Organizing Chaos Via Constructing Almost Invariant Surfaces

Dr. Stuart Hudson *and Prof. R.L. Dewar*

1st Dewar Symposium on (Equilibrium,) Stability and Nonlinear dynamics of plasmas, 31 October 2009, Atlanta, GA, USA

Motivation

- toroidal magnetic fields are equivalent to $1\frac{1}{2}$ dim. Hamiltonian flows, which (in the absence of symmetry) are generally chaotic.

- for perfectly integrable systems, action-angle coordinates can be constructed by adapting the coordinates to the invariant surfaces.

- for nearly-integrable systems, the KAM theorem suggests that *approximate* action-angle coordinates will be useful.
  (i.e. phase space *does not* encounter a catastrophic collapse, but rather *slowly* disintegrates)

- we need to construct *almost*-invariant surfaces to act as coordinate surfaces in regions of chaos.

→ This talk will highlight the successes of constructing almost-invariant surfaces ←

Quadratic-flux minimizing (QFMin) surfaces are a natural extension of flux surfaces, defined for chaotic fields

- toroidal coordinates (ψ,θ,ζ), given magnetic field $\mathbf{B} = \nabla\times(\psi\;\nabla\theta - \chi\;\nabla\zeta)$, where $\chi = \chi(\psi,\theta,\zeta)$,

- a toroidal surface may be described $\psi = P(\theta,\zeta)$, normal $\mathbf{N} \equiv (\mathbf{e}_\theta + P_\theta \mathbf{e}_s)\times(\mathbf{e}_\zeta + P_\zeta \mathbf{e}_s)$, $\nu \equiv \mathbf{B}\bullet\mathbf{N}$

- tangential dynamics described according to

  angle dynamics from field, $\dot{\theta} = B^\theta / B^\zeta$, and radial dynamics constrained to lie on surface $\dot{\psi} = P_\theta \dot{\theta} + P_\zeta$

  i.e. <u>*pseudo-field*</u> $\mathbf{B}_\nu = \mathbf{B} - \nu\;\nabla\theta\times\nabla\zeta$

- quadratic flux $\varphi_2 = \dfrac{1}{2}\displaystyle\int\!\!\int (\mathbf{B}\bullet\mathbf{N})^2\, d\theta d\zeta$,

Note → coordinate dependence;
extra Jacobian factor appears

- allowing the surface to vary to extremize φ_2, obtain Euler-Lagrange equation $\mathbf{B}_\nu \bullet\nabla\nu = 0$

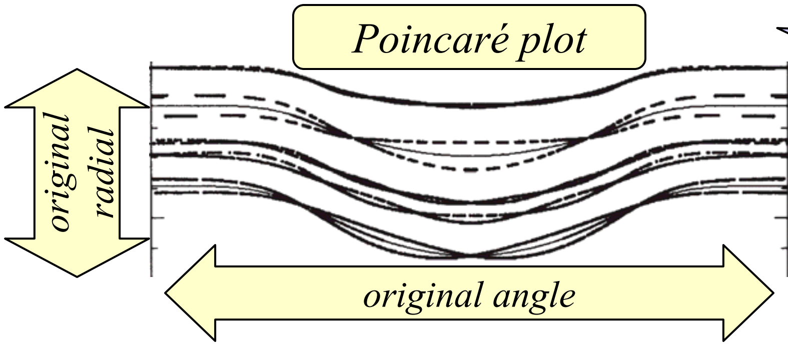
  pseudo-field-lines determined by following *pseudo*-field

1) normal-field, v, constant along pseudo-field lines;
2) rational surface = family of periodic pseudo-field lines;
3) o.d.e. integration suitable for low-order periodic surfaces;

