, and the modified coefficient is the correct one.

A Stokes structure consisting of more than an isolated singularity is more complicated, in that the Stokes constants are modified by the proximity of the other singularities. However, the modification is normally exponentially small, and one can in most cases use the values of the Stokes constants obtained for isolated singularities also for complex structures. In the following, for the sake of completeness, we will derive the Stokes constants for structures involving two singularities to explicitly display this behaviour. The complete expressions are essential to understand for example the behaviour of the Stokes phenomena when two singularities approach one another, in which case the mutual influence of the singularities on one another is not negligible.

However, as will be seen, the values of the Stokes constants for the case of a bound state, the most common problem encountered in searching for instabilities, are exactly those given by the isolated singularities, and for scattering problems the use of the values for isolated singularities produces only exponentially small errors. Thus for essentially all practical calculations the values given by the isolated singularities can be used.

III. CAUSALITY

To properly understand the choice of boundary conditions in solving differential equations it is necessary to invoke causality, namely to require an outward group velocity in directions from which there is assumed to be no incoming wave. Assume for simplicity that Q tends to a constant k^2 for large $|x|$. The WKBJ solutions take the form $\Psi_\pm = e^{\pm ikx}$. In a physics problem the time dependence is also relevant, and the wave has a frequency $\omega(k)$ with the k dependence given by properties of the physical system. Form a wave packet

$$\Psi_+(x, t) = \int dk f(k) e^{i(kx - \omega t)} \tag{18}$$

where $f(k)$ is localized about k_0. Expand $\omega(k) \simeq \omega_0 + d\omega/dk(k - k_0)$, giving

$$\Psi_+(x, t) = e^{i(k_0 x - \omega_0 t)} \int dk f(k) e^{i(k-k_0)(x - td\omega/dk)}. \tag{19}$$

The integral is very small unless $x \simeq (d\omega/dk)t$, giving the usual identification of $d\omega/dk$ as group velocity, and thus Ψ_+ is outgoing for $x > 0$ if $d\omega/dk > 0$. Now consider the spatial dependence of Ψ_+ with a complex frequency. If the mode frequency is $\omega = \omega_r + i\gamma$ expanding $e^{ik(\omega)x}$ gives

$$\Psi_+ \simeq e^{(ik_0 x - \gamma x dk/d\omega)}. \tag{20}$$

We then obtain the rules for the asymptotic behaviour of the solution. If $\gamma > 0$ and $d\omega/dk > 0$ the solution is decreasing (subdominant) for large x. Physically the spatial behaviour can be simply understood in terms of information propagation. If the mode is growing the news of its growth propagates outward at the group velocity, so the mode is largest for small $|x|$, *i.e.* it is damped in the direction of propagation. In the following we will use the introduction of a small dissipation to clarify the choice of the solution*.

IV. BOUND STATES - INSTABILITIES

The bound state problem, or the search for instability, is generically given by a function $Q(z)$ which is real on the real axis, with two first order zeros at points a and b, and Q is positive between a and b and negative otherwise. The Stokes diagram is shown in Fig. 4. Denote the Stokes constants as S_k, where k refers to a line bordering domain k. We will find that the boundary conditions immediately give the Bohr-Sommerfeld condition, which determines the energy of the bound state, or equivalently, the growth rate of the instability, independent of the values of the Stokes constants. We further find the six Stokes constants can be represented by a single magnitude and one phase δ but with a sign which must be distributed as shown in Fig. 4. In addition, the bound state boundary conditions fix the Stokes constants to be equal to the value for isolated singularities.

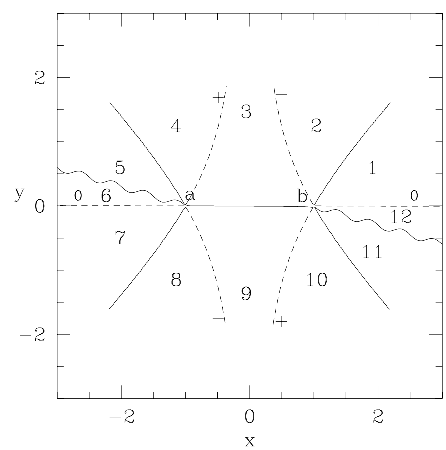

FIG. 4. Stokes plot for the bound state problem.

