# Chaotic fields, pressure, and MHD equilibria

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*Abstract*

→ The ideal MHD equilibrium equations, $\mathbf{J} \times \mathbf{B} = \nabla p$,
imply that pressure is constant along a field line, $\mathbf{B} \cdot \nabla p = 0$.

→ For chaotic fields, this means that the structure of the pressure cannot be resolved using standard numerical methods.

→ To avoid these problems, we instead consider the pressure to be determined by

$$\nabla \cdot \left( \kappa_{\|} \, \mathbf{b} \, \mathbf{b} \cdot \nabla p + \kappa_{\perp} \nabla_{\perp} p \right) = S.$$

where $\kappa_{\|}, \kappa_{\perp}$ are diffusion coefficients, $\kappa_{\|} >> \kappa_{\perp}$, and $S$ is a source.

→ A numerical algorithm for solving this equation is being implemented in the HINT code.

# Dr. Stuart Hudson

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|---|---|---|---|
| 1996 | : Ph.D. | Australian National University | (with Prof. Dewar) |
| 1997 – 1998 | : Postdoc | JAERI | (with Prof. Tokuda) |
| 1999 – 2000 | : Postdoc | University of Wisconsin | (with Prof. Hegna) |
| 2001 – | : | Princeton Plasma Physics Laboratory | |

Research interests include:

1) Chaotic fields
   1) magnetic coordinates for chaotic fields
   2) heat transport, transport barriers in chaotic fields

2) Ideal MHD stability in stellarators
   1) ballooning stability in stellarators
   2) marginal stability diagrams for stellarators

3) MHD equilibrium calculation
   1) design of NCSX
   2) island healing algorithm

# Part 1 : Review of Recent Work

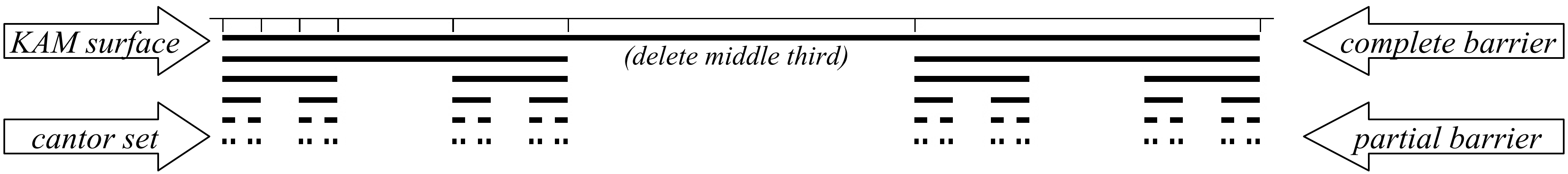
In the following slides I will try to show that :

→ chaotic fields are not random; near critical fields most relevant;
→ some flux surfaces may still exist (KAM surfaces) that **stop** field line transport;
→ other structures exist which **restrict** field line transport e.g. cantori;
→ ghost-surfaces are almost invariant surfaces = replacement KAM surfaces;
→ construct chaotic magnetic coordinates for chaotic fields;