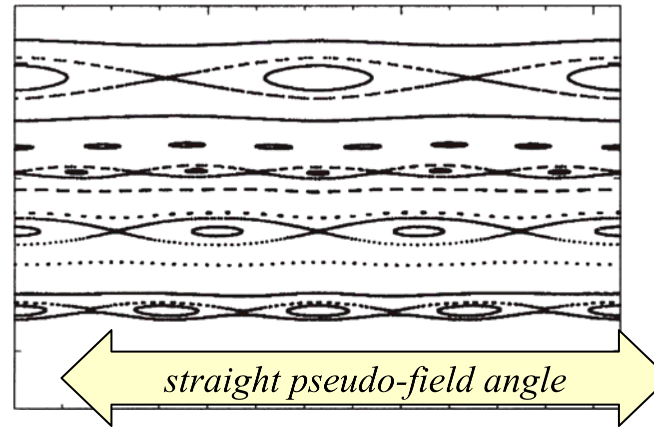
*Dewar & Meiss, Physica D, **57**, 476 (1992); Dewar, Hudson & Price, Phys. Lett. A **194**, 49 (1994); Dewar & Khorev, Physica D, **85**, 66 (1995); Hudson & Dewar, J. Plasma Phys., **56**, 361 (1996); Hudson & Dewar, Phys. Lett. A (2009)*

A family of low-order QFMin surfaces defines the *nearest*, smooth, nearby-integrable field

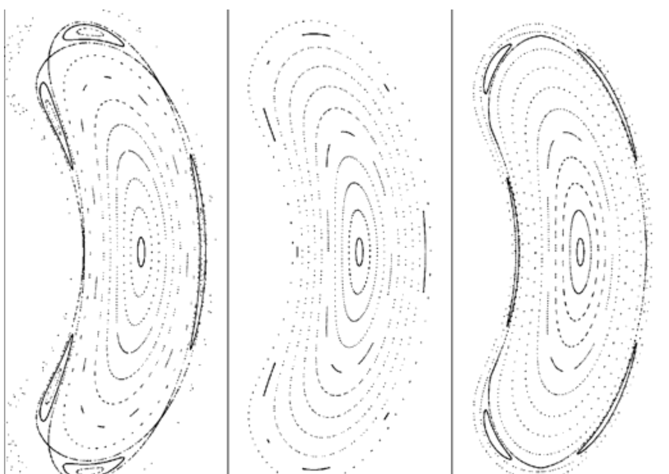
If an arbitrary field **B** is given, how do we best decompose $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}$?



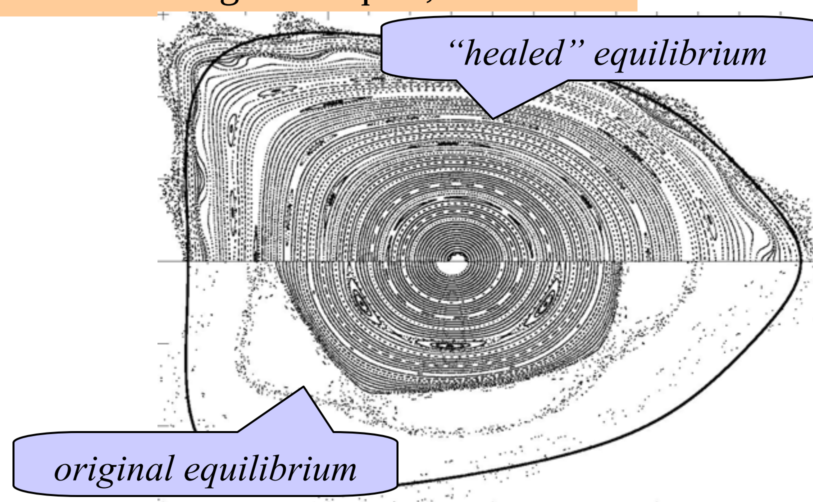
fit "coordinate" QFMin-surfaces through islands



Hamiltonian is efficiently written $\chi=\chi_0(\psi)+\sum\chi_{mn}(\psi)\cos(m\theta-n\zeta)$, where χ_{mn} are small;
determination of resonant perturbations allows efficient island healing techniques;



