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Loss of the second stability regime for ideal MHD ballooning modes in 3-D equilibria

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Abstract

The effect of three-dimensional geometry on the stability of ideal MHD ballooning modes is addressed. By using a class of 'local 3-D equilibria,' the effects of plasma shaping, profile variations and symmetry on local stability properties can be studied. A class of local helical axis equilibria are constructed which model the properties of a quasi-helically symmetric stellarator. Changes in the 3-D shaping allow for manipulation of the harmonic content of the local shear within the magnetic surface of interest. The presence of symmetry breaking components in the local shear are shown to have a dramatic impact on the ballooning stability boundaries. In particular, these symmetry breaking components are shown to lower first ballooning stability thresholds and potentially eliminate the second stability regime.

I. Introduction

The physical mechanism that limits the allowable stored energy in a non-symmetric magnetic confinement system is an open physics question in stellarator research. In many theoretical assessments of particular stellarator designs, ideal MHD ballooning modes are thought to provide the most stringent instability limit to the plasma β. Ballooning instabilities have eigenfunctions which are localized to regions in the plasma with unfavorable magnetic field line curvature. Instability ensues when the pressure/curvature drive exceeds the stabilizing magnetic field line bending energy. In this work, we review recent calculations which emphasize the role of three dimensional geometry on the linear stability properties of ideal MHD ballooning modes [1,2].

Ballooning modes are characterized by having short perpendicular wavelengths relative to parallel wavelenghts. Using this ratio as an asymptotic expansion parameter, one derives to lowest order a local mode behavior that is described by an ordinary differential equation along a field line [3]. Solutions of this equation yield eigenvalues of the local problem which in general are functions of a three-dimensional space labeled by magnetic surface, field line label and radial wavenumber (η_k). Solutions to the global eigenvalue problem require solutions of the ray tracing problem subject the periodicity properties of the problem [3,4]. Due to the presence of magnetic field line dependent local eigenvalues, which are unique to three-dimensional equilibria, construction of the global eigenvalues are problematic [3].

The subject of this work deals with the properties of the local eigenvalues. We introduce methods for contructing the local stability properties of 3-D equilibria as functions of profile and shaping parameters [5,6]. Previously, this technique was utilized to address the ballooning stability properties of an example of the local equilibria [1] as a model for discussing generic stability properties of three-dimensional equilibra [2]. Specifically, we discuss the role of symmetry breaking effects in the geometric properties of the magnetic field and how these effects impact the stability boundaries [1,2]. Symmetry breaking effects introduce truly three-dimensional equilibria effects which yield field-line dependent ballooning eigenvalues.

In the following section, we review the notion of local equilibria [6]. In Section III, we describe some ideal MHD ballooning stability properties in the context of a particular example local equilibria. Finally, we conclude with a brief discussion in Section IV.

II. Local Equilibria

The difficult aspect of deducing the role of three-dimensional shaping on local stability criteria is the generation of 3-D MHD equilibria with nested toroidal flux surfaces. Since no rigorous Grad-Shafranov theory exist for 3-D systems, one needs to rely on numerically generated solutions which are unwieldy if not impossible to use in calculations involving parameter scans of shaping parameters. However, a method was developed to generate sequences of 3-D MHD equilibria in a region localized to a particular magnetic surface [6]. By application of this technique, the stability properties of various local modes can be studied as functions of profile and 3-D shaping parameters. This work parallels similar efforts to examine the effects of axisymmetric shaping and profile parameters on local stability properties of tokamaks [7,8].

A local 3-D equilibrium is specified by two free profile functions and the coordinate mapping $\mathbf{X}(\theta, \zeta)$ and rotational transform ι_o on the magnetic surface $\psi = \psi_o$, where θ and ζ are any straight field poloidal and toroidal angle ($d\theta/d\zeta = \iota_o$). The choice of $\mathbf{X}$ is constrained by the physical requirement that there by no normal component of the plasma current. The constraint requires that a first order partial differential equation on the magnetic surface have non-secularly growing solutions and that parallel currents are finite on the magnetic surface [6].

It is useful to distinguish the role of magnetic geometry, as specified by $\mathbf{X}$ and ι_o, from the two profile functions. In particular, fundametal geometric information such as the normal and geodesic curvatures are described by the magnetic geometry and given by

$$\kappa_n = \hat{\mathbf{n}} \cdot (\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}}, \tag{1}$$

$$\kappa_g = \hat{\mathbf{b}} \times \hat{\mathbf{n}} \cdot (\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}}. \tag{2}$$

These quanities play crucial roles in describing the free energy source and are derivable from $\mathbf{X}$ and ι_o using $\hat{\mathbf{b}} = (\partial\mathbf{X}/\partial\zeta + \iota\partial\mathbf{X}/\partial\zeta)/|(\partial\mathbf{X}/\partial\theta + \iota\partial\mathbf{X}/\partial\zeta)|$ and $\hat{\mathbf{n}} = (\partial\mathbf{X}/\partial\theta \times \partial\mathbf{X}/\partial\zeta)/|(\partial\mathbf{X}/\partial\theta \times \partial\mathbf{X}/\partial\zeta)|$.