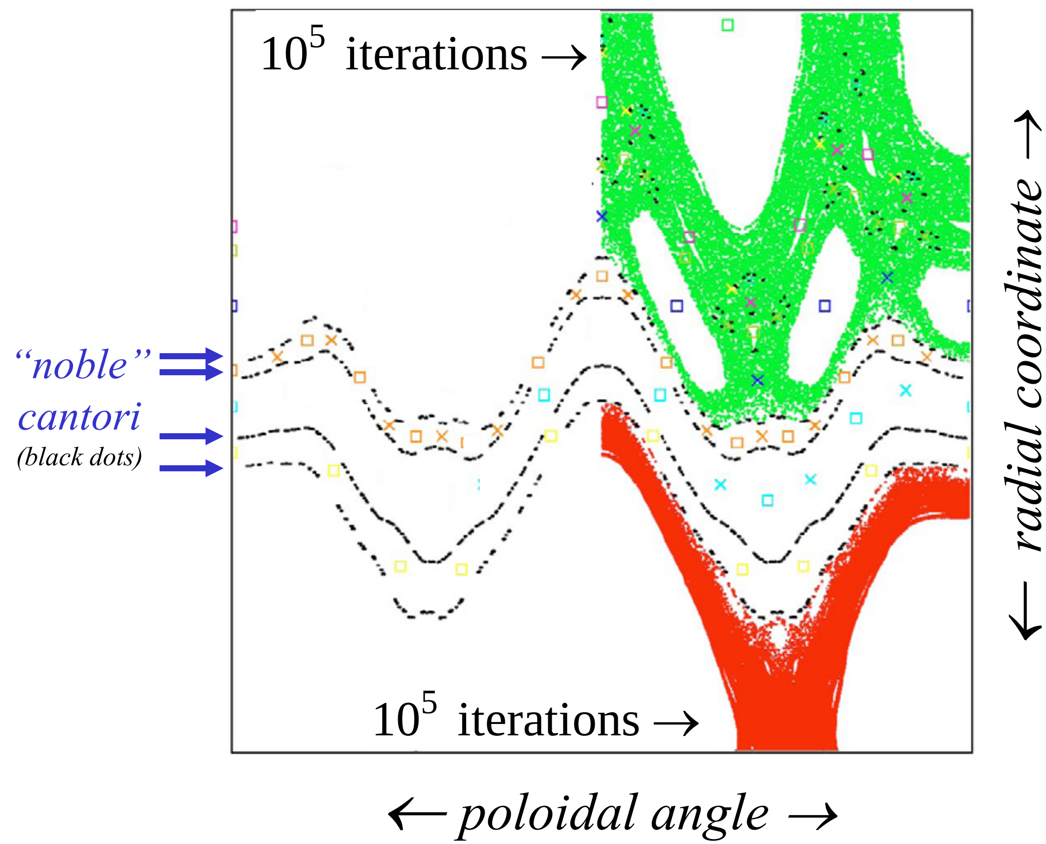
→ for heat transport in chaotic fields, temperature becomes a surface function;
→ semi-analytic expression for temperature gradient can be derived;

# For chaotic fields, field line transport is restricted by cantori

→ *the irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict field line transport even after the onset of chaos.*




*Poincaré plot (model field → next slide)*




→ *KAM surfaces* **stop** *radial field line transport*

→ *broken KAM surfaces ≡cantori* **do not stop, but do slow down** *radial field line transport*

*Hudson, Phys. Rev. E., 2006*

# Cantori are approximated by high-order periodic orbits;

$\rightarrow$ *high-order (minimizing) periodic orbits are located using variational methods;*

- Magnetic field lines, $\mathbf{B} = \nabla \times \mathbf{A}$, are stationary curves $C$ of the action integral $S = \int_C \mathbf{A} \cdot \mathbf{dl}$,

  where $\mathbf{A} = \psi \, \nabla \theta - \chi \, \nabla \phi$ and $\chi(\psi, \theta, \phi) = \psi^2 / 2 + \sum k_{mn}(\psi) \cos(m\theta - n\phi)$.

- Setting $\delta S = 0$ gives $\dot{\theta} = B^\theta / B^\phi = \dot{\theta}(\psi, \theta, \phi)$ and $\dot{\psi} = B^\psi / B^\phi$.

- A piecewise linear, $\theta(\phi) = \theta_i + (\theta_{i+1} - \theta_i) / \Delta\phi$, trial curve

  allows analytic evaluation of the action integral, $S = S(\theta_0, \theta_1, \theta_2 \, \ldots) \rightarrow \textit{fast}!$

- To find $(p, q)$ periodic curves, use Newton's method to find $\partial S / \partial \theta_i = 0 \rightarrow \textit{robust}!$

  with constraint $\phi_N = 2\pi q$, $\theta_N = \theta_0 + 2\pi p$.

