Appendix 6
Plasma Flows, Radial Electric Fields, and Radial Currents in a 3D Electron-Ion Plasma.

A6.0: Introduction

This appendix lays out the method of calculating the neoclassical heat and particle flows in an unbiased plasma. The development is based on the works by Shaing[1], Shaing and Callen,[2] and Coronado and Talmadge,[3] but most closely follows the work of Coronado and Wobig.[4,5] Indeed, reference 4 specifies the solution to the problem in very general terms. Easily computed expressions for the viscosity are found in Shaing, Hirshman, and Callen.[6]

The purpose of this discussion is to specify the derivation for an electron-ion plasma in HSX, and cast the equations in a form that can be easily computed and compared to measurements. Some further points regarding neoclassical theory will be clarified ion the process. It is assumed that the gradients in the plasma potential, electron and ion pressure, and electron and ion temperature are all measured quantities; the modeling predicts the particle and heat flows based upon these measured gradients. It is also assumed that the Hamada spectrum and basis vectors are known for the configurations of interest, so that calculation of the contravariant components of the flow is sufficient to specify the flows.

This section is laid out as follows. Section A6.1 details the consequences of incompressibility of the heat and particle flows; the contravariant flows are written in terms of **E**x**B** and diamagnetic parts, and unknown force free parallel parts. Section A6.2 specifies the forms of the viscosity and collisional friction. Section A6.3 solves the parallel momentum and heat flux balance equations to derive a set of coupled differential equations for the force free flows. These equations can be solved for the force free parallel flows, solving the problem. Section A6.5 provides the relationship between the gradients and radial currents. Sections A6.5 calculates the viscosity coefficients needed in the formulation.

The previous developments on this subject have generally used the volume as a flux surface label and let the Hamada toroidal (ζ) and poloidal (α) angles vary from zero to one. The development here will leave the flux surface label as an arbitrary ρ, and allow the angle to vary as $0<\alpha,\zeta<2\pi$. It will be assumed throughout that the neutral density is a flux surface constant. This restriction could presumably be removed via the techniques described in chapter 6. This derivation will assume that the reader is familiar with Hamada coordinates and the manipulations that were performed in the chapters 5 and 6.

A6.1: Source of the Force Free Heat and Particle Flows.

For any given species 'a', the lowest order heat and particle flows are determined by the lowest order components of the continuity, momentum balance, pressure, and heat flux equations[1,2,4]

$$\nabla \cdot \mathbf{U}_a = 0 \tag{A6.1}$$

$$\frac{\mathbf{U}_a \times \mathbf{B}}{c} = \nabla\Phi + \frac{1}{e_a n_a}\nabla p_a \tag{A6.2}$$

$$\nabla \cdot \mathbf{q}_a = 0 \tag{A6.3}$$

$$\mathbf{q}_a \times \mathbf{B} = \frac{5cp_a}{2e_a}\nabla T_a \tag{A6.4}$$

The terms on the left of these equations are the flows; the terms on the right are the driving gradients. It can be shown[4,5] that these expressions lead to poloidal and toroidal heat and particle fluxes of the form

$$U_a^\alpha = \frac{c}{B^\zeta\sqrt{g}}\left(\frac{A_1}{A_1+A_2}\right)\left(\frac{\partial\Phi}{\partial\rho} + \frac{1}{e_a n_a}\frac{\partial p_a}{\partial\rho}\right) + \lambda_a B^\alpha \tag{A6.5a}$$

$$U_a^\zeta = -\frac{c}{B^\alpha\sqrt{g}}\left(\frac{A_2}{A_1+A_2}\right)\left(\frac{\partial\Phi}{\partial\rho} + \frac{1}{e_a n_a}\frac{\partial p_a}{\partial\rho}\right) + \lambda_a B^\zeta \tag{A6.5b}$$

$$q_a^\alpha = \frac{c}{B^\zeta \sqrt{g}} \left(\frac{B_1}{B_1 + B_2} \right) \left(\frac{5p_a}{2e_a} \frac{\partial T_a}{\partial \rho} \right) + p_a \sigma_a B^\alpha \tag{A6.6a}$$

$$q_a^\zeta = -\frac{c}{B^\alpha \sqrt{g}} \left(\frac{B_2}{B_1 + B_2} \right) \left(\frac{5p_a}{2e_a} \frac{\partial T_a}{\partial \rho} \right) + p_a \sigma_a B^\zeta \tag{A6.6b}$$

A set of equations very similar to these was derived in Section 5.4.4. In general, one is free to pick any value for A_1, A_2, B_1, and B_2, as long as $A_1+A_2 \neq 0$ and $B_1+B_2 \neq 0$. Coronado suggests the following choice,[4] which will be used throughout this derivation:

$$A_1 / A_2 = B_1 / B_2 = \frac{\langle B_\zeta B^\zeta \rangle}{\langle B_\alpha B^\alpha \rangle} = \frac{\langle \mathbf{B} \cdot \mathbf{B}_T \rangle}{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle} \tag{A6.7}$$

Note that this choice is not unique. In particular, the Coronado and Talmadge development uses $A_1=1$ and $A_2=0$. With the choice made in (A6.7) and collapsing the gradients into the constants

$$G_a = \left(\frac{\partial \Phi}{\partial \rho} + \frac{1}{e_a n_a} \frac{\partial p_a}{\partial \rho} \right) \tag{A6.8a}$$

$$\overline{G}_a = \left(\frac{5p_a}{2e_a} \frac{\partial T_a}{\partial \rho} \right) \tag{A6.8b}$$

the heat and particles fluxes are given by

$$U_a^\alpha = \frac{cG_a}{B^\zeta \sqrt{g}} \frac{\langle \mathbf{B} \cdot \mathbf{B}_T \rangle}{\langle \mathbf{B} \cdot \mathbf{B} \rangle} + \lambda_a B^\alpha \tag{A6.9a}$$

$$U_a^\zeta = -\frac{cG_a}{B^\alpha \sqrt{g}} \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle \mathbf{B} \cdot \mathbf{B} \rangle} + \lambda_a B^\zeta \tag{A6.9b}$$

$$q_a^\alpha = \frac{c\overline{G}_a}{B^\zeta \sqrt{g}} \frac{\langle \mathbf{B} \cdot \mathbf{B}_T \rangle}{\langle \mathbf{B} \cdot \mathbf{B} \rangle} + p_a \sigma_a B^\alpha \tag{A6.10a}$$

$$q_a^\zeta = -\frac{c\overline{G}_a}{B^\alpha \sqrt{g}} \frac{\langle \mathbf{B} \cdot \mathbf{B}_P \rangle}{\langle \mathbf{B} \cdot \mathbf{B} \rangle} + p_a \sigma_a B^\zeta \tag{A6.10b}$$

These expressions will be used throughout this derivation. Note that all of the terms are known except for the force free parts λ_i, λ_e, σ_i, and σ_e.