Begin at large positive x with a subdominant solution. Define $[a, b] = e^{iW}$. Along the real axis between a and b Q is real and positive and thus W is real and positive. With the choice of cuts as shown, and using Q positive and real for $a < x < b$ we have for $x > b$ and for $x < a$ that $Q^{1/2} \sim e^{-i\pi/2}$, $Q^{1/4} \sim e^{-i\pi/4}$. Thus for $b < z$ the solution (b, z) is dominant. Begin with a subdominant solution at large positive x and continue:
(1) $(z, b)_s$
(2) $(z, b)_d$
(3) $(z, b)_d + S_2(b, z)_s = e^{iW}(z, a)_d + e^{-iW}S_2(a, z)_s$
(4) $e^{iW}(z, a)_d + [e^{iW}S_4 + e^{-iW}S_2](a, z)_s$
(5) $e^{iW}(z, a)_s + [e^{iW}S_4 + e^{-iW}S_2](a, z)_d$
(6) $-ie^{iW}(a, z)_s - i[e^{iW}S_4 + e^{-iW}S_2](z, a)_d$
The differential equation is real, and thus the complex phase of the solution is constant along the real axis. $(z, a)_d$

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*Note that the convention $e^{-i\omega t}$, is opposite from that used by Heading, which switches upper and lower half planes in some arguements.

in domain 6 (z real) has the same complex phase as $(z,b)_s$ in domain 1 (z real), thus the coefficient of the dominant solution must be real for any W, giving $S_4 = iSe^{i\delta}$, $= S_2 = iSe^{-i\delta}$, with S, δ real. We then have
(6) $-ie^{iW}(a,z)_s + 2Scos(W+\delta)(z,a)_d$.
If the dominant part is nonzero, the subdominant correction is not relevant and we cannot require it to be real. Requiring the dominant solution to be zero gives $W+\delta = (n+1/2)\pi$, in which case the subdominant part must be real, giving $W = (n+1/2)\pi$, and therefore $\delta = 0$. S remains undetermined. Note however that bound state boundary conditions, namely the requirement that the dominant solution vanish to the left, gives $W = (n+1/2)\pi$, the Bohr-Sommerfeld quantization condition,

$$\int_a^b Q^{1/2}dx = (n+1/2)\pi, \tag{21}$$

independent of the value of the Stokes constant. In the following we continue the solution completely around the complex plane in an attempt to determine S. This turns out to be only possible if the Bohr-Sommerfeld condition is imposed.

Continuing we find
(7) $2Scos(W+\delta)(z,a)_d + [-ie^{iW} + 2Scos(W+\delta)S_6](a,z)_s$
(8) $2Scos(W+\delta)(z,a)_s + [-ie^{iW} + 2Scos(W+\delta)S_6](a,z)_d$.
Beginning again in domain 1 and continuing in the lower half plane we find
(12) $(z,b)_s$
(11) $i(b,z)_s$
(10) $i(b,z)_d$
(9) $i(b,z)_d - iS_{10}(z,b)_s = ie^{-iW}(a,z)_d - iS_{10}e^{iW}(z,a)_s$
(8) $ie^{-iW}(a,z)_d - i[S_{10}e^{iW} + S_8e^{-iW}](z,a)_s$
(7) $ie^{-iW}(a,z)_s - i[S_{10}e^{iW} + S_8e^{-iW}](z,a)_d$
Now compare the solutions obtained by continuing in each direction. The matching can be carrried out in both domains (7) and (8), where dominant and subdominant terms switch roles. Comparing we find $S_{10} = iSe^{i\delta}$, $S_8 = iSe^{-i\delta}$, and $S_6 = iS$,

$$S = \sqrt{\frac{cos(W)}{cos(W+\delta)}}. \tag{22}$$

The relative distribution of positive, negative, and zero phases δ are thus determined and shown in Fig. 4 but matching determines neither the phase δ nor the magnitude S. However, at each bound state solution $\delta = 0$ and thus $S = 1$, *i.e.* the Stokes constant is equal to its value for an isolated first order zero. The general value is of interest only for the limit $a \to b$, in which case the Stokes constant differs significantly from the isolated singularity value.

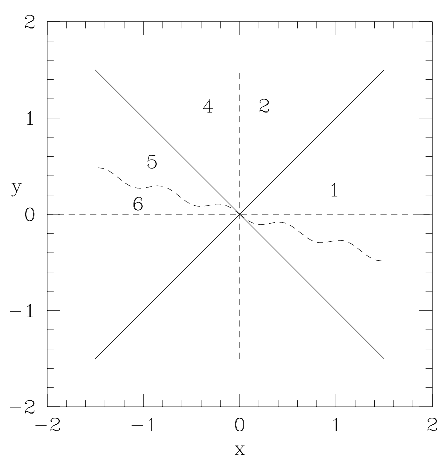

FIG. 5. Stokes plot for the bound state, $a \to b$.

The limit $a \to b$ is singular in that each pair of anti-Stokes lines extending vertically merges to become one, and thus the Stokes constants associated with two lines become that of one. The Stokes diagram obtained in this limit is shown in Fig. 5. For $a \to b$ the continuation is given by the above with $W \to 0$.