- Two types of periodic orbit: $O$ : stable, action-minimax

  $X$ : unstable, action-minimizing $\rightarrow$ *cantori as* $p/q \rightarrow$ *irrational*

# Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.

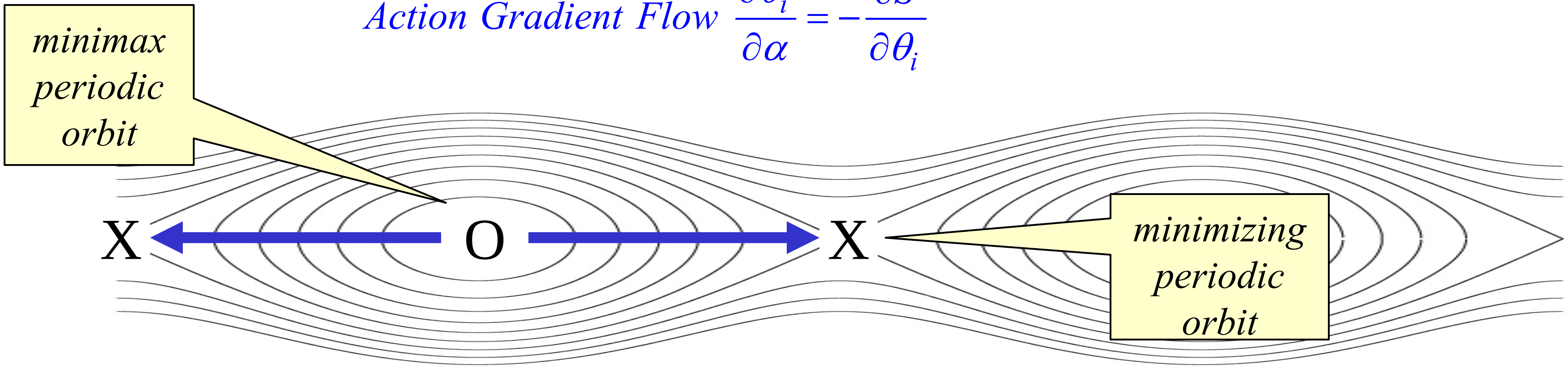
C. Golé, J. Differ. Equations **97**, 140 1992., R. S. MacKay and M. R. Muldoon, Phys. Lett. A **178**, 245, 1993.

- At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\partial^2 S / \partial^2 \theta_{ij}$, with negative eigenvalue indicates the direction in which the action integral decreases.

- Pushing trial curve from minimax (stable) $p/q$ orbit down action-gradient flow to minimizing (unstable) $p/q$ orbit defines *ghost-surfaces*,

- Ghost-surfaces may be thought of as rational coordinate surfaces that pass through island chains.

*Action Gradient Flow* $\dfrac{\partial \theta_i}{\partial \alpha} = -\dfrac{\partial S}{\partial \theta_i}$



# Steady state temperature is solved numerically; isotherms coincide with ghost-surfaces.

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|---|
| $\rightarrow$ *ghost-surfaces "fill in the gaps" in the irrational cantori; quasi-KAM surfaces for chaos!* |
| $\rightarrow$ *ghost-surfaces and isotherms are almost indistinguishable;* |

- anisotropic diffusion $\kappa_{\parallel} / \kappa_{\perp} = 10^{10}$

$$\nabla \cdot \left( \kappa_{\parallel} \mathbf{b}\, \mathbf{b} \cdot \nabla T + \kappa_{\perp} \nabla T \right) = \cancel{Source}$$

*field line derivative*

parallel diffusion

$$\nabla \cdot \left( \mathbf{b}\, \mathbf{b} \cdot \nabla T \right) = B^{\phi} \frac{\partial}{\partial \phi} \left( \frac{B^{\phi}}{B^2} \frac{\partial T}{\partial \phi} \right)$$

perpendicular diffusion

$$\nabla \cdot \nabla T \approx \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