To solve for these terms, it is necessary to look at the next order terms in the fluid equations. Consider the 1st order momentum and heat balance equations: [4,1]

$$m_i N_i \frac{\partial}{\partial t}\mathbf{U}_a = N_a e_a\left(\mathbf{E} + \frac{1}{c}\mathbf{U}_a \times \mathbf{B}\right) - \nabla\cdot\mathbf{D}_a - \nabla p_a - m_a N_a \upsilon_{an}\mathbf{U}_a + \mathbf{F}_{a1} \qquad \text{(A6.11)}$$

$$\frac{\partial}{\partial t}\mathbf{q}_a = \frac{e_a}{m_a}\frac{1}{c}\mathbf{q}_a \times \mathbf{B} - \frac{T_a}{m_a}\nabla\cdot\grave{\mathbf{E}}_a + \frac{5}{2}\frac{T_a}{m_a}\nabla p_a - \upsilon_{an}\mathbf{q}_a + \frac{T_a}{m_a}\mathbf{F}_{a2} \qquad \text{(A6.12)}$$

The required parallel momentum and heat balance equations can be derived by calculating the scalar products of (A6.11) and (A6.12) with **B** and taking a flux surface average, assuming that there is no inductive electric field. After this has been done, the steady state version of these equations becomes

$$< \mathbf{B}\cdot\nabla\cdot\mathbf{D}_a > + m_a N_a \upsilon_{an} < \mathbf{B}\cdot\mathbf{U}_a > - \left\langle \mathbf{B}\cdot\mathbf{F}_{1a}\right\rangle = 0 \qquad \text{(A6.13)}$$

$$< \mathbf{B}\cdot\nabla\cdot\grave{\mathbf{E}}_a > + \frac{m_a \upsilon_{an}}{T_a} < \mathbf{B}\cdot\mathbf{q}_a > - \left\langle \mathbf{B}\cdot\mathbf{F}_{2a}\right\rangle = 0 \qquad \text{(A6.14)}$$

For the electron-ion plasma being considered here, there are the four unknowns (λ_i, λ_e, σ_i, and σ_e) and four equations ((A6.13) and (A6.14), for both electrons and ions). The task is to write the equations (A6.13) and (A6.14) in terms of the unknown variables. To do this, it is necessary to specify the friction and viscous forces.

A6.2: Specification of the Friction Forces and Viscosities.

This section specifies the form of the friction and viscous forces needed to solve equations (A6.13) and (A6.14) for the parallel heat and particle flows.

To begin with, the viscosities can be written in general as

$$\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{D}_a\right\rangle = \mu_{a1\alpha}U_a^{\alpha} + \mu_{a1\zeta}U_a^{\zeta} + \mu_{a2\alpha}\frac{q_a^{\alpha}}{p_a} + \mu_{a2\zeta}\frac{q_a^{\zeta}}{p_a} \qquad \text{(A6.15a)}$$

$$\left\langle \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_a \right\rangle = \mu^{(P)}_{a1\alpha} U^{\alpha}_a + \mu^{(P)}_{a1\zeta} U^{\zeta}_a + \mu^{(P)}_{a2\alpha} \frac{q^{\alpha}_a}{p_a} + \mu^{(P)}_{a2\zeta} \frac{q^{\zeta}_a}{p_a} \tag{A6.15b}$$

$$\left\langle \mathbf{B} \cdot \nabla \cdot \grave{\mathbf{E}}_a \right\rangle = \gamma_{a1\alpha} U^{\alpha}_a + \gamma_{a1\zeta} U^{\zeta}_a + \gamma_{a2\alpha} \frac{q^{\alpha}_a}{p_a} + \gamma_{a2\zeta} \frac{q^{\zeta}_a}{p_a} \tag{A6.16}$$

The forms for the heat and particle flows given in (A6.9) and (A6.10) can be substituted into these expressions, yielding

$$\begin{aligned} \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{D}_a \right\rangle = \quad & \lambda_a \left(\mathbf{i}_{a1} \cdot \mathbf{B}\right) + G_a \frac{c}{\sqrt{g}} \left(\frac{\mu_{a1\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\mu_{a1\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \\ & \sigma_a \left(\mathbf{i}_{a2} \cdot \mathbf{B}\right) + \frac{\overline{G}_a}{p_a} \frac{c}{\sqrt{g}} \left(\frac{\mu_{a2\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\mu_{a2\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \end{aligned} \tag{A6.17a}$$

$$\begin{aligned} \left\langle \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_a \right\rangle = \quad & \lambda_a \left(\mathbf{i}^{(P)}_{a1} \cdot \mathbf{B}\right) + G_a \frac{c}{\sqrt{g}} \left(\frac{\mu^{(P)}_{a1\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\mu^{(P)}_{a1\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \\ & \sigma_a \left(\mathbf{i}^{(P)}_{a2} \cdot \mathbf{B}\right) + \frac{\overline{G}_a}{p_a} \frac{c}{\sqrt{g}} \left(\frac{\mu^{(P)}_{a2\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\mu^{(P)}_{a2\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \end{aligned} \tag{A6.17b}$$

$$\begin{aligned} \left\langle \mathbf{B} \cdot \nabla \cdot \grave{\mathbf{E}}_a \right\rangle = \quad & \lambda_a \left(\tilde{\mathbf{a}}_{a1} \cdot \mathbf{B}\right) + G_a \frac{c}{\sqrt{g}} \left(\frac{\gamma_{a1\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\gamma_{a1\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \\ & \sigma_a \left(\tilde{\mathbf{a}}_{a2} \cdot \mathbf{B}\right) + \frac{\overline{G}_a}{p_a} \frac{c}{\sqrt{g}} \left(\frac{\gamma_{a2\alpha}}{B^{\zeta}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_T \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} - \frac{\gamma_{a2\zeta}}{B^{\alpha}} \frac{\left\langle \mathbf{B} \cdot \mathbf{B}_P \right\rangle}{\left\langle \mathbf{B} \cdot \mathbf{B} \right\rangle} \right) \end{aligned} \tag{A6.18}$$

These expressions show that the viscosities can be written as a term directly proportional to the gradients and a term proportional to the force free heat and particle flows. In the future, these expressions will be written as

$$\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{D}_a \right\rangle = \lambda_a \left(\mathbf{i}_{a1} \cdot \mathbf{B}\right) + G_a \frac{c}{\sqrt{g}} V_a + \sigma_a \left(\mathbf{i}_{a2} \cdot \mathbf{B}\right) + \frac{\overline{G}_a}{p_a} \frac{c}{\sqrt{g}} \overline{V}_a \tag{A6.19a}$$

$$\left\langle \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_a \right\rangle = \lambda_a \left(\mathbf{i}^{(P)}_{a1} \cdot \mathbf{B}\right) + G_a \frac{c}{\sqrt{g}} V^{(P)}_a + \sigma_a \left(\mathbf{i}^{(P)}_{a2} \cdot \mathbf{B}\right) + \frac{\overline{G}_a}{p_a} \frac{c}{\sqrt{g}} \overline{V}^{(P)}_a \tag{A6.19b}$$

$$\left\langle \mathbf{B}\cdot\nabla\cdot\grave{\mathbf{E}}_a \right\rangle = \lambda_a\left(\tilde{\mathbf{a}}_{a1}\cdot\mathbf{B}\right) + G_a\frac{c}{\sqrt{g}}H_a + \sigma_a\left(\tilde{\mathbf{a}}_{a2}\cdot\mathbf{B}\right) + \frac{\overline{G}_a}{p_a}\frac{c}{\sqrt{g}}\overline{H}_a \tag{A6.20}$$

The terms V and H are geometric factors related to the viscosity coefficients that can be found by comparing the different expressions above.