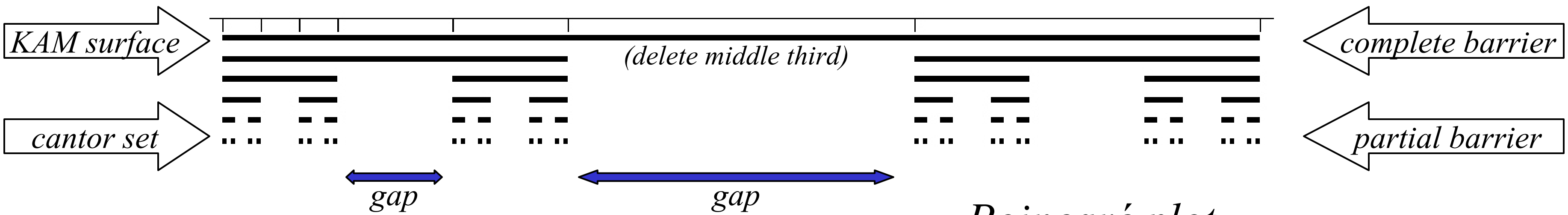
island-phase control in H1 stellarator
Hudson & Dewar, Phys. Lett. A 226,85 (1997)



island-healing in NCSX
Hudson et al., Phys. Rev. Lett. 89,275003 (2002)

Field-line transport is restricted by irrational field-lines

→ *the irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict field-line transport even after the onset of chaos.*



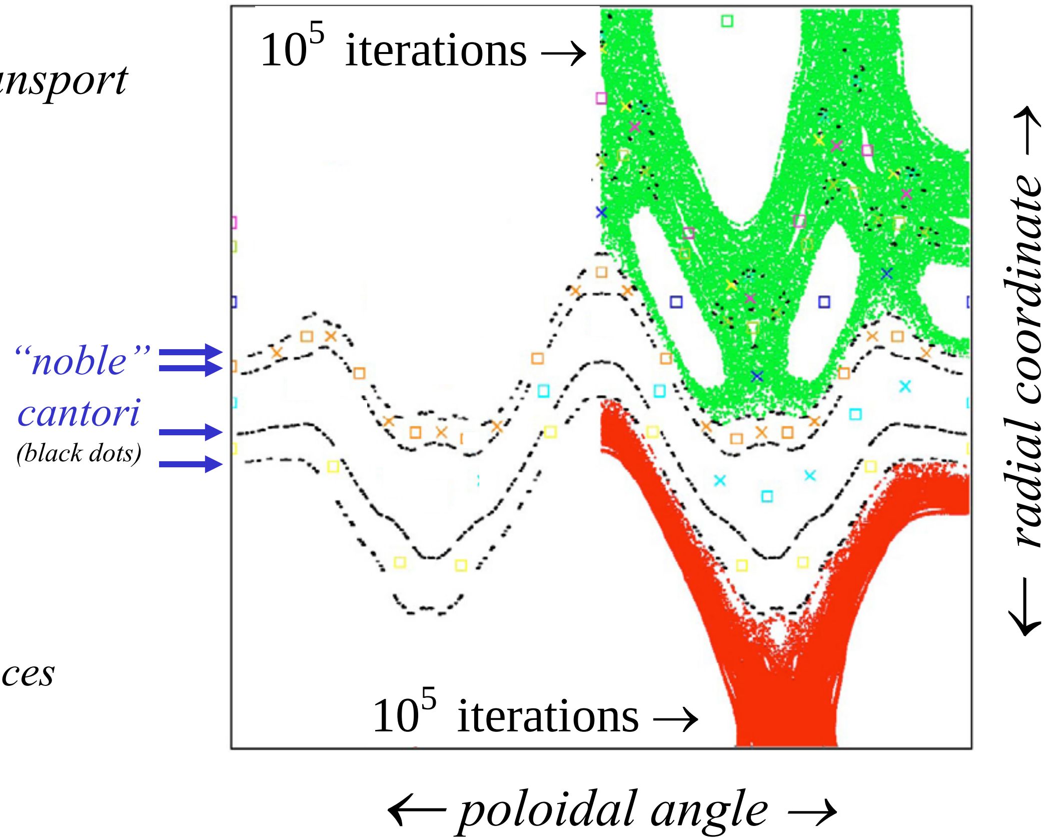
→ *KAM surfaces* ***stop*** *radial field-line transport*

→ *broken KAM surfaces ≡cantori*
do not stop, but do slow down
radial field-line transport