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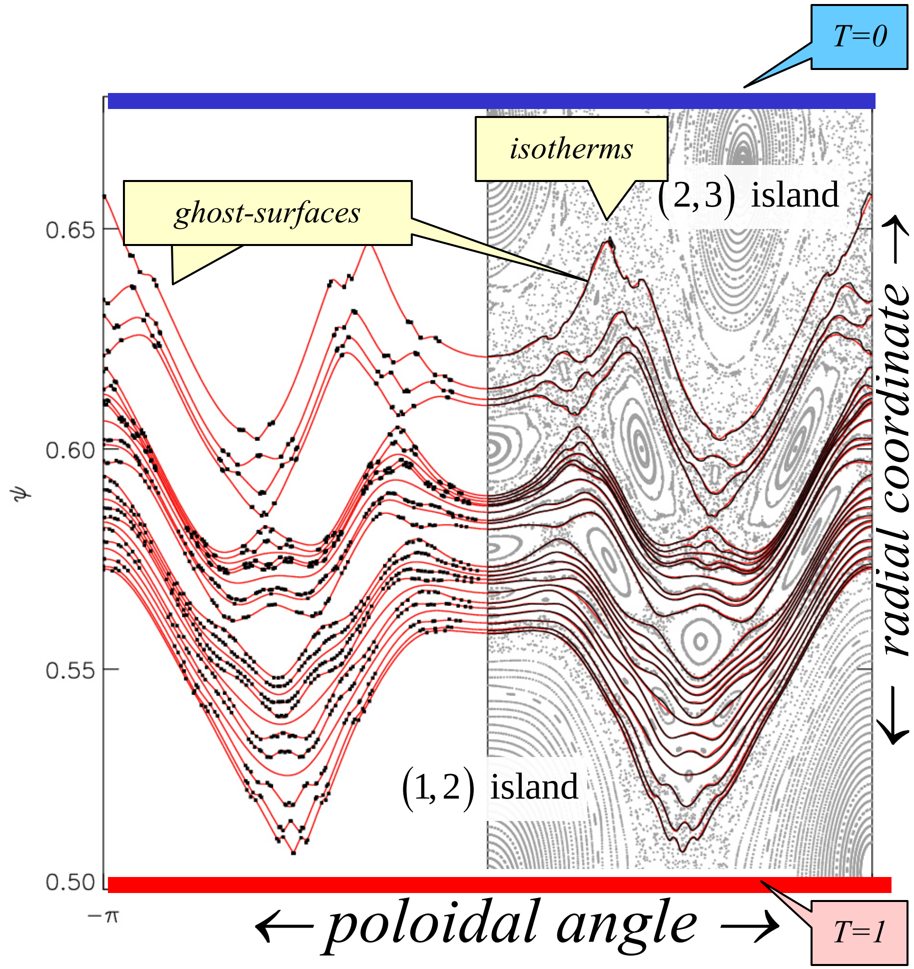
| *Hudson & Breslau, Phys. Rev. Lett., 2008* |



# Chaotic-coordinates simplifies temperature profile

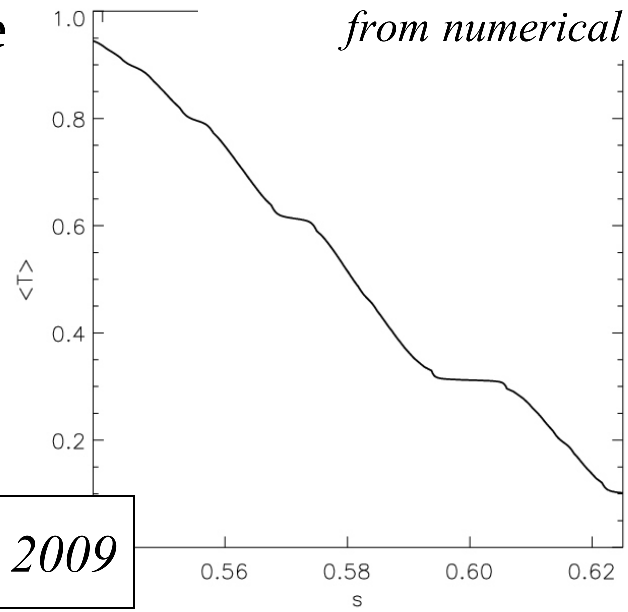
*→ ghost-surfaces can be used as radial coordinate surfaces → chaotic-coordinates $(s,\theta,\phi)$*