(1) $(z,0)_s$
(2) $(z,0)_d$
To pass to domain 4 we cross two Stokes lines. Substituting the values for S_2 and S_4 we find
(4) $(z,0)_d + 2i\sqrt{cos(\delta)}(0,z)_s$
(5) $(z,0)_s + 2i\sqrt{cos(\delta)}(0,z)_d$
.

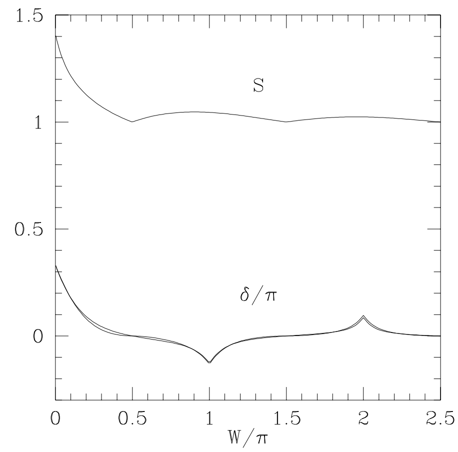

FIG. 6. δ and S for the bound state.

Whereas in the singular limit $a = b$ the Stokes constant is $\sqrt{2}i$ and the continuation is
(1) $(z,0)_s$
(2) $(z,0)_d$
(4) $(z,0)_d + \sqrt{2}i(0,z)_s$
(5) $(z,0)_s + \sqrt{2}i(0,z)_d$
Comparing we find $\delta = \pi/3$. The Stokes constants for the two cases are not identical, with $S_k = e^{i\pi/3}\sqrt{2}i$, $e^{-i\pi/3}\sqrt{2}i$, $\sqrt{2}i$ in the case $a \to b$ (six Stokes lines) and $S_k = \sqrt{2}i$ for $a = b$ (four Stokes lines), with the two Stokes constants associated with the merging vertical lines, $S_k = e^{\pm i\pi/3}\sqrt{2}i$ adding to give $\sqrt{2}i$.

A numerical determination of the phase δ and an analytic fit as a function of W for the case $Q = b^2 - z^2$ are shown in Fig. 6. The two curves are almost indistinguishable. Note that at all bound states $W = (n + \frac{1}{2})\pi$ the phase is zero and $S = 1$, and at W = 0 $S = \sqrt{2}$. The approximate fit to S satisfying these conditions and giving a very good fit to the numerical δ is given by $S = \sqrt{1 + |cos(W)|/(1 + 3W)}$ and shown in the figure, with of course S and δ related by Eq. 22.

V. OVERDENSE BARRIER - SCATTERING

The overdense barrier is given by a function $Q(z)$ which is real on the real axis, with two first order zeros at real points a and b, and Q is negative between a and b and positive otherwise. The Stokes diagram is shown in Fig. 7, and propagating oscillatory solutions exist for large positive and negative x.

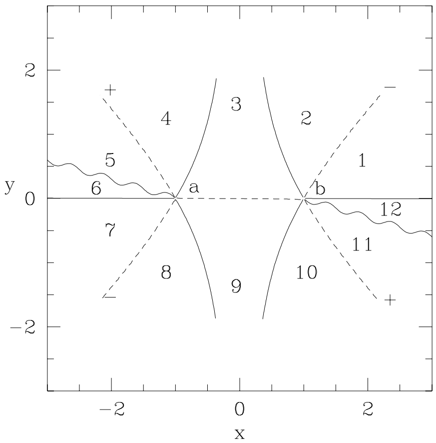

FIG. 7. Stokes plot for the overdense barrier.

We consider an incident wave from the left, the problem being to determine the reflected and transmitted waves. In classical physics problems the absolute square of these coefficients gives the reflected and transmitted power, and in quantum mechanical problems the probability of reflection and transmission. Thus causality requires outgoing boundary conditions at the far right. Define W through $[a, b] = e^{-W}$. Take Q to be real and positive for $b < x$, then with the choice of the cuts as shown in Fig. 7 along the real axis between a and b Q has phase $Q \sim e^{i\pi}$ and W is real and positive, and for $x < a$ Q is again real and positive. Requiring outgoing wave conditions for large positive x gives boundary conditions of a subdominant solution in domain (1). Continuation through the upper half plane gives:
(1) $(b, z)_s$
(2) $(b, z)_s$
(3) $(b, z)_d = e^W (a, z)_s$
(4) $e^W (a, z)_d$
(5) $e^W (a, z)_d + S_4 e^W (z, a)_s$
(6) $-ie^W (z, a)_d - iS_4 e^W (a, z)_s$

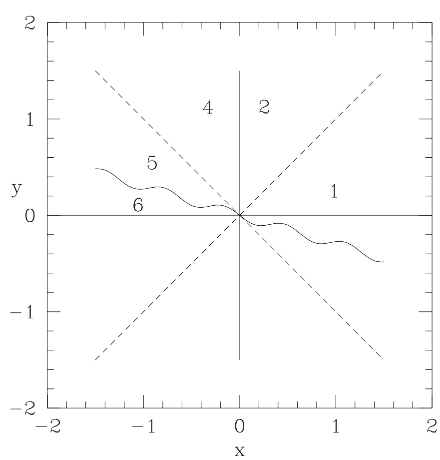

FIG. 8. $Q = z^2$, Stokes plot.