→ *cantori are approximated by high-order, unstable, periodic-orbits;*

→ *chaotic-coordinates are fit to the cantori,*
we need to (i) locate the cantori
(ii) fill-in-the-gaps to make surfaces

Poincaré plot (model field)



Alternative construction of QFMin surfaces employs constrained action-integral techniques

- Magnetic field-lines are curves, $C:\theta=\theta(\zeta),\ \psi=\psi(\zeta)$, that extremize the action $S=\int_C \mathbf{A}\cdot d\mathbf{r}$

$\rightarrow$ Euler-Lagrange equation $\mathbf{B}\times\delta\mathbf{r}=0$

$\rightarrow$ variational integration faster, robust to chaos; suited for finding high-order periodic orbits in chaos;

$\rightarrow$ for numerical implementation: discretize infinite-dimensional curves, $S=S(\theta_0,\theta_1,\theta_2,\ldots,\theta_N)$

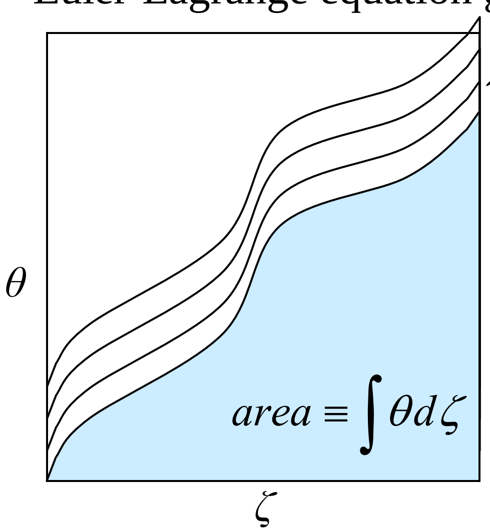
enforce periodicity constraint $\theta_N=\theta_0+2\pi p,\ \zeta_N=\zeta_0+2\pi q$

find zero of action-gradient vector, $\partial S/\partial\theta_i=0$, using Hessian $\partial^2 S/\partial^2\theta_{ij}$,

- A constrained variational principle for pseudo-field-lines $S=\int_C \mathbf{A}\cdot d\mathbf{r}-\nu\left(\int\theta\,\nabla\zeta\cdot d\mathbf{r}-a\right)$

recall, to find minimum of $f(x)$, subject to constraint $g(x)=g_0$, minimize $F(x,\lambda)=f(x)-\lambda\,[g(x)-g_0]$

- Euler-Lagrange equation gives pseudo-field $\mathbf{B}_\nu\equiv\mathbf{B}-\nu\,\nabla\theta\times\nabla\zeta$



↑ *increasing area constraint gives* ***family*** *of periodic pseudo-curves*

1) *Lagrangian variational methods suitable for high-order periodic orbits (q ~ 10,000) in strongly chaotic fields;*
2) *High-order periodic orbits approximate KAM surfaces and broken KAM surfaces (cantori);*

Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.

C. Golé, J. Differ. Equations **97**, 140 1992., R. S. MacKay and M. R. Muldoon, Phys. Lett. A **178**, 245, 1993.

- At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\partial^2 S / \partial^2 \theta_{ij}$, with negative eigenvalue indicates the direction in which the action integral decreases.

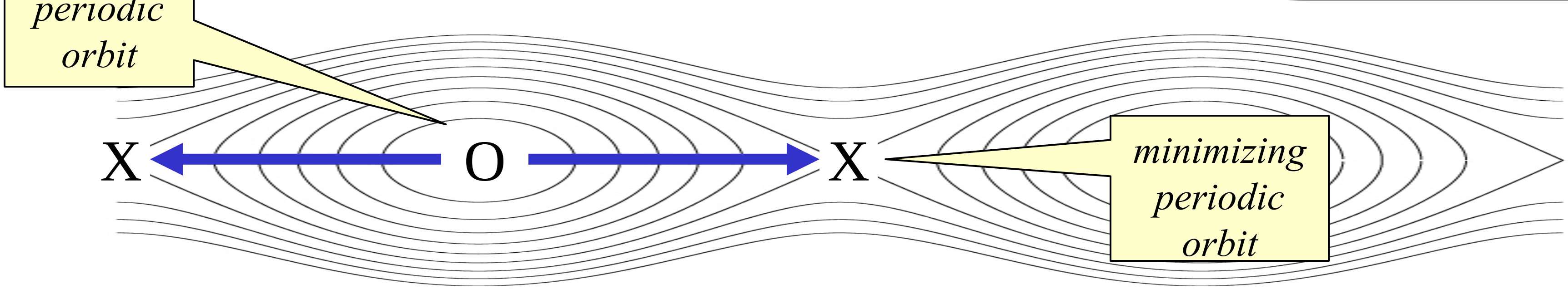
- Pushing trial curve from minimax (stable) p/q orbit down action-gradient flow to minimizing (unstable) p/q orbit defines *ghost-surfaces*,