The complete expressions for the collisional friction are given by Coronado and Wobig.[4] The friction is given by

$$\mathbf{F}_{1a} = \sum_b\left(\ell_{11}^{ab}\mathbf{U}_b - \frac{2}{5}\ell_{12}^{ab}\frac{\mathbf{q}_b}{p_b}\right) \tag{A6.21}$$

$$\mathbf{F}_{2a} = \sum_b\left(-\ell_{21}^{ab}\mathbf{U}_b + \frac{2}{5}\ell_{22}^{ab}\frac{\mathbf{q}_b}{p_b}\right) \tag{A6.22}$$

The quantities l can be calculated as

$$\ell_{11}^{ab} = -n_a m_a\left(\nu_a\delta_{ab} - \nu_{ab}^*\right) \tag{A6.23}$$

$$\ell_{21}^{ab} = \ell_{12}^{ba} = -n_a m_a\left(\Gamma_a\delta_{ab} - \Gamma_{ab}^*\right) \tag{A6.24}$$

$$\ell_{22}^{ab} = -n_a m_a\left\{\begin{array}{l}\delta_{ab}\sum_k \nu_{ak}^*\left(\dfrac{13/4 + 4x_{ak}^2 + 15x_{ak}^4/2}{\left(1+x_{ak}^2\right)\left(1+m_a/m_k\right)}\right) - \\ \nu_{ab}^*\left(\dfrac{\left(m_a/m_b\right)\left(27/4\right)}{\left(1+x_{ab}^2\right)\left(1+m_a/m_b\right)}\right)\end{array}\right\} \tag{A6.25}$$

where $x_{ab}=v_{th,a}/v_{th,b}$ is the ratio of thermal velocities, $\nu_a=\Sigma_k\nu_{ak}^*$, $\Gamma_a=\Sigma_k\Gamma_{ak}^*$, and δ_{ab} is the usual Kronecker delta. The remaining quantities are defined as

$$\Gamma_{ab}^* = \tfrac{3}{2}\nu_{ab}^*/\left(1+x_{ab}^2\right) \tag{A6.26}$$

$$\nu_{ab}^* = \frac{4}{3\sqrt{\pi}}\frac{4\pi n_b e_a^2 e_b^2\ln\Lambda}{m_a^2 v_{th,a}^3}\frac{1+m_a/m_b}{\left(1+x_{ab}^2\right)^{3/2}} \tag{A6.27}$$

A6.3: Solution for the Parallel Flows and Heat Fluxes.

The next step in the derivation is to write down the coupled linear algebraic equations for the parallel heat and particle flows. In evaluating the parallel heat flux and momentum balance equations, it will be necessary to evaluate the <**B**·**U**> and <**B**·**q**>. These can be calculated by first noticing that

$$\left\langle \mathbf{B}\cdot\mathbf{U}_a \right\rangle = \frac{U^\zeta}{B^\zeta}\left\langle \mathbf{B}\cdot\mathbf{B}_T \right\rangle + \frac{U^\alpha}{B^\alpha}\left\langle \mathbf{B}\cdot\mathbf{B}_P \right\rangle \tag{A6.28}$$

Substituting in expressions for the flows in (A6.9) and (A6.10) allows (A6.28) to be simplified as

$$\left\langle \mathbf{B}\cdot\mathbf{U}_a \right\rangle = \lambda_a\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.29}$$

In a similar manner, the term <**B**·**q**> can be written as

$$\left\langle \mathbf{B}\cdot q_a \right\rangle = p_a\sigma_a\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.30}$$

With these expressions, the friction terms in the parallel equations (A6.11) and (A6.12) can be written as

$$\left\langle \mathbf{B}\cdot\mathbf{F}_{1i} \right\rangle = \ell_{11}^{ii}\lambda_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \frac{2}{5}\ell_{12}^{ii}\sigma_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \ell_{11}^{ie}\lambda_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \frac{2}{5}\ell_{12}^{ie}\sigma_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.31}$$

$$\left\langle \mathbf{B}\cdot\mathbf{F}_{1e} \right\rangle = \ell_{11}^{ei}\lambda_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \frac{2}{5}\ell_{12}^{ei}\sigma_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \ell_{11}^{ee}\lambda_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \frac{2}{5}\ell_{12}^{ee}\sigma_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.32}$$

$$\left\langle \mathbf{B}\cdot\mathbf{F}_{2i} \right\rangle = -\ell_{21}^{ii}\lambda_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \frac{2}{5}\ell_{22}^{ii}\sigma_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \ell_{21}^{ie}\lambda_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \frac{2}{5}\ell_{22}^{ie}\sigma_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.33}$$

$$\left\langle \mathbf{B}\cdot\mathbf{F}_{2e} \right\rangle = -\ell_{21}^{ei}\lambda_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \frac{2}{5}\ell_{22}^{ei}\sigma_i\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle - \ell_{21}^{ee}\lambda_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle + \frac{2}{5}\ell_{22}^{ee}\sigma_e\left\langle \mathbf{B}\cdot\mathbf{B} \right\rangle \tag{A6.34}$$

The electron-neutral collision frequency will be neglected in this development, but charge exchange collisions will be considered in the ion equations. Hence, the charge exchange terms in the ion momentum and heat flux equations will be written as

$$m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{U}_i > = m_i n_i \upsilon_{in}\lambda_i < \mathbf{B}\cdot\mathbf{B} > \tag{A6.35}$$

$$\frac{m_i \upsilon_{in}}{T_i} < \mathbf{B} \cdot \mathbf{q}_i >= \frac{m_i \upsilon_{in}}{T_i} p_i \sigma_i \langle \mathbf{B} \cdot \mathbf{B} \rangle \tag{A6.36}$$