Now consider the flux. In domain 1 we have $flux = Im(\psi^* \psi') = 1$. In domain 6 we find $flux = S_4 S_4^* e^{2W} - e^{2W}$. Without loss of generality write

$$S_4 = i\sqrt{1 + e^{-2W}} e^{i\delta}, \qquad (23)$$

with δ undetermined. The solution in domain (6) becomes
(6) $\sqrt{1 + e^{-2W}} e^{i\delta} e^W (a, z)_s - ie^W (z, a)_d$.
In domain (6) the subdominant solution is right-moving, and the dominant solution is thus the reflected wave. Using

the edges of the propagation domains, a and b as the reference points for phase changes we find reflection and transmission coefficients of

$$r = -\frac{ie^{-i\delta}}{\sqrt{1+e^{-2W}}} \tag{24}$$

$$t = \frac{e^{-W-i\delta}}{\sqrt{1+e^{-2W}}}. \tag{25}$$

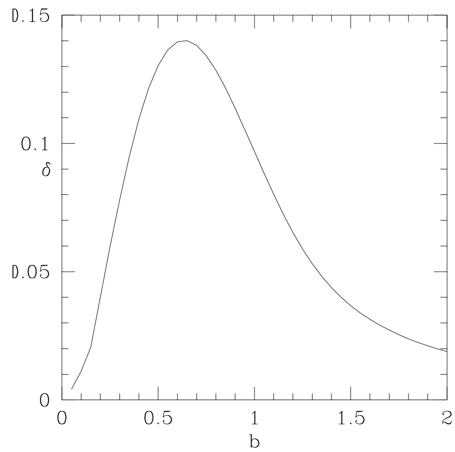

FIG. 9. Phase δ for overdense scattering, $Q = z^2 - b^2$.

We thus find the probabilities for reflection and transmission to be

$$|r|^2 = \frac{1}{1+e^{-2W}} \tag{26}$$

$$|t|^2 = \frac{e^{-2W}}{1+e^{-2W}}. \tag{27}$$

The phase δ is not given in general by WKB analysis, but can be determined in two limits. First examining the limit $a \to b$. The Stokes diagram collapses to a single second order zero, and the Stokes plot is shown in Fig. 8. We retain the cuts to make the results coincide exactly with those resulting from the limit $a \to b$ in Fig. 7. The Stokes constants have a continuous limit, all lines remaining distinct. In this case the Stokes constant is $\sqrt{2}i$ and the continuation is given by the above with $W = 0$, $\delta = 0$, giving for the reflection and transmission coefficients $r = -i/\sqrt{2}$, $t = -1/\sqrt{2}$, and $|r|^2 = |t|^2 = 1/2$. In the limit $|a - b| \to \infty$ the singularities are isolated and $\delta = 0$, $S = 1$. Note that the analytic continuation carried out for large $|z|$ is valid for all W, large W is not a requirement for the WKB analysis.

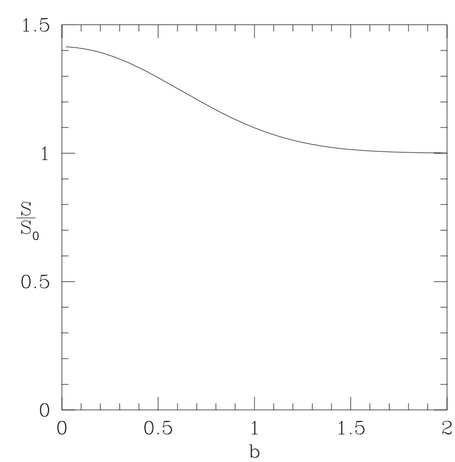

FIG. 10. Stokes constant for overdense scattering, $Q = z^2 - b^2$.

Similarly by continuation through the lower half plane it can be shown that all Stokes constants can be written in the form $S_k = iSe^{\pm i\delta}$, $S = \sqrt{1+e^{-2W}}$, for the line bordering domain k, with the sign of the phase δ distributed as shown in Fig. 7.

The phase can be determined for a particular functional form of $Q(z)$ by an analytical continuation of the solution (Bender-Orszag 1978, Ford 1959a 1959b, Soop 1965), a numerical integration of the differential equation, or by a matching of the solution to a numerical evaluation of the two Taylor series solutions. Shown in Figs. 9, 10 is the phase δ and the Stokes constant for the overdense barrier scattering problem with $Q = z^2 - b^2$, determined using the Taylor series to find the reflection and transmission coefficients. Note that the Stokes constant deviates significantly from the value given by the isolated singularity analysis, $S = i$ only for $b < 1$.