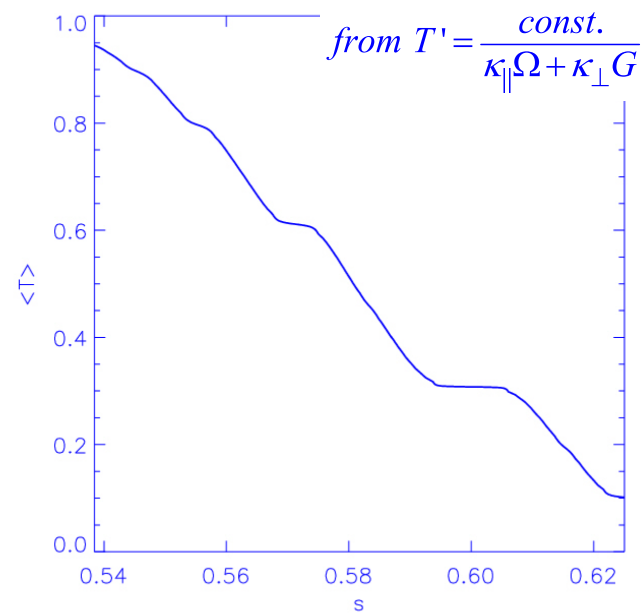
- From $0=\dfrac{\partial}{\partial s}\int_V \nabla\cdot\mathbf{q}\,dV=\dfrac{\partial}{\partial s}\int_{\partial V}\mathbf{q}\cdot\mathbf{n}\,d\sigma$ assume $T=T(s)$ to derive $T'=\dfrac{const.}{\kappa_{\parallel}\Omega+\kappa_{\perp}G}$

  for quadratic-flux $\Omega=\int d\sigma g^{ss}(B_n/B)^2$, and metric $G=\int d\sigma g^{ss}$, where $g^{ss}=\nabla s\cdot\nabla s$, $B_n=\mathbf{B}\cdot\nabla s/|\nabla s|$
- in the "ideal limit" $\kappa_{\perp}\to 0$, $T'\to\infty$ on irrational KAM surfaces where $\Omega=0$;
- non-zero $\kappa_{\perp}$ ensures $T(s)$ is smooth, $T'$ peaks on minimal-$\Omega$ surfaces (noble cantori).

Temperature Profile

$(\kappa_{\parallel}/\kappa_{\perp}=10^{10})$




*Hudson, Phys. Plasmas, 2009*

# Part 2 : Outline of work at NIFS

In the following slides I will try to show that:

$\rightarrow$ for a chaotic field, the equation $\mathbf{B}\cdot\nabla p = 0$ has the solutions

(i) $p' =$ zero , or

(ii) $p'$ is discontinuous almost everywhere;

$\rightarrow$ discuss the modifications required to the HINT code

(replace $\mathbf{B}\cdot\nabla p = 0$ with anisotropic pressure diffusion $\nabla\cdot\left(\kappa_{\|}\,\mathbf{b}\,\mathbf{b}\cdot\nabla p + \kappa_{\perp}\nabla_{\perp} p\right) = S$)

$\rightarrow$ give some early results indicating numerical convergence for a toy magnetic field

# Chaotic Magnetic Fields are Fractal

KAM theorem: a finite measure of irrational flux surfaces will remain

* the rotational transform, $\iota$, must be sufficiently irrational

$$\exists r > 0,\ \exists k > 2\ \ s.t.\ \forall m, n \in \mathbb{Q},\ \ \left|\iota - n/m\right| > \frac{r}{m^k}$$