With all of these pieces now assembled, it is possible to derive the coupled linear equations for the force free particle and heat flows. In particular, the parallel ion momentum balance equation becomes

$$\begin{aligned} &\lambda_i \left(\boldsymbol{\mu}_{i1} \cdot \mathbf{B} \right) + G_i \frac{c}{\sqrt{g}} V_i + \sigma_i \left(\boldsymbol{\mu}_{i2} \cdot \mathbf{B} \right) + \frac{\overline{G}_i}{p_i} \frac{c}{\sqrt{g}} \overline{V}_i + m_i n_i \upsilon_{in} \lambda_i < \mathbf{B} \cdot \mathbf{B} > - \\ &\left(\ell_{11}^{ii} \lambda_i \langle \mathbf{B} \cdot \mathbf{B} \rangle - \frac{2}{5} \ell_{12}^{ii} \sigma_i \langle \mathbf{B} \cdot \mathbf{B} \rangle + \ell_{11}^{ie} \lambda_e \langle \mathbf{B} \cdot \mathbf{B} \rangle - \frac{2}{5} \ell_{12}^{ie} \sigma_e \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) = 0 \end{aligned} \tag{A6.37}$$

Collecting terms yields a final form for the ion momentum balance equation

$$\begin{aligned} &\lambda_i \left(\boldsymbol{\mu}_{i1} \cdot \mathbf{B} - \ell_{11}^{ii} \langle \mathbf{B} \cdot \mathbf{B} \rangle + m_i n_i \upsilon_{in} < \mathbf{B} \cdot \mathbf{B} > \right) - \lambda_e \ell_{11}^{ie} \langle \mathbf{B} \cdot \mathbf{B} \rangle \\ &\sigma_i \left(\boldsymbol{\mu}_{i2} \cdot \mathbf{B} + \frac{2}{5} \ell_{12}^{ii} \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) + \sigma_e \frac{2}{5} \ell_{12}^{ie} \langle \mathbf{B} \cdot \mathbf{B} \rangle = -G_i \frac{c}{\sqrt{g}} V_i - \frac{\overline{G}_i}{p_i} \frac{c}{\sqrt{g}} \overline{V}_i \end{aligned} \tag{A6.38}$$

In a similar manner, the electron momentum balance can be written as

$$\begin{aligned} &-\lambda_i \ell_{11}^{ei} \langle \mathbf{B} \cdot \mathbf{B} \rangle + \lambda_e \left(\boldsymbol{\mu}_{e1} \cdot \mathbf{B} - \ell_{11}^{ee} \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) + \sigma_i \frac{2}{5} \ell_{12}^{ei} \langle \mathbf{B} \cdot \mathbf{B} \rangle + \\ &\sigma_e \left(\boldsymbol{\mu}_{e2} \cdot \mathbf{B} + \frac{2}{5} \ell_{12}^{ee} \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) = -G_e \frac{c}{\sqrt{g}} V_e - \frac{\overline{G}_e}{p_e} \frac{c}{\sqrt{g}} \overline{V}_e \end{aligned} \tag{A6.39}$$

The ion parallel heat balance equation can be simplified to read

$$\begin{aligned} &\lambda_i \left(\boldsymbol{\gamma}_{i1} \cdot \mathbf{B} + \ell_{21}^{ii} \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) + \lambda_e \ell_{21}^{ie} \langle \mathbf{B} \cdot \mathbf{B} \rangle + \\ &\sigma_i \left(\frac{m_i \upsilon_{in}}{T_i} p_i \langle \mathbf{B} \cdot \mathbf{B} \rangle + \boldsymbol{\gamma}_{i2} \cdot \mathbf{B} - \frac{2}{5} \ell_{22}^{ii} \langle \mathbf{B} \cdot \mathbf{B} \rangle \right) - \sigma_e \frac{2}{5} \ell_{22}^{ie} \langle \mathbf{B} \cdot \mathbf{B} \rangle \\ &= -G_i \frac{c}{\sqrt{g}} H_i - \frac{\overline{G}_i}{p_i} \frac{c}{\sqrt{g}} \overline{H}_i \end{aligned} \tag{A6.40}$$

The electron parallel heat balance equation can be simplified to read

$$\lambda_i \ell_{21}^{ei}\langle \mathbf{B}\cdot\mathbf{B}\rangle + \lambda_e\left(\ell_{21}^{ee}\langle \mathbf{B}\cdot\mathbf{B}\rangle + \tilde{\mathbf{a}}_{e1}\cdot\mathbf{B}\right) - \sigma_i \frac{2}{5}\ell_{22}^{ei}\langle \mathbf{B}\cdot\mathbf{B}\rangle +$$
$$\sigma_e\left(\frac{2}{5}\ell_{22}^{ee}\langle \mathbf{B}\cdot\mathbf{B}\rangle + \tilde{\mathbf{a}}_{e2}\cdot\mathbf{B}\right) = -G_e \frac{c}{\sqrt{g}} H_e - \frac{\overline{G}_e}{p_e}\frac{c}{\sqrt{g}}\overline{H}_e \qquad \text{(A6.41)}$$

Combining equation (A6.38)-(A6.42) yields a system of four coupled equations of the form

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\begin{bmatrix} \lambda_i \\ \lambda_e \\ \sigma_i \\ \sigma_e \end{bmatrix} = \begin{bmatrix} S_1 & S_2 & 0 & 0 \\ 0 & 0 & S_3 & S_4 \\ S_5 & S_6 & 0 & 0 \\ 0 & 0 & S_7 & S_8 \end{bmatrix}\begin{bmatrix} \frac{\partial\Phi}{\partial\rho} + \frac{1}{e_i n_i}\frac{\partial p_i}{\partial\rho} \\ \frac{5p_i}{2e_i}\frac{\partial T_i}{\partial\rho} \\ \frac{\partial\Phi}{\partial\rho} + \frac{1}{e_e n_e}\frac{\partial p_e}{\partial\rho} \\ \frac{5p_e}{2e_e}\frac{\partial T_e}{\partial\rho} \end{bmatrix} \qquad \text{(A6.42)}$$

The equations here are, from top to bottom, ion momentum, electron momentum, ion heat balance, and electron heat balance. The constants a_i and S_i can be determined by comparing equations (A6.38)-(A6.41) to the expressions above. These constants are geometric quantities which can be calculated if the Hamada spectrum and basis vectors are known. The term on the far right hand side are the forces that drive the flows. When these coupled linear equations are solved for the parallel heat and particle flows, the problem is essentially solved.

While it is possible to analytically invert this system of equations and derive expressions for each of the flows in terms of the forces, the algebra is daunting. Rather, it will be assumed for now that some method, either analytic or numerical, is used to invert this matrix.