VI. UNDERDENSE BARRIER - SCATTERING

The underdense barrier problem is given by a function $Q(z)$ which is real on the real axis, with two first order zeros at points a and b which are pure imaginary, and Q is positive everywhere on the real axis. The Stokes diagram is shown in Fig. 11. Again define W through $[a, b] = e^{-W}$. Along the imaginary axis between a and b Q is real and positive and W is real and positive. Consider an incident wave from the left, giving again outgoing boundary conditions at the far right. For this problem Heading discusses two methods which he discards as approximate. The first method is continuation in the lower half plane alone. This gives a transmission coefficient of 1 and zero reflection. It is used by Berry (1990) in an analysis of the birth point of reflected waves.

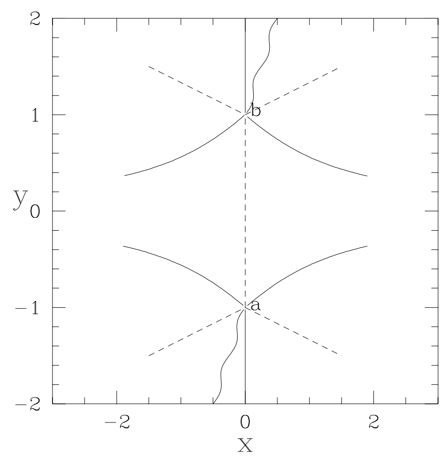

FIG. 11. Stokes plot for the underdense barrier.

Continuation in the upper half plane gives a transmission coefficient of 1 and a reflection coefficient of $r = -ie^{-W}$. This continuation involves the singular point $z = b$ only, and the correct Stokes constant is that associated with an isolated first order zero, $S = i$. This is because, as far as the continuation is concerned, the second singularity at $z = a$ does not exist, and the Stokes diagram consists only of the structure in the upper half plane.

However, neither of these continuations give solutions which conserve flux and neither can be considered correct. A third method is given by Heading, who describes it as more accurate, but does not give a justification for its use. A justification for its use comes from the consideration of causality. Not only is it necessary to consider outgoing wave boundary conditions, a small dissipation must be considered so as to cause waves to damp in the direction of propagation. This is done by multiplying $Q(z)$ by $e^{i\nu}$ with ν small and positive. Since Stokes lines are given by $dz \sim 1/\sqrt{Q(z)}$ this results in a rotation of the Stokes plot. Shown in Fig. 12 is the resulting plot (the rotation has been exaggerated for clarity). Now the only possible continuation from large real positive z to large real negative z is clear. It is necessary to begin in domain 1 and continue to domain 7, a process which necessarily involves both singular points.

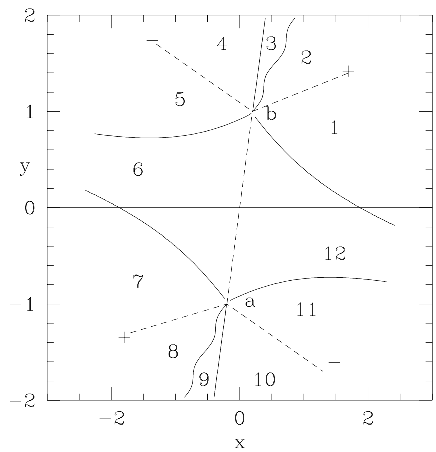

FIG. 12. Underdense barrier, rotated Stokes plot.

Continuation then gives:
(1) $(b,z)_s$
(2) $(b,z)_s$
(3) $i(z,b)_s$
(4) $i(z,b)_d$
(5) $i(z,b)_d + iS_-(b,z)_s$
(6) $i(z,b)_s + iS_-(b,z)_s$
(6) $ie^{-W}(z,a)_d + iS_- e^{2W}(a,z)_s$
(7) $i(z,a) + iS_-(0,z)$
The incoming solution is $(a,z) = e^{-W}(b,z)$, and the reflected wave is $(z,a) = e^{-W}(z,b)$. We then have

$$r = \frac{e^{-W-i\delta}}{S} \tag{28}$$

$$t = \frac{-ie^{-i\delta}}{S}. \tag{29}$$

and equating flux for $x \to -\infty$ $x \to \infty$ gives again $S = i\sqrt{1+e^{-2W}}$, but again δ is undetermined. This gives

$$|r|^2 = \frac{e^{-2W}}{1+e^{-2W}} \tag{30}$$

$$|t|^2 = \frac{1}{1+e^{-2W}}. \tag{31}$$

A comparison with numerical integration for the case $Q = z^2 + b^2$ is shown in Fig 13, obtained numerically at $|x| = 5$ using 100 terms in the Taylor Series, with the constants in the series fixed by requiring outgoing wave conditions at $x = 5$. Solid points show the data from the Taylor expansion and open points the data from the WKBJ continuation using Eqs. 30, 31.