While the curvature plays a fundamental role in describing the free energy source for ballooning modes, another geometric object, the normal torsion τ_n, plays an important role in describing the stabilizing effect of field line bending. Specifically, the field line bending energy associated with a local ideal MHD perturbation is related to the local shear of the field line. The local shear is defined by $s = (\hat{\mathbf{b}} \times \hat{\mathbf{n}}) \cdot \nabla \times (\hat{\mathbf{b}} \times \hat{\mathbf{n}})$ and is related to the normal torsion and parallel current by the identity

$$s = \frac{\mathbf{J} \cdot \mathbf{B}}{B^2} - 2\tau_n. \tag{3}$$

The normal torsion of the magnetic field is also completely specified by the magnetic geometry and given by

$$\tau_n = -\hat{\mathbf{n}} \cdot (\hat{\mathbf{b}} \cdot \nabla)\hat{\mathbf{b}} \times \hat{\mathbf{n}}. \tag{4}$$

Note from Eq. (3), that variations in the normal torsion and Pfirsch-Schlüter currents within the flux surface the local shear. The parallel current satisfies the quasineutrality equation which yields a solution that is given by

$$\frac{\mathbf{J} \cdot \mathbf{B}}{B^2} = \sigma + \frac{dp}{d\psi}\lambda \tag{5}$$

where $\sigma =< \mathbf{J} \cdot \mathbf{B} > / < B^2 >$ is the total flux surface averaged current, $dp/d\psi$ is the pressure gradient and λ is the Pfirsch-Schlüter coefficient determined from the solution to

$$\mathbf{B} \cdot \nabla\lambda = 2\frac{|\nabla\psi|}{B}\kappa_g, \tag{6}$$

where $|\nabla\psi|$, B and the Jacobian can determined from the geometric input [6]. While magnetic geometry influence the local shear through the quantities τ_n and λ, the two free profile quantities σ and $dp/d\psi$ (or equivilantly $d\iota/d\psi$ and $dp/d\psi$) affect the local shear as well.

In the following ballooning stability studies we will use a parametrization which crudely models a quasihelical system [9]. In this approximation, the normal and geodesic curvatures are dominated by a single Fourier harmonic, $\kappa_n \sim cos(N\zeta-\theta), \kappa_g \sim sin(N\zeta-\theta)$ with N denoting the toroidal periodicity. In particular, the inverse mapping quantity $\mathbf{X}(\theta,\zeta)$ is specified in cylindrical coordinates $[R,\phi,Z] = [R(\theta,\zeta),-\zeta,Z(\theta,\zeta)]$ by

$$R = R_o + \rho_o\cos(\theta) + \Delta\cos(N\zeta) + \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\sin(\theta), \tag{7}$$

$$Z = \rho_o\sin(\theta) + \Delta\sin(N\zeta) - \frac{2R_o\rho_o}{N^2\Delta}\sin(N\zeta)\cos(\theta), \tag{8}$$

where the lengthscales Δ, R_o and ρ_o and the toroidal periodicity parameter N are input to the local equilibria geometry. While more general cases exist, we concentrate on the particular orderings $N^2\Delta/R_o \gg 1 > N\Delta/R_o$ and $\rho_o \sim \Delta$. In this limit, the magnetic surface is circular to lowest order with the center of the circle rotating with the helical pitch $N\zeta-\theta$. The last terms in $R(\theta,\zeta)$ and $Z(\theta,\zeta)$ give small mirror-like corrections that beat with the helical symmetric angle to cancel out the toroidal curvature to leading order. This has the effect of producing a curvature vector, magnetic field spectrum and Pfirsch-Schlüter current spectrum that are dominated by the helical angle $N\zeta-\theta$, as one would find in a quasi-helically symmetric configuration.