* KAM surfaces can support pressure gradient
* rational surfaces are destroyed $\rightarrow$ replaced by local irregular region

* Let $S$ be the set $s \in S$ if KAM surface with $\iota = s$ exists
* Let $I(s)$ be an indicator function: $I(s) = 1$ if $s \in S$, $I(s) = 0$ if $s \notin S$

$\mathbf{B} \cdot \nabla p = 0$ requires that $p'(s) = I(s) P(s)$

* for $p(s)$ to be continuous, $P(s)$ must be bounded

* $I(s) = 0$ for $s = n/m$, $\quad p'(s) = 0$ for $s = n/m$
* for p(s) to be non-trivial, $p'(s) \neq 0$ for $s \neq n/m$

p'(s) is discontinuous almost everywhere

# The HINT code seeks approximation to MHD equilibria

$\rightarrow$ the (old) HINT code seeks solutions to

(1) $\mathbf{B}\cdot\nabla p = 0$

(2) $\mathbf{J}\times\mathbf{B}-\nabla p = 0$

(3) $\nabla\times\left(\mathbf{v}\times\mathbf{B}-\eta\mathbf{J}\right)=0$

$\rightarrow$ the (new) HINT code (under development) will solve

(1) $\nabla\cdot\left(\kappa_{||}\nabla_{||}p+\kappa_{\perp}\nabla_{\perp}p\right)=S,$

(2) $\mathbf{J}\times\mathbf{B}-\nabla p+\mu\nabla\cdot\nabla\mathbf{v}= \cancel{\rho\mathbf{v}\cdot\nabla\mathbf{v}}$

(3) $\nabla\times\left(\mathbf{v}\times\mathbf{B}-\eta\mathbf{J}\right)=0$

where $\nabla_{||}=\mathbf{B}\,\dfrac{\mathbf{B}\cdot\nabla}{B^2}\equiv$ *directional* derivative along the field

$\kappa_{||} >> \kappa_{\perp}$ are the parallel and perpendicular diffusion coefficients;

# Numerical solution to anisotropic diffusion almost completed

→A field aligned grid is constructed by following field lines

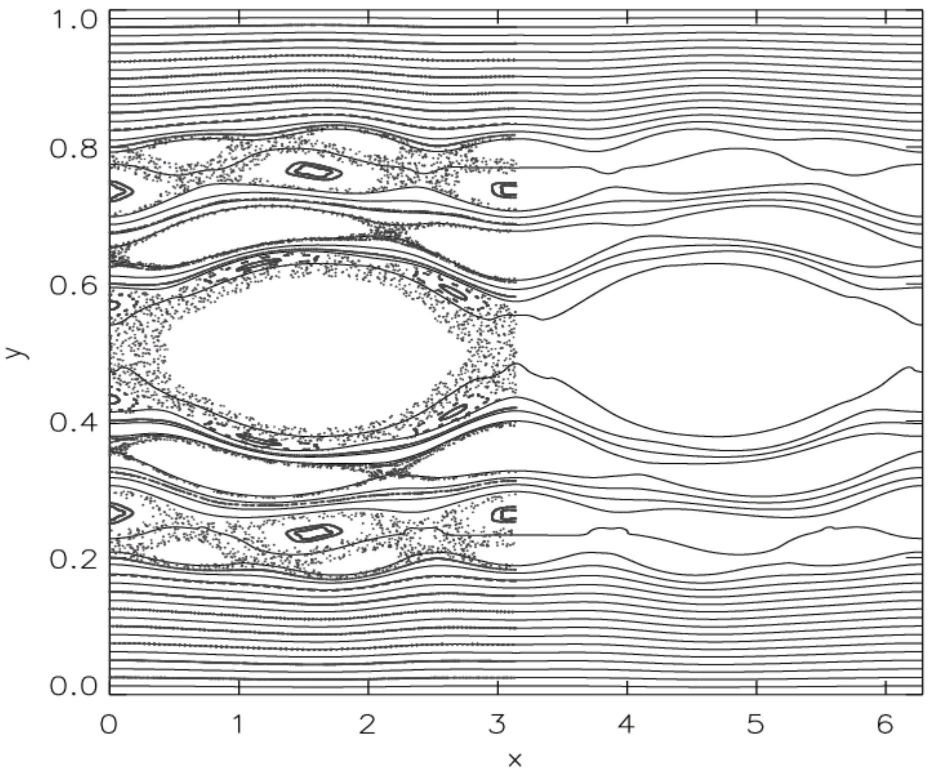
→Similar to existing HINT algorithm for pressure relaxation but faster;