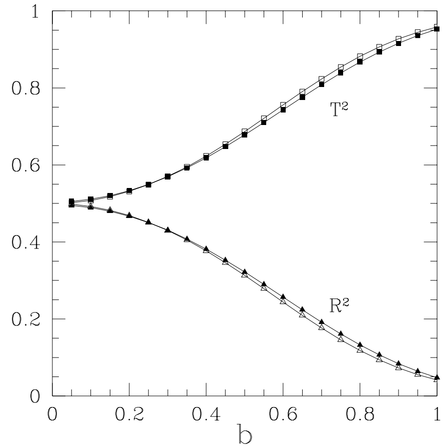

FIG. 13. $Q = z^2 + b^2$, transmission and reflection vs b.

The phase δ can again be determined in two limits. First examine the limit $a \to b$. The Stokes diagram collapses to a single second order zero, and the Stokes plot is shown in Fig. 14, the same as in the case of the overdense barrier Fig. 8 except for cut placement. We retain the cuts to make the results coincide exactly with those resulting from the limit $a \to b$ in Fig. 12. Again the Stokes constants have a continuous limit. In this case the Stokes constant is $\sqrt{2}i$ and the continuation is given by the above with $W = 0$, giving for the reflection and transmission coefficients $r = -i/\sqrt{2}$, $t = -1/\sqrt{2}$, and $|r|^2 = |t|^2 = 1/2$.

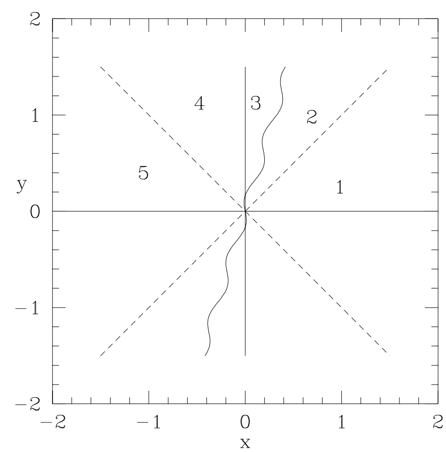

FIG. 14. $Q = z^2$, Stokes plot.

The phase δ is found numerically to be the same as in the overdense case, which is not surprising since the two Stokes diagrams are equivalent under rotation.

VII. EIGENVALUE PROBLEMS

Normally in plasma physics problems the function Q is a complex function of the frequency of the mode ω. For an arbitrary guess for ω the Stokes plot will be quite complicated. To find an unstable mode one must identify turning points and check whether there exists a value of ω giving a solution which satisfies the required boundary conditions. This is normally best accomplished numerically (White, 1979). Unless ω is chosen to be the eigenvalue, pairs of turning points will not exhibit the canonical attractive well Stokes structure shown in Fig. 4. A typical example of what is seen is shown in Fig. 15. In addition there may be other nearby singularities and zeros of $Q(z, \omega)$, making it difficult to recognize possible unstable mode structures. This is especially true if the physical problem possesses several parameters which must be explored.

As an example we give a problem which arose in the theory of the collisionless drift wave. The differential equation is given on the real axis by

$$Q = -\frac{\omega L}{2}\left(K^2 - 1 - \frac{1}{\omega}\right) + \frac{x^2}{4} + \frac{L}{2}(1-\omega)AZ(A) \tag{32}$$

where L, K, R are real parameters, $A = (\omega RL)^{1/2}/|x|$ and Z is the plasma dispersion function. A more complete description of the derivation and solutions of this equation has been reported elsewhere, Chen *et al.* (1978), Pearlstein and Berk (1969), Tsang *et al.* (1978), Ross and Mahajan (1978). The expression $|x|$ arises in a derivation valid on the real axis, and the analytic continuation into the complex plane is provided by $|x| = (z^2)^{1/2}$ with an appropriate choice of cuts.

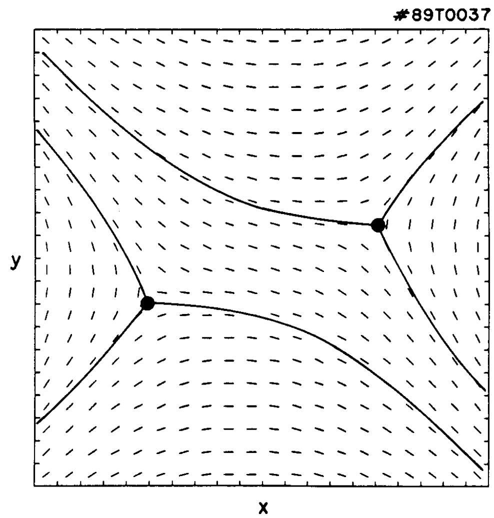

FIG. 15. Stokes diagram showing two turning points.