From Eqs. (1)-(2), the normal and geodesic curvatures are dominated by a single harmonic to leading order and given by

$$\kappa_n = -\frac{N^2\Delta}{R_o^2}\cos(N\zeta-\theta), \tag{9}$$

$$\kappa_g = -\frac{N^2\Delta}{R_o^2}\sin(N\zeta-\theta)\sin(N\zeta). \tag{10}$$

Additionally, the normal torsion is given by

$$\tau_n = -\frac{2}{N\Delta}\cos(N\zeta) + \frac{\iota_o}{R_o} - \frac{N^3\Delta^2}{R_o^3}\cos^2(N\zeta-\theta). \tag{11}$$

Unlike components of the curvature vector, the normal torsion is not dominated by a single harmonic. The first term in Eq. (11) describes a dominant "mirror" term that has a helicity which is incommensurate with the basic helical angle which dominates the curvature vector. Since there is no reason to believe that the torsion vector should have any particular symmetry property (unlike the magnetic field spectrum), this feature is generic for all quasi-symmetric configurations. Further details on this choice for the magnetic field geometry can be found in Ref. [2].

III. Ballooning Stability

Local stability properties are determined from the ballooning equation which for the model equilibria described in Sec. II is given by

$$\frac{\partial}{\partial\eta}(1+\Lambda^2)\frac{\partial\xi}{\partial\eta}+\alpha[cos(\eta)+\Lambda sin(\eta)]\xi=-\Omega^2(1+\Lambda^2)\xi \tag{12}$$

where the coordinate $\eta = N\zeta - \theta$ labels points along the magnetic field line, the integrated local shear is

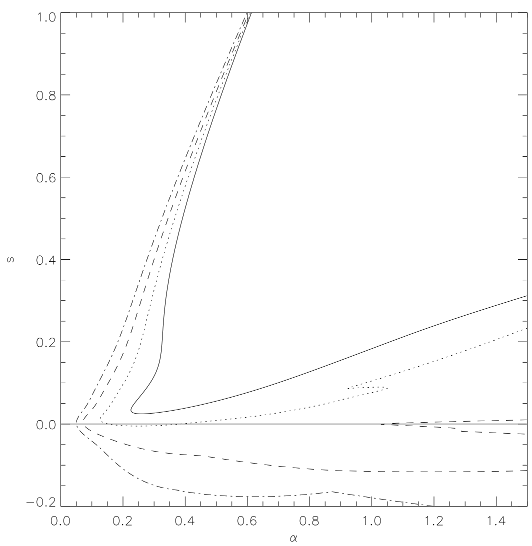
$$\Lambda=\int_{\eta_k}^{\eta} d\eta[\hat{s}-\alpha cos(\eta)+\tau_o cos(2\eta)+\delta cos(k\eta+k\chi)], \tag{13}$$

$\alpha = -(dp/d\psi)N^2\Delta\rho_o\hat{V}'/R_o(N-\iota)^2$ is dimensionless measure of the pressure gradient, $\tau_o = N^3\Delta^2/2R_o^2(N-\iota)$, $\delta = N\Delta/R_o(N-\iota)$, and $\hat{V}' = \oint dl/B(4\pi^2)^{-1}$ is an normalizing factor. The last term in $\Lambda(\eta,\chi)$ represents the three-dimensional property of the helical axis equilibria. This term arises from the first term of the expression for the normal torsion, Eq. (11). Note that this term is explicitly field line (χ) dependent and has incommensurate helicity with the helical symmetry angle.

The ideal MHD stability boundaries are present when the local eigenvalue satisfies Ω=0 [note, on the magnetic surface $\Omega^2 = \Omega^2(\eta_k,\chi)$]. In the above ballooning equation, five parameters describe the equilibria. The parameters $\hat{s}$ and α are dimensionless measures of the two profile quantities required for the specification of the local equilibria. The quantities δ, τ_o and k, are due to the magnetic geometry specification. In the limit $\delta = \tau_o = 0$, the ballooning equation is mathematically equivalent to the shifted circle equilibrium used in tokamak studies [10]. In what follows, we consider the special case $\tau_o = 0$. Non-zero values of τ_o alter the stability boundaries quantitatively, however, the same general features are present in the $\tau_o = 0$ case since this term does not introduce symmetry breaking terms into the ballooning equation. The factor $k = N/(N-\iota_o)$ is only relevant when $\delta \neq 0$ and is irrational if the rotational transform is irrational.

The most important modification due to three-dimensional geometry is described in the term in the local shear proportional to δ. In the following figure, we plot stability boundaries as functions of the profile quantities $\hat{s}$ and α for a range of values for δ. Ideal MHD stability boundaries with $\tau_o = 0$ and $k = \pi^2/8$ are shown for different values of the symmetry breaking factor δ. The solid, dotted, dashed, and dash-dotted correspond to $\delta =$

$0, 0.15, 0.30, 0.45$, respectively. If at least one field line with one value of η_k has an unstable local eigenvalue, the region of $\hat{s} - \alpha$ parameter space is considered to be unstable.