- Ghost-surfaces may be thought of as rational coordinate surfaces that pass through island chains.

$\rightarrow$ shown to have desirable properties, such as non-intersection, . . .

Action Gradient Flow $\dfrac{\partial \theta_i}{\partial \alpha} = -\dfrac{\partial S}{\partial \theta_i}$

problem: action-gradient flow is small near integrable limit



Ghost-surfaces are almost identical to QFMin surfaces.

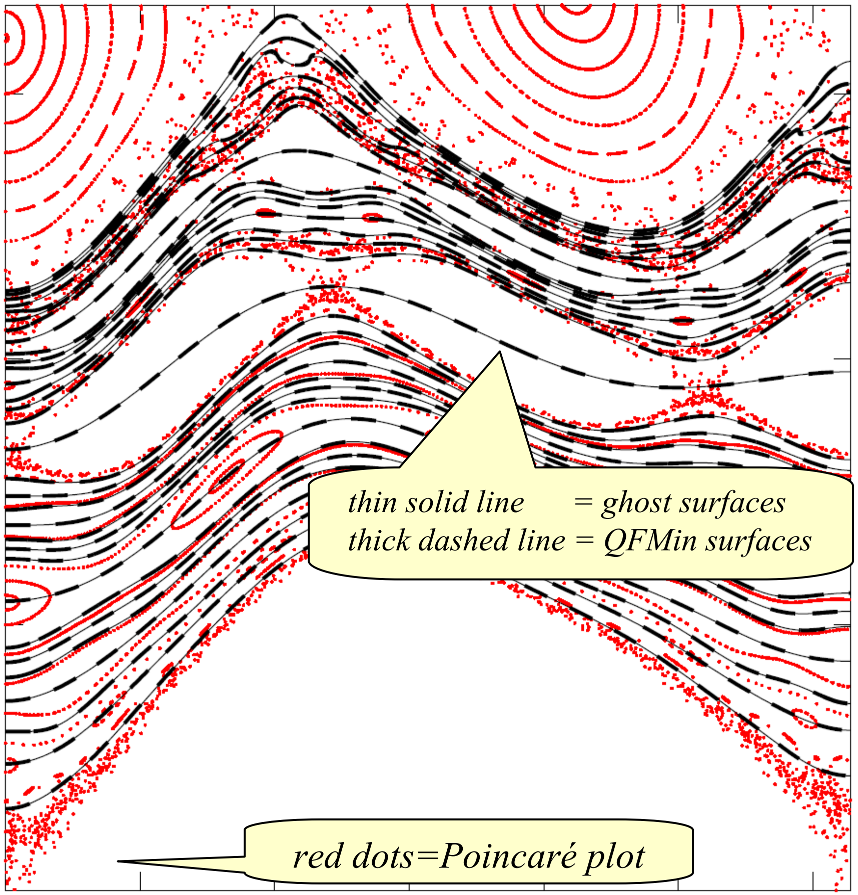
!!Ghost surfaces are defined by action-gradient flow;
!! QFMin surfaces defined by minimizing quadratic-flux;
→ no obvious reason why these different definitions should give the same surfaces

- Numerical evidence suggests ghost-surfaces and QFMin surfaces are *almost* the same;

- confirmed to1st-order using perturbation theory;

→ to higher order, need to exploit coordinate dependence of QFMin surfaces and ghost surfaces . . .