The physical real axis is determined by $|x| \geq 0$, and thus the physical plane is divided into two parts by branch cuts which can be taken along the imaginary axis. Thus in the physical plane for $Rez > 0$, $|x|$ is replaced by z, and for $Rez < 0$, $|x|$ is replaced by $-z$. The continuation of these functions through the cuts defines a second plane which we will refer to as the nonphysical plane. It is also divided into two distinct parts. For values of K much less than a critical value K_c which depends on R and L (for $R = 1/1837$ and $L = 50$, $K_c = 0.36$) there are two turning points located approximately along the line $x = -y$ in the physical plane. As K increases, these turning points begin to migrate toward the imaginary axis. In Fig. 16 is shown the location of the turning points (v and $-v$) for a mode with $L = 50$, $K = 0.26$, and $R = 1/1837$. Note the presence of another pair of turning points (g and $-g$), located in the nonphysical plane. Assume a decaying mode, in which case the solution for $x \to \infty$ must be dominant, $\psi(z) = (z, v)_d$. Continuing in toward $x = 0$ we cross a Stokes line associated with v and thus the solution becomes $\psi(z) = (z, v)_d + i(v, z)_s$. The differential equation and the turning points under consideration are symmetric about $z = 0$, thus we can choose a solution with a particular parity with respect to z. For an odd solution we require $\psi(0) = 0$, or $(0, v) + i(v, 0) = 0$. This has the solution $2\int_0^v Q^{1/2}dz = [n + 1/2]\pi$, n odd. For the even solutions we require that $d\psi/dz$ vanish at $z = 0$, or $i(0, v) + (v, 0) = 0$. This has the solution $2\int_0^v Q^{1/2}dz = [n + 1/2]\pi$, n even. Thus we obtain the standard connection formula between $v, -v$ which for the parameters given above gives $\omega = 0.908 - 0.008i$. The mode is decaying in agreement with our initial assumption of a purely dominant solution for large x. It is readily verified that the anti-Stokes line emanating from v toward positive x does not cross the real axis.

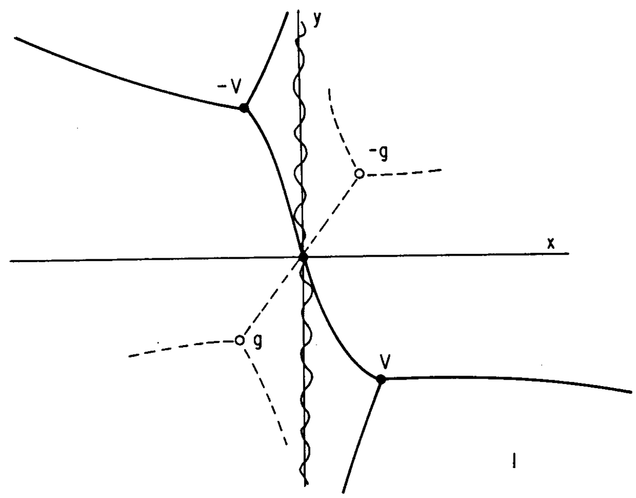

FIG. 16. Stokes structure for $K < K_c$.

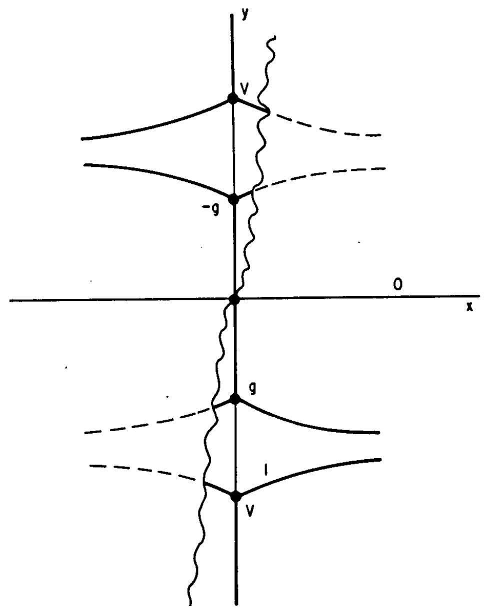

FIG. 17. Stokes structure for $K > K_c$.

As the parameter K increases, provided $L > 3R^{-1/4}$, the turning points $v, -v$ as well as the turning points $g, -g$ approach the imaginary axis and coalesce for $K = K_c$ after which the Stokes diagram takes the form of Fig. 17. The role of turning point for the determination of the solution has been passed on from v to g. Once again the analysis presented above carries through, only now the connection formula is determined by performing the integral from g to $-g$. We then discover that the growth rate $Im(\omega)$ is zero. If $L < 3R^{-1/4}$ this coalescense does not occur for any value of K, and the mode continues to be determined by the turning point v.

