EQUILIBRIUM AND STABILITY IN RELAXED MAGNETOHYDRODYNAMICS OF TOROIDAL PLASMAS

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Summary The calculation of three-dimensional magnetohydrodynamic (MHD) toroidal plasma equilibria is a challenging one conceptually because of the problems of Hamiltonian field-line chaos and singular currents. Combined with the need to achieve high accuracy for plasma stability calculations, this makes the problem very difficult, but important for advanced fusion experiments. A new approach based on multiple relaxed regions separated by ideal-MHD toroidal barrier tori is presented.

INTRODUCTION

It is standard practice to analyze the equilibrium and stability of plasma confinement designs for fusion energy experiments using magnetohydrodynamics (MHD) as a first approximation, either with zero viscosity and resistivity (ideal MHD) or with arbitrarily small resistivity to allow magnetic reconnection (resistive MHD). In axisymmetric systems, such as tokamaks (when discrete coil effects are ignored), where the magnetic field $\mathbf{B}(\mathrm{r})$ forms an integrable Hamiltonian system, equilibria with arbitrary pressure profiles exist. These can be calculated straightforwardly and then tested for stability by normal-mode analysis.

However, machines of the stellarator type, such as the H-1NF helical-magnetic-axis (heliac) device at the Australian National University or the NCSX experiment being built at Princeton (see Fig. 1), are *intrinsically* nonaxisymmetric. For these the magnetic field is generically nonintegrable and always has magnetic islands associated with rational values of the rotational transform (winding number of the field lines), with chaotic regions in their separatrix regions. This makes problematical the strict mathematical existence of equilibria [1] but, in designing NCSX, physicists have been able to compute (at least approximately and with considerable effort) configurations with finite plasma pressure β (ratio of kinetic to magnetic pressure). These have been optimized to reduce island widths [2] and thus increase the measure of "flux surfaces" [invariant tori on which field lines stay forever, the existence of which, for sufficiently irrational rotational transforms, is supported by Kolmogorov–Arnol'd–Moser (KAM) theory]. Little has been done, however, on the stability question in the presence of islands and chaotic regions.

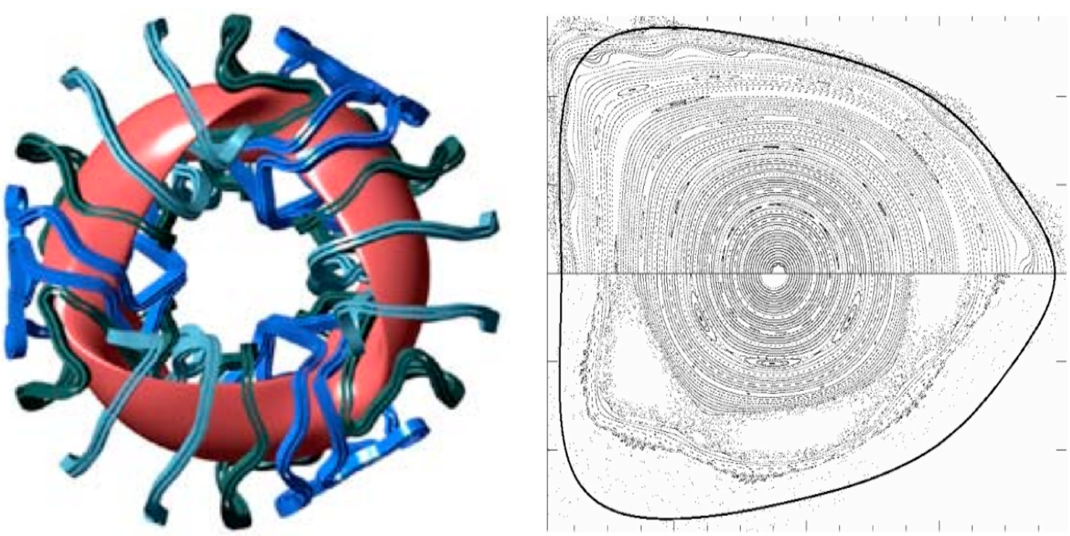

Figure 1. Left: A schematic showing the NCSX plasma (red), and the toroidal conductors (green) in which external currents producing the magnetic field flow. Right: Magnetic-field Poincaré plots for a $\beta = 4.1\%$ plasma before (lower) and after (upper) optimization of flux surfaces using external coils [2].