It is interesting to clarify which terms of these expressions are kept in the Coronado and Talmadge model presented and tested in this work. This ion parallel momentum equation is given by (A6.38) with the heat fluxes and temperature gradients neglected, yielding

$$\lambda_i\left(\grave{\mathbf{i}}_{i1}\cdot\mathbf{B} - \ell_{11}^{ii}\langle \mathbf{B}\cdot\mathbf{B}\rangle + m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{B} >\right) - \lambda_e \ell_{11}^{ie}\langle \mathbf{B}\cdot\mathbf{B}\rangle = -G_i \frac{c}{\sqrt{g}} V_i \qquad \text{(A6.43)}$$

The electron parallel momentum balance is given by (A6.39) with the electron viscosity and the heat fluxes neglected, yielding an expression

$$\ell_{11}^{ei}\lambda_i\langle \mathbf{B}\cdot\mathbf{B}\rangle + \ell_{11}^{ee}\lambda_e\langle \mathbf{B}\cdot\mathbf{B}\rangle = 0 \tag{A6.44}$$

The heat flux balance equations are not considered in this limit. The coupled differential equations then reduce to

$$\begin{bmatrix} \mathbf{\grave{\imath}}_{\,i1}\cdot\mathbf{B} - \ell_{11}^{ii}\langle \mathbf{B}\cdot\mathbf{B}\rangle + \\ m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{B} > & -\ell_{11}^{ie}\langle \mathbf{B}\cdot\mathbf{B}\rangle \\ \ell_{11}^{ei}\langle \mathbf{B}\cdot\mathbf{B}\rangle & \ell_{11}^{ee}\langle \mathbf{B}\cdot\mathbf{B}\rangle \end{bmatrix} \begin{bmatrix} \lambda_i \\ \lambda_e \end{bmatrix} = \begin{bmatrix} -\frac{c}{\sqrt{g}}V_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial\Phi}{\partial\rho} + \frac{1}{e_i n_i}\frac{\partial p_i}{\partial\rho} \\ 0 \end{bmatrix} \tag{A6.45}$$

These equations can be solved by simply adding them, yielding the single equation

$$\begin{gathered} \left(\mathbf{\grave{\imath}}_{\,i1}\cdot\mathbf{B} - \ell_{11}^{ii}\langle \mathbf{B}\cdot\mathbf{B}\rangle + m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{B} > \right)\lambda_i - \ell_{11}^{ie}\langle \mathbf{B}\cdot\mathbf{B}\rangle\lambda_e + \ell_{11}^{ei}\langle \mathbf{B}\cdot\mathbf{B}\rangle\lambda_i + \\ \ell_{11}^{ee}\langle \mathbf{B}\cdot\mathbf{B}\rangle\lambda_e = -\frac{c}{\sqrt{g}}V_i\left(\frac{\partial\Phi}{\partial\rho} + \frac{1}{e_i n_i}\frac{\partial p_i}{\partial\rho}\right) \end{gathered} \tag{A6.46}$$

Then noting that the friction terms cancel, the force free parallel ion flows can be found as

$$\lambda_i = -\frac{c}{\sqrt{g}}\left(\frac{\partial\Phi}{\partial\rho} + \frac{1}{e_i n_i}\frac{\partial p_i}{\partial\rho}\right)\frac{\frac{\mu_{a1\alpha}}{B^{\xi}}\frac{\langle \mathbf{B}\cdot\mathbf{B}_T\rangle}{\langle \mathbf{B}\cdot\mathbf{B}\rangle} - \frac{\mu_{a1\xi}}{B^{\alpha}}\frac{\langle \mathbf{B}\cdot\mathbf{B}_P\rangle}{\langle \mathbf{B}\cdot\mathbf{B}\rangle}}{\left(\mathbf{\grave{\imath}}_{\,i1}\cdot\mathbf{B} + m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{B} >\right)} \tag{A6.47}$$

This expression is different than that in Coronado and Talmadge. The difference can be traced back to the different choice of A_1 and A_2 made in this derivation compared to their model. Indeed, if expressions (A6.9) are used in the direct Coronado and Talmadge development, then expression (A6.47) results. This ambiguity in the contravariant flow definition has been discussed in detail by Coronado and Wobig.[5]

The electron force free flow in this limit can be derived from (44) as

$$\lambda_i = -\frac{\ell_{11}^{ee}}{\ell_{11}^{ei}}\lambda_e = \lambda_e \tag{A6.48}$$

The electron bootstrap current does not appear because the electron viscosity has been neglected. Instead, the electrons are simply dragged along by the ions.

A6.4: Relationship Between the Gradients and Radial Fluxes.

At this point, the force free heat and particle flows have been completely solved for in terms of the electric field, pressure gradients, and temperature gradients. All of the plasma heat and particle flows can now be calculated from these expressions. The remaining task is to relate these potential, pressure, and temperature gradients to the radial current flowing through the plasma.

As noted by Shaing and Callen,[2] taking the scalar product of the 1st order momentum equation (A6.11) with $\mathbf{B}_P$ (or $\mathbf{B}_T$) and flux surface averaging gives rise to expressions for the radial flux of particles. These expressions are known as the flux friction relationships. Specifying to the poloidal component of the momentum equation and defining $\Gamma_a = N_a \langle \mathbf{U}_a \cdot \nabla\rho \rangle$, the radial flux of species 'a' can be calculated as

$$\tilde{\mathbf{A}}_a = \frac{c}{e_a B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_a > + \frac{c m_a N_a \upsilon_{an}}{e_a B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \mathbf{U}_a > + \frac{c}{e_a B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \mathbf{F}_{a1} > \qquad \text{(A6.49)}$$

Defining the radial current as $\langle \mathbf{J}_a \cdot \nabla\rho \rangle = \Sigma e_a \Gamma_a$ and neglecting electron-neutral collisions, the radial current can be calculated as

$$< \mathbf{J} \cdot \nabla\rho > = \frac{c}{B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_i > + \frac{c}{B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_e > + \frac{c m_i n_i \upsilon_{in}}{B^\xi B^\alpha \sqrt{g}} < \mathbf{B}_P \cdot \mathbf{U}_i > + \frac{c}{B^\xi B^\alpha \sqrt{g}} \left(< \mathbf{B}_P \cdot \mathbf{F}_{i1} > + < \mathbf{B}_P \cdot \mathbf{F}_{ei1} > \right) \qquad \text{(A6.50)}$$

The term in parenthesis on the right hand side is exactly zero, due to conservation of momentum between the ion and electron fluids (or, alternatively, because the classical and Pfirsch-Schlueter fluxes are intrinsically ambipolar[2]). Hence, we derive

$$\frac{\sqrt{g}B^{\xi}B^{\alpha}}{c} < \mathbf{J}\cdot\nabla\rho > = < \mathbf{B}_P \cdot \nabla\cdot \mathbf{D}_i > + < \mathbf{B}_P \cdot \nabla \cdot \mathbf{D}_e > + m_i n_i \upsilon_{in} < \mathbf{B}_P \cdot \mathbf{U}_i > \tag{A6.51}$$

Expression (A6.51) is exactly that used in Coronado and Talmadge, except that the electron viscosity is neglected in that model.