→A 4th-order finite-difference solution has been implemented (for a toy field)

→the expected convergence is obtained;

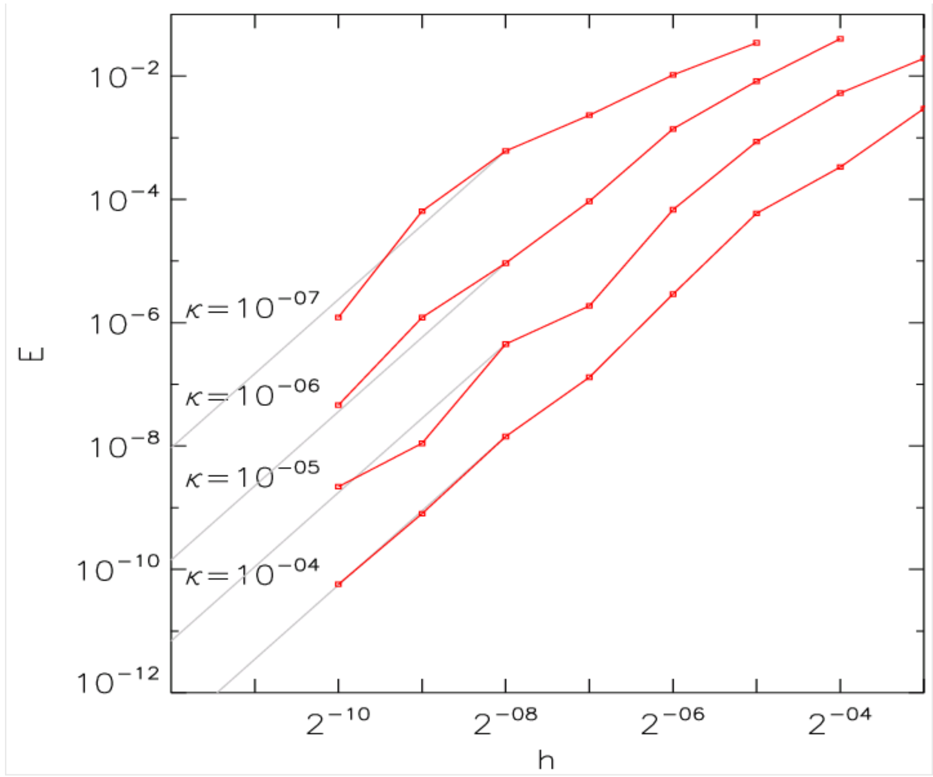
Poincare plot

Showing islands, chaos and pressure;




Error vs grid resolution

error ~ $h^4$



# Conclusion

$\rightarrow$ Chaotic magnetic fields have a fractal structure;

* field line transport is not diffusive

$\rightarrow$ The chaotic structure can be exploited

* irrational cantori severely restrict radial field line transport
* magnetic coordinates for chaotic fields can be constructed
* the temperature becomes a surface function

$\rightarrow$ If ignored, the chaotic structure causes problems for numerical algorithms

* the solution to $\mathbf{B} \cdot \nabla p = 0$ is pathological
* instead we must use $\nabla \cdot \left( \kappa_{\|} \nabla_{\|} p + \kappa_{\perp} \nabla_{\perp} p \right) = S$