For K less than the critical value; *i.e.*, with Stokes structure as shown in Fig. 16, imposing the connection formula between g and $-g$ gives rise to a growing mode $Im(w) > 0$. However, this ghost mode remains in the nonphysical plane, interpretable as an outwardly propagating solution only in this plane. The critical value of K is seen to be the coalescense of the turning points of a nonphysical growing mode and a physical damped mode. The nonphysical turning point then dominates to produce a marginally stable mode for all larger values of K.

VIII. PROBLEMS

1. Sketch the Anti-Stokes lines associated with $d^2\psi/dz^2 + Q\psi = 0$ for $|z| < 4$

$$Q = \frac{z - i\pi}{z + i\pi} sinz$$

2. Sketch the Anti-Stokes and Stokes lines and find the Stokes constant T for a second order zero, $Q = z^2$, by directly continuing the solution around the turning point and requiring the solution to be single valued. Check your result using Eq. 16.

3. Consider the eigenvalue problem $y'' + Ecos(x)y = 0$ with $y(\pi) = 0$, and $y'(0) = 0$, $E > 0$. Sketch the Stokes plot for y.

a. Use WKB to find the three smallest eigenvalues E, approximating Stokes constants as those from isolated singularities.

Hint 1: The bounday condition $y(\pi) = 0$ is satisfied ON a Stokes line.

Hint 2: Continue the WKB solution to $x = 0$ and require $y'(0) = 0$, giving a transcendental equation for $w = \int_0^{\pi/2} \sqrt{Ecos(x)}dx$. Solve this iteratively. Note

$$\int_0^{\pi/2} \sqrt{cos(x)}dx = \int_{\pi/2}^{\pi} \sqrt{-cos(x)}dx \simeq 1.19814025$$

b. Integrate the equation numerically using the WKB values as first approximations to E. Adjust E to give $y(\pi) = 0$. What is the accuracy of the WKB eigenvalues?

4. Consider the differential equation

$$\frac{d^2y}{dx^2} + \left(\omega^2 - \frac{x^2}{4}\right) y = 0$$

a. Find the asymptotic behaviour of the two solutions for $x \to \pm\infty$.

b. Write the solutions as $y \sim e^S W(x)$ and find a series representation for $W(x)$ for the solution tending to zero at ∞. Find the radius of convergence of the series.

c. Draw the Stokes diagram. Use WKB analysis to find ω such that the solution tends to zero at $\pm\infty$. These are the eigenvalues given by the WKB analysis.

d. What happens to the series for $W(x)$ for these values of ω?

e. What is the accuracy of the eigenvalues and eigenfunctions given by the WKB method?

5. Consider the differential equation

$$\frac{d^2\psi}{dx^2} + B(x^2 - x^4 - 1/4 + E)\psi = 0,$$

boundary conditions $\psi(-\infty) = 0, \psi(\infty) = 0$, with $B = 10^4$, $0 < E << 1$.

a. Sketch the Stokes plot for ψ.

Neglecting exponentially small tunneling effects there are two states with the smallest value of E (ground states), one symmetric and one antisymmetric in x.

b. Sketch the ground state eigenfunctions and write an expression determining the energy. Use $E \ll 1$ to expand the potential to second order in x around the minima and calculate the ground state energy.

6. Use WKB theory and require that the two ground states be exactly symmetric and antisymmetric to find an expression for the changes in the energy from the degenerate value of problem 5. Use Stokes constants for isolated

singularities. HINT- It is not necessary to follow the solution through the whole Stokes diagram, for symmetry use $\psi'(0) = 0$ and for antisymmetry use $\psi(0) = 0$.

7. A normally incident plane polarized electromagnetic wave enters a magnetized plasma $\vec{B}_0 = B_0\hat{z}$ where $\omega_p(x)$ increases with x. If $\vec{k}$ and $\vec{E}$ of the wave are perpendicular to $\vec{B}_0$, $k_y = k\hat{y}$, $\vec{E} = E\hat{y}$ the y component of $\vec{E}$ satisfies

$$\frac{d^2E_y}{dx^2} + k_0^2\epsilon(x)E_y = 0$$

The dielectric function has a cutoff, $\epsilon = 0$, and resonance $\epsilon = \infty$. Model this dielectric through $k^2\epsilon(x) = 1 + a/x$. Draw the Stokes diagram and find the reflection and transmission coefficients for a wave incident from the left using phase integral methods. This is called the Budden problem, see Budden (1979), White and Chen, Plasma Physics 16, 565 (1974). Note that $|R|^2 + |T|^2 \neq 1$. Why?

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