To evaluate this expression, note that

$$m_i n_i \upsilon_{in} < \mathbf{B}_P \cdot \mathbf{U} > = m_i n_i \upsilon_{in}\left(\frac{U^{\alpha}}{B^{\alpha}}\left\langle \mathbf{B}_P \cdot \mathbf{B}_P \right\rangle + \frac{U^{\xi}}{B^{\xi}}\left\langle \mathbf{B}_P \cdot \mathbf{B}_T \right\rangle\right) \tag{A6.52}$$

Using (A6.9) and (A6.10) allows this term to be written simply as

$$m_i n_i \upsilon_{in} < \mathbf{B}_P \cdot \mathbf{U} > = m_i n_i \upsilon_{in}\left(\begin{array}{c}\dfrac{G_i c}{B^{\alpha}B^{\xi}\sqrt{g}}\left(\dfrac{\left\langle \mathbf{B}_P \cdot \mathbf{B}_P \right\rangle\left\langle \mathbf{B}\cdot\mathbf{B}_T \right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle} - \dfrac{\left\langle \mathbf{B}_P \cdot \mathbf{B}_T \right\rangle\left\langle \mathbf{B}\cdot\mathbf{B}_P \right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle}\right) + \\ \lambda_i\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle\end{array}\right) \tag{A6.53}$$

In the interest of simplification, this expression will be written as

$$m_i n_i \upsilon_{in} < \mathbf{B}_P \cdot \mathbf{U} > = m_i n_i \upsilon_{in}\left(\frac{G_i c}{B^{\alpha}B^{\xi}\sqrt{g}}F + \lambda_i\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle\right) \tag{A6.54}$$

with

$$F = \frac{\left\langle \mathbf{B}_P \cdot \mathbf{B}_P \right\rangle\left\langle \mathbf{B}\cdot\mathbf{B}_T \right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle} - \frac{\left\langle \mathbf{B}_P \cdot \mathbf{B}_T \right\rangle\left\langle \mathbf{B}\cdot\mathbf{B}_P \right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle} \tag{A6.55}$$

Using (A6.54) and (A6.17b) in (A6.51), an expression relating the radial current to the electric field and thermodynamic gradients can be derived.

$$\begin{aligned}\frac{\sqrt{g}B^{\xi}B^{\alpha}}{c} < \mathbf{J}\cdot\nabla\rho > = {} & \lambda_i\left(\mathbf{i}_{i1}^{(P)}\cdot\mathbf{B}\right) + G_i\frac{c}{\sqrt{g}}V_i^{(P)} + \sigma_i\left(\mathbf{i}_{i2}^{(P)}\cdot\mathbf{B}\right) + \frac{\overline{G}_i}{p_i}\frac{c}{\sqrt{g}}\overline{V}_i^{(P)} + \\ & \lambda_e\left(\mathbf{i}_{e1}^{(P)}\cdot\mathbf{B}\right) + G_e\frac{c}{\sqrt{g}}V_e^{(P)} + \sigma_e\left(\mathbf{i}_{e2}^{(P)}\cdot\mathbf{B}\right) + \frac{\overline{G}_e}{p_e}\frac{c}{\sqrt{g}}\overline{V}_e^{(P)} + \\ & m_i n_i \upsilon_{in}\left(\frac{G_i c}{B^{\alpha}B^{\xi}\sqrt{g}}F + \lambda_i\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle\right)\end{aligned} \tag{A6.56}$$

This expression relates the radial current to the gradients in the potential, pressure, and temperature. Note that the force free components of the flow are explicitly written in this expression. It is necessary to solve (A6.42) for the force free flows and insert them into (A6.56) to write the radial current in term of the gradients only.

In the limit of the Coronado and Talmadge formulation, the terms that go like the heat flux and the electron viscosity are eliminated and this expression simply becomes.

$$-\frac{\sqrt{g}B^{\xi}B^{\alpha}}{c} < \mathbf{J}\cdot\nabla\rho > = \begin{pmatrix} \lambda_i\left(\mathbf{i}_{i1}^{(P)}\cdot\mathbf{B} + m_i n_i \upsilon_{in}\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle\right) + \\ G_i\left(\frac{c}{\sqrt{g}}V_i^{(P)} + m_i n_i \upsilon_{in}\frac{cF}{B^{\alpha}B^{\xi}\sqrt{g}}\right) \end{pmatrix} \tag{A6.57}$$

Using the definition for λ given in expressions (A6.45) allows this expression to be written as

$$< \mathbf{J}\cdot\nabla\rho > = \left(\frac{\partial\Phi}{\partial\rho} + \frac{1}{e_a n_a}\frac{\partial p_a}{\partial\rho}\right)\frac{(-1)c}{\sqrt{g}B^{\xi}B^{\alpha}}\begin{pmatrix} -\frac{c}{\sqrt{g}}\frac{\frac{\mu_{a1\alpha}}{B^{\xi}}\frac{\left\langle \mathbf{B}\cdot\mathbf{B}_T\right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle} - \frac{\mu_{a1\xi}}{B^{\alpha}}\frac{\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle}}{\left(\mathbf{i}_{i1}\cdot\mathbf{B} + m_i n_i \upsilon_{in} < \mathbf{B}\cdot\mathbf{B} >\right)} \times \\ \left(\mathbf{i}_{i1}^{(P)}\cdot\mathbf{B} + m_i n_i \upsilon_{in}\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle\right) + \\ \begin{pmatrix} \frac{c}{\sqrt{g}}\left(\frac{\mu_{a1\alpha}^{(P)}}{B^{\xi}}\frac{\left\langle \mathbf{B}\cdot\mathbf{B}_T\right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle} - \frac{\mu_{a1\xi}^{(P)}}{B^{\alpha}}\frac{\left\langle \mathbf{B}\cdot\mathbf{B}_P\right\rangle}{\left\langle \mathbf{B}\cdot\mathbf{B}\right\rangle}\right) + \\ m_i n_i \upsilon_{in}\frac{cF}{B^{\alpha}B^{\xi}\sqrt{g}} \end{pmatrix} \end{pmatrix} \tag{A6.58}$$