The progress described above gives reason to believe that a well-posed mathematical formulation of the MHD equilibrium problem should be possible, and one hopes this will improve the calculational efficiency and also allow meaningful discussion of stability. In fact we [3] have recently proposed a variational approach that holds promise of achieving this. This paper briefly introduces the idea and discusses progress on implementing it.

PARTIALLY CONSTRAINED ENERGY MINIMIZATION

The variational principle on which we base our approach lies between that of Kruskal & Kulsrud [4]—minimization of total energy $W = \int [B^2/2 + p/(\gamma - 1)]$ (where p is plasma pressure, γ the ratio of specific heats) under the uncountable infinity of constraints provided by applying ideal MHD within each fluid element—and the relaxed MHD of Woltjer [5] and Taylor [6]—minimization of W holding only the two global toroidal and poloidal magnetic fluxes, and the single global ideal-MHD helicity invariant $K \equiv \int \mathbf{A} \cdot \mathbf{B}$, constant. We extend the latter approach by dividing the plasma into subregions separated by infinitely thin ideal-MHD toroidal barriers between which Woltjer–Taylor relaxation is assumed. Expansion of W up to second order in the amplitude of fluid displacements provides both a generalized force from the first variation and a Jacobian operator from the second variation, thus allowing an accelerated steepest descent minimization and also providing stability information from the eigenvalues of the Jacobian operator.

FORCE BALANCE ACROSS IDEAL-MHD BARRIERS

The first variation of the positions of the ideal barriers yields the requirement that the total pressure, $B^2/2 + p$, be continuous across the ideal-MHD barriers. This gives rise to a Hamilton–Jacobi problem that relates the existence of the barriers to KAM theory [7, 8].

STABILITY

If the second variation is positive definite, $\delta^2 W > 0$, then an equilibrium configuration ($\delta W = 0$) is stable against the wide class of variations allowed in relaxed MHD, which includes ideal MHD and resistive MHD. We recast this as a generalized eigenvalue problem by defining a Lagrangian $L = \delta^2 W - \lambda N$, with N a positive definite normalization. The stability condition is $\lambda \geq 0$ for all eigenvalues. Using a normalization concentrated on the ideal-MHD barrier interfaces, the perturbed field in plasma regions is computed to be Beltrami ($\nabla \times \mathbf{B} = \mu \mathbf{B}$), with the same Lagrange multiplier μ as the equilibrium field. The interface equations produce an eigenvalue problem.

In cylindrical geometry with axial periodicity, the displacement is Fourier decomposed, and displacements of the form $e^{i(m\theta+\kappa z)}$ sought, where m is the poloidal mode number, and κ the axial wave number. Hole *et al.* [9] have studied the stability of these configurations as a function of mode number and number of ideal barriers, and bench-marked these results to earlier single interface studies. Hole *et al.* also revealed a paradox: the stability of a two-interface plasma with continuous rotational transform in the limit of barrier separation approaching zero differs from the stability of a single-interface barrier configuration with the same internal and vacuum rotational transform profile.

The discrepancy has been resolved by Mills [10], who studied the stability of configurations in which the inter-barrier region was ideal. In this case, the ideal stability of resonances in the inter-barrier region was handled explicitly, as opposed to the Woltjer–Taylor relaxed treatment, in which resonances do not explicitly feature. The ideal inter-barrier plasmas studies showed similar stability to the single interface configuration. Mills concluded it is the different treatment of resonances, which are implicit in Woltjer–Taylor relaxed plasmas, but explicit when computing ideal MHD stability that is responsible for reconciling the vanishing interface separation paradox. In more recent work, we have also shown that the tearing mode stability of the plasmas is equivalent to the frustrated Woltjer–Taylor plasmas studied here. In ongoing work, we are also studying whether quantization in the toroidal direction leaves a stable residue of configurations. If so, these constrained minimum energy states may be related to internal transport barrier configurations, which are plasma configurations with good confinement properties that form at sufficiently high heating power.

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