- opens possibility of using fast, robust construction of *unified* almost-invariant surfaces & chaotic coordinates



Hudson & Dewar, Phys. Lett. A, 2009.

Anisotropic transport is solved by chaotic-coordinates.

→ *ghost-surfaces for high-order periodic orbits "fill-in-the-gaps" in the irrational cantori;*
→ *ghost-surfaces and isotherms are almost indistinguishable; suggests T=T(s);*

- heat transport in plasmas is strongly anisotropic

$$\kappa_{\|}\nabla_{\|}^{2}T+\kappa_{\perp}\nabla_{\perp}^{2}T=S, \qquad (S \text{ is source})$$

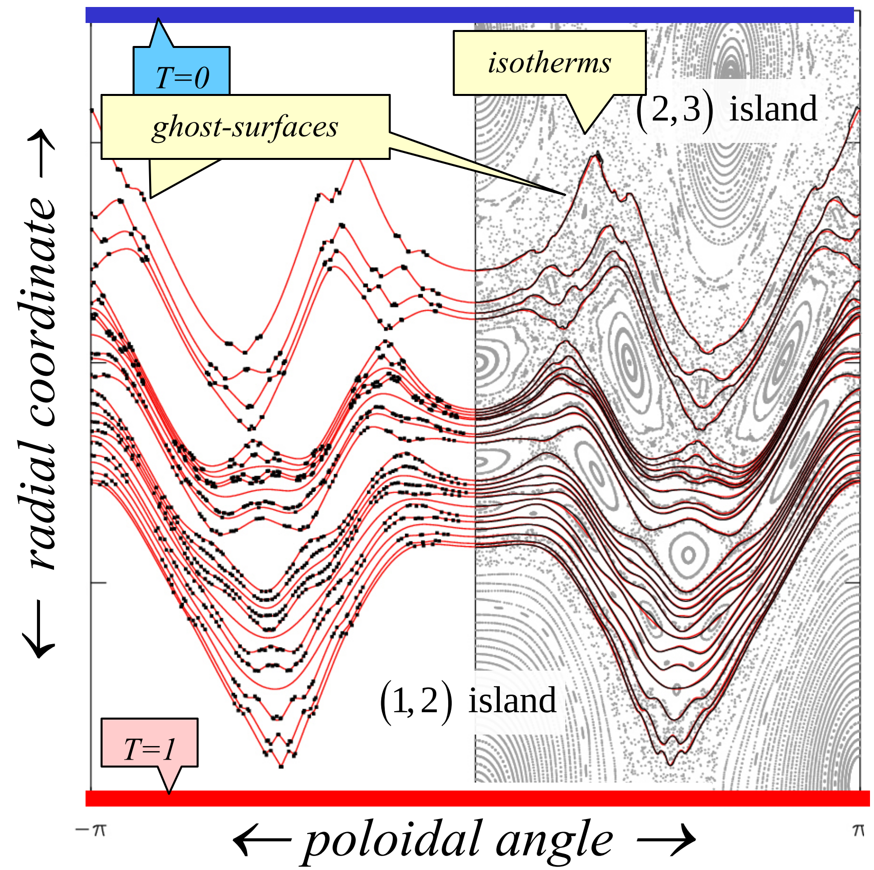
$\kappa_{\perp}/\kappa_{\|}\sim 10^{-10}$, solved numerically on grid $2^{12}\times 2^{12}$

→ parallel diffusion dominates perpendicular diffusion

- structure of temperature is dominated by the structure of the magnetic field;

→ structure of coordinates ≡ structure of field

- temperature adapts to almost-invariant surfaces; we obtain $T=T(s)$, where s labels ghost surfaces;



Hudson & Breslau, Phys. Rev. Lett. **100**, 095001 (2008)