Hence, just as in the basic Coronado and Talmadge formulation, a relationship is derived between the radial current and the electric field. The radial conductivity in this case is given by

$$\sigma_{\perp} = \frac{(-1)c}{\sqrt{g}B^{\xi}B^{\alpha}\langle\nabla\rho\cdot\nabla\rho\rangle}\left(\begin{array}{l} -\dfrac{c}{\sqrt{g}}\dfrac{\dfrac{\mu_{a1\alpha}}{B^{\xi}}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{T}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}-\dfrac{\mu_{a1\zeta}}{B^{\alpha}}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}}{\left(\mathbf{i}_{i1}\cdot\mathbf{B}+m_{i}n_{i}\upsilon_{in}<\mathbf{B}\cdot\mathbf{B}>\right)}\times \\ \left(\mathbf{i}_{i1}^{(P)}\cdot\mathbf{B}+m_{i}n_{i}\upsilon_{in}\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle\right)+ \\ \left(\begin{array}{l}\dfrac{c}{\sqrt{g}}\left(\dfrac{\mu_{a1\alpha}^{(P)}}{B^{\xi}}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{T}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}-\dfrac{\mu_{a1\zeta}^{(P)}}{B^{\alpha}}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}\right)+ \\ m_{i}n_{i}\upsilon_{in}\dfrac{cF}{B^{\alpha}B^{\xi}\sqrt{g}}\end{array}\right)\end{array}\right) \tag{A6.59}$$

This can be rewritten in terms of viscous frequencies[3] to yield

$$\begin{array}{l}\sigma_{\perp} = \dfrac{(-1)c^{2}m_{i}n_{i}\langle B_{P}^{2}\rangle}{\left(\sqrt{g}\right)^{2}\left(B^{\xi}B^{\alpha}\right)^{2}\langle\nabla\rho\cdot\nabla\rho\rangle\left(\iota\upsilon_{\alpha}+\upsilon_{\zeta}+\upsilon_{in}\right)}\times \\ \left(\begin{array}{l}-\left(\iota\upsilon_{\alpha}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{T}\rangle}{<B^{2}>}-\upsilon_{\zeta}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{<B^{2}>}\right)\left(\upsilon_{\alpha}^{(P)}+\iota^{-1}\upsilon_{\zeta}^{(P)}+\upsilon_{in}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{\langle B_{P}^{2}\rangle}\right)+ \\ \left(\upsilon_{\alpha}^{(P)}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{T}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}-\upsilon_{\zeta}^{(P)}\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{\iota\langle\mathbf{B}\cdot\mathbf{B}\rangle}+\upsilon_{in}\left(\dfrac{\langle\mathbf{B}\cdot\mathbf{B}_{T}\rangle}{\langle\mathbf{B}\cdot\mathbf{B}\rangle}-\dfrac{\langle\mathbf{B}_{P}\cdot\mathbf{B}_{T}\rangle\langle\mathbf{B}\cdot\mathbf{B}_{P}\rangle}{\langle B_{P}^{2}\rangle\langle\mathbf{B}\cdot\mathbf{B}\rangle}\right)\right)\left(\iota\upsilon_{\alpha}+\upsilon_{\zeta}+\upsilon_{in}\right)\end{array}\right)\end{array} \tag{A6.60}$$

This expression is entirely equivalent to the radial conductivity derived in the Coronado and Talmadge model, although the arrangement of the terms is somewhat different.

A6.5: Evaluation of the Viscosities Coefficients in the Plateau Regime.

The development here requires a more complicated treatment of the viscosities than is necessary for the Coronado and Talmadge model. To derive expressions for the viscosity coefficients, the plateau expressions in Shaing, Hirshman and Callen[6] can be used.

In that reference, the viscosities are written as

$$\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{D}_a \right\rangle = 3\left(\mu_{a1}^{P}U_a^{\alpha} + \mu_{a1}^{T}U_a^{\zeta} + \frac{2}{5}\mu_{a2}^{P}\frac{q_a^{\alpha}}{p} + \frac{2}{5}\mu_{a2}^{T}\frac{q_a^{\zeta}}{p} \right), \tag{A6.61a}$$

$$\left\langle \mathbf{B}_P\cdot\nabla\cdot\mathbf{D}_a \right\rangle = 3\left(\mu_{Pa1}^{P}U_a^{\alpha} + \mu_{Pa1}^{T}U_a^{\zeta} + \frac{2}{5}\mu_{Pa2}^{P}\frac{q_a^{\alpha}}{p} + \frac{2}{5}\mu_{Pa2}^{T}\frac{q_a^{\zeta}}{p} \right), \tag{A6.61b}$$

$$\left\langle \mathbf{B}_P\cdot\nabla\cdot\grave{\mathbf{E}}_a \right\rangle = 3\left(\mu_{a1}^{P}U_a^{\alpha} + \mu_{Pa1}^{T}U_a^{\zeta} + \frac{2}{5}\mu_{Pa2}^{P}\frac{q_a^{\alpha}}{p} + \frac{2}{5}\mu_{Pa2}^{T}\frac{q_a^{\zeta}}{p} \right), \tag{A6.62}$$

Using the definitions of the μ coefficients presented in the paper, these expressions can be simplified to read

$$\begin{aligned}\left\langle \mathbf{A}\cdot\nabla\cdot\mathbf{D}_a \right\rangle = \; & \frac{\sqrt{\pi}P_aC_1B_o}{v_{ta}B^{\zeta}}\sum_{n,m\neq 0}\frac{1}{\left|n-m\bar{\iota}\right|}\left\langle \frac{\mathbf{A}\cdot\nabla\mathbf{B}}{B}\mathbf{U}_a\cdot\nabla B_{nm} \right\rangle + \\ & \frac{2\sqrt{\pi}C_2B_o}{5v_{ta}B^{\zeta}}\sum_{n,m\neq 0}\frac{1}{\left|n-m\bar{\iota}\right|}\left\langle \frac{\mathbf{A}\cdot\nabla\mathbf{B}}{B}\mathbf{q}_a\cdot\nabla B_{nm} \right\rangle\end{aligned} \tag{A6.63}$$

with $\mathbf{A}$ equal to $\mathbf{B}$, $\mathbf{B}_p$, or $\mathbf{B}_T$, and

$$\begin{aligned}\left\langle \mathbf{A}\cdot\nabla\cdot\grave{\mathbf{E}}_a \right\rangle = \; & \frac{\sqrt{\pi}P_aC_2B_o}{v_{ta}B^{\zeta}}\sum_{n,m\neq 0}\frac{1}{\left|n-m\bar{\iota}\right|}\left\langle \frac{\mathbf{A}\cdot\nabla\mathbf{B}}{B}\mathbf{U}_a\cdot\nabla B_{nm} \right\rangle + \\ & \frac{2\sqrt{\pi}C_3B_o}{5v_{ta}B^{\zeta}}\sum_{n,m\neq 0}\frac{1}{\left|n-m\bar{\iota}\right|}\left\langle \frac{\mathbf{A}\cdot\nabla\mathbf{B}}{B}\mathbf{q}_a\cdot\nabla B_{nm} \right\rangle\end{aligned} \tag{A6.64}$$

The factors C_i are given by $C_1=\Gamma(3)=2$, $C_2=\Gamma(4)-5\Gamma(3)/2$, and $C_3=\Gamma(5)-5\Gamma(4)-(25/4)\Gamma(3)$, and v_{ta} is the thermal velocity of species 'a'.