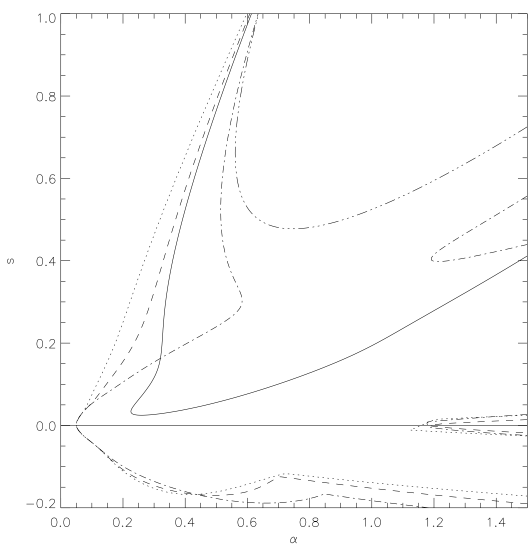


The stability curves for the $\delta = 0$ case is equivalent to the standard symmetric tokamak case where for $\hat{s} > 0$, there are two marginal stability points at fixed $\hat{s}$ that demark the first and second stability regimes. Generally, the first stability boundary degrades as δ increases. Additionally, as δ increases there is a significant deterioration of the second stability regime, and for large enough δ, there is only one ballooning stability boundary for a given $\hat{s}$. Unlike the tokamak case, ballooning instability can occur at $\hat{s} = 0$. This effect is solely due to the presence of the symmetry breaking contributions to the local shear. These terms largely determine the stability properties in the small shear region.

The stability boundary is weakly dependent on δ in the large-$\hat{s}$ [$\hat{s} \sim \mathcal{O}(1)$] region. In this region, the mode is highly localized to a narrow region in η

where the curvature is unfavorable. Here the average shear, $\hat{s}$, dominates other contributions to the local shear and is responsible for the localization along the field line. Hence, at large $\hat{s}$, the ballooning stability boundaries are weakly dependent upon δ.

In the small $\hat{s}$ region of symmetric tokamak-like configurations, the mode extent along η is large compared to 2π. Using a multiple scale analysis, one finds that the mode structure is described by an oscillation along the field line, which describes the ballooning effect, modulating an extended envelope which has width $\mathcal{O}(1/\hat{s})$. An important aspect of this analysis is a description of the second stability region at large α. When symmetry breaking contributions to the local shear enter, the extended envelope feature of the mode shape is disrupted [1]. In the presence of non-zero δ, the mode tends to localized along η in the bad curvature regions. This has the property of lowering the first stability region and eliminating the second stability region.



As seen in the above figure, another aspect of three-dimensional equilibria is that the local eigenvalues are field line dependent. In this figure, the solid curve represents the $\delta = 0$ case corresponding to the symmetric case. The eigenvalues for this case are independent of field line label. The remainder of the stability curves correspond to the common value of $\delta = 0.45$ (the magnetic geometry is fixed) for different magnetic field lines on the magnetic surface as labeled by the value of χ. The dotted, dashed, dash-dotted, and dash-triple-dotted curves correspond to $\chi = 0, 0.85, 1.70, 2.55$, respectively. Note that for the field line choice $\chi = 2.55$, all equilibria with $\hat{s} < 0.5$ have stable ballooning eigenvalues. However, for the same magnetic geometry at the field line choice $\chi = 0$, at sufficiently large α nearly every equilibrium is unstable for $\hat{s} > -0.2$. Generally speaking, in small $\hat{s}$ regions every magnetic surface contains a mixture of field lines with both stable and unstable local eigenvalues.

IV. Discussion

By employing a technique for generating local three-dimensional equilibria, generic properties of ideal MHD ballooning mode instabilities are examined. This allows one to manipulate profile and shaping properties of 3-D configurations for applications to ideal MHD ballooning mode stability. An example local helical axis model is introduced to emphasize the technique. In this model, the curvature and magnetic field spectrum are dominated by a single helical harrmonic. However, the local shear does not have the same spectral purity. In the model, a parameter can be adjusted to control the strength of symmetry breaking terms in the local shear.

Ballooning stability boundaries are demarked for the helically symmetric equilibria and demonstrates a sensitivity to symmetry breaking terms in the local shear. The presence of these terms produce a deterioration of the first stability boundary and a reduction or elimination of the second stability regions. This suggest that the lack of symmetry in the local shear can greatly impact the ballooning stability properties of quasi-symmetric systems. This property can explain the fairly low predicted critical β for ideal ballooning mode onset in the HSX quasi-helical stellarator [9]. Additionally, this work illustrates a unique property of 3-D systems, the eigenvalues of the local theory are field-line dependent.

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