Chaotic-coordinates simplifies temperature profile

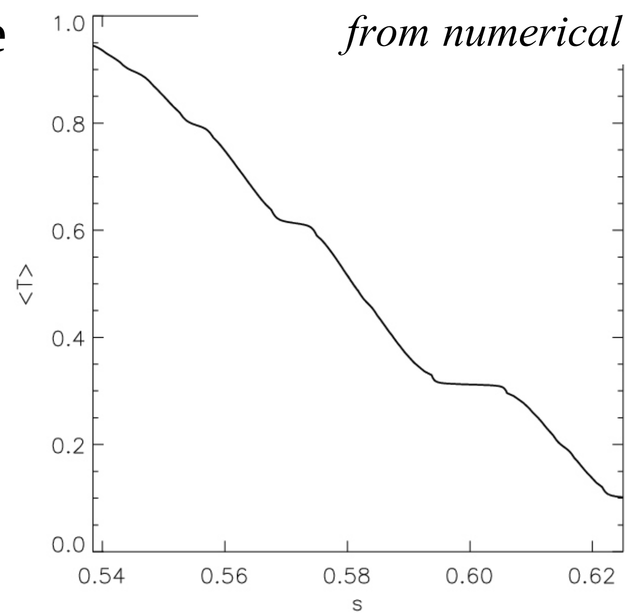
$\rightarrow$ *ghost-surfaces can be used as radial coordinate surfaces* $\rightarrow$ *chaotic-coordinates* (s,θ,ϕ)

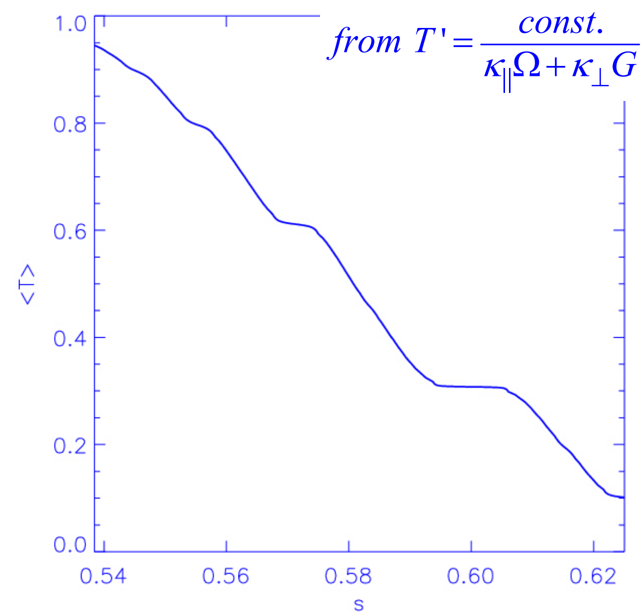
- From $0=\dfrac{\partial}{\partial s}\int_V \nabla\cdot\mathbf{q}\, dV = \dfrac{\partial}{\partial s}\int_{\partial V}\mathbf{q}\cdot\mathbf{n}\, d\sigma$ assume $T=T(s)$ to derive $T' = \dfrac{const.}{\kappa_{\parallel}\Omega + \kappa_{\perp} G}$

  for quadratic-flux $\Omega = \int d\sigma g^{ss}(B_n/B)^2$, and metric $G = \int d\sigma g^{ss}$, where $g^{ss} = \nabla s\cdot\nabla s$, $B_n = \mathbf{B}\cdot\nabla s/|\nabla s|$
- in the "ideal limit" $\kappa_{\perp}\rightarrow 0$, $T' \rightarrow \infty$ on irrational KAM surfaces where $\Omega = 0$;
- non-zero $\kappa_{\perp}$ ensures $T(s)$ is smooth, T' peaks on minimal-Ω surfaces (noble cantori).

Temperature Profile

$(\kappa_{\perp}/\kappa_{\parallel} = 10^{-10})$





Conclusions

→ constructing almost-invariant surfaces is useful for understanding chaotic systems

→ *e.g. for constructing nearby integrable fields, understanding transport through chaos*

→ chaotic magnetic coordinates show the promise of (approximately) extending the simplicity of action-angle coordinates to chaotic flows

→*e.g. the temperature becomes a surface function $T=T(s)$*

→ Ghost-surfaces have desirable properties (non-intersecting . .), and are very similar to quadratic-flux minimzing surfaces

→ *but QFMin surfaces are much easier to construct*