To evaluate these expressions, first note that the terms in the flux surface average can be written as

$$\left\langle \frac{\mathbf{A}\cdot\nabla B}{B}\mathbf{U}\cdot\nabla B_{nm} \right\rangle = B_o\left(\begin{array}{l} A^{\zeta}U^{\zeta}\left\langle \dfrac{S_1S_3}{B} \right\rangle - A^{\zeta}U^{\alpha}\left\langle \dfrac{S_1S_4}{B} \right\rangle \\ -A^{\alpha}U^{\zeta}\left\langle \dfrac{S_2S_3}{B} \right\rangle + A^{\alpha}U^{\alpha}\left\langle \dfrac{S_2S_4}{B} \right\rangle \end{array} \right), \tag{A6.65}$$

where

$$S_1 = \sum_{n,m\neq 0} n\varepsilon_{nm}\, \mathbf{sin}(m\alpha - n\zeta), \tag{A6.66a}$$

$$S_2 = \sum m\varepsilon_{nm}\, \mathbf{sin}(m\alpha - n\zeta), \tag{A6.66b}$$

$$S_3 = n'\varepsilon_{n'm'}\, \mathbf{sin}(m'\alpha - n'\zeta), \tag{A6.66c}$$

$$S_4 = m'\varepsilon_{n'm'}\, \mathbf{sin}(m'\alpha - n'\zeta). \tag{A6.66d}$$

The next step is to evaluate the flux surface averages in the expression above. As an example, the quantity $\langle S_1S_3/B\rangle$ will be evaluated below. This quantity can be written as

$$\left\langle \frac{S_1S_3}{B} \right\rangle = \left\langle \frac{\sum_{n,m\neq 0} n\varepsilon_{nm} \sin(m\alpha - n\zeta) n'\varepsilon_{n'm'} \sin(m'\alpha - n'\zeta)}{B_o\left(1 + \sum_{n'',m''\neq 0} \varepsilon_{n''m''} \cos(m''\alpha - n''\zeta)\right)} \right\rangle. \tag{A6.67}$$

For $\varepsilon_{n,m} << 1$, the sum in the denominator can be neglected. Utilizing the definition of the flux surface average the yields

$$\left\langle \frac{S_1S_3}{B} \right\rangle = \frac{1}{4\pi^2 B_o} \int_0^{2\pi}\int_0^{2\pi} \sum_{n,m\neq 0} n\varepsilon_{nm}\, \mathbf{sin}(m\alpha - n\zeta) n'\varepsilon_{n'm'}\, \mathbf{sin}(m'\alpha - n'\zeta) d\zeta d\alpha\,. \tag{A6.68}$$

The orthogonality relationship between sines,

$$\int_0^{2\pi}\int_0^{2\pi} \mathbf{sin}(m\alpha - n\zeta)\mathbf{sin}(m'\alpha - n'\zeta) = \begin{cases} 2\pi^2 \\ 0 \end{cases}, \tag{A6.69}$$

allows the flux surface average to be evaluated as

$$\left\langle \frac{S_1S_3}{B} \right\rangle = \frac{n^2\varepsilon_{nm}^2}{2B_o}. \tag{A6.70a}$$

The other terms in brackets on the RHS of (A6.65) yield

$$\left\langle \frac{S_1S_4}{B} \right\rangle = \frac{nm\varepsilon_{nm}^2}{2B_o}. \tag{A6.70b}$$

$$\left\langle \frac{S_2 S_3}{B} \right\rangle = \frac{nm\varepsilon_{nm}^2}{2B_o}. \tag{A6.70c}$$

$$\left\langle \frac{S_2 S_3}{B} \right\rangle = \frac{m^2\varepsilon_{nm}^2}{2B_o}. \tag{A6.70d}$$

These expressions are to be inserted into (A6.67), taking care of the flux surface averages. After the following definitions are made,

$$\alpha_T = \sum_{n,m\neq 0} \frac{n^2\varepsilon_{nm}^2}{|n - m\iota\!\!\!-|}, \tag{A6.71a}$$

$$\alpha_C = -\sum_{n,m\neq 0} \frac{nm\varepsilon_{nm}^2}{|n - m\iota\!\!\!-|}, \tag{A6.71b}$$

$$\alpha_P = \sum_{n,m\neq 0} \frac{m^2\varepsilon_{nm}^2}{|n - m\iota\!\!\!-|}, \tag{A6.71c}$$

it is found that the viscosities can be written as

$$\begin{aligned} \langle \mathbf{A} \cdot \nabla \cdot \mathbf{D}_a \rangle = \quad & \kappa_{1a}\left(U_a^\alpha \left(A^\zeta \alpha_T + A^\alpha \alpha_C\right) + U_a^\zeta \left(A^\alpha \alpha_P + A^\zeta \alpha_C\right)\right) \\ & \kappa_{2a}\left(q_a^\alpha \left(A^\zeta \alpha_T + A^\alpha \alpha_C\right) + q_a^\zeta \left(A^\alpha \alpha_P + A^\zeta \alpha_C\right)\right) \end{aligned} \tag{A6.72}$$

$$\begin{aligned} \langle \mathbf{A} \cdot \nabla \cdot \grave{\mathbf{E}}_a \rangle = \quad & \kappa_{3a}\left(U_a^\alpha \left(A^\zeta \alpha_T + A^\alpha \alpha_C\right) + U_a^\zeta \left(A^\alpha \alpha_P + A^\zeta \alpha_C\right)\right) \\ & \kappa_{4a}\left(q_a^\alpha \left(A^\zeta \alpha_T + A^\alpha \alpha_C\right) + q_a^\zeta \left(A^\alpha \alpha_P + A^\zeta \alpha_C\right)\right) \end{aligned} \tag{A6.73}$$

The constants κ are defined as

$$\kappa_{1a} = \frac{\sqrt{\pi} P_a B_o}{v_{ta} B^\zeta}. \tag{A6.74a}$$

$$\kappa_{2a} = \frac{\sqrt{\pi} C_2 B_o}{5 v_{ta} B^\zeta}. \tag{A6.74b}$$

$$\kappa_{3a} = \frac{\sqrt{\pi} P_a C_2 B_o}{2 v_{ta} B^\zeta}. \tag{A6.74c}$$

$$\kappa_{4a} = \frac{\sqrt{\pi} C_3 B_o}{5 v_{ta} B^{\zeta}}. \qquad \text{(A6.74d)}$$

Comparing equations (A6.15) and (A6.16) to (A6.72) and (A6.73) shows that the viscosities have now been calculated. In particular, the viscosity coefficients can be found by comparing the expressions.

The viscosity coefficients in the Pfirsch-Schlueter regime can be calculated using very similar techniques based on the expressions in Shaing and Callen.[2]

[1] K.C. Shaing, Phys. Fluids **29**, 2231 (1986).

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