Action-Based Definitions of Almost-Invariant Tori in Close-To-Integrable Hamiltonian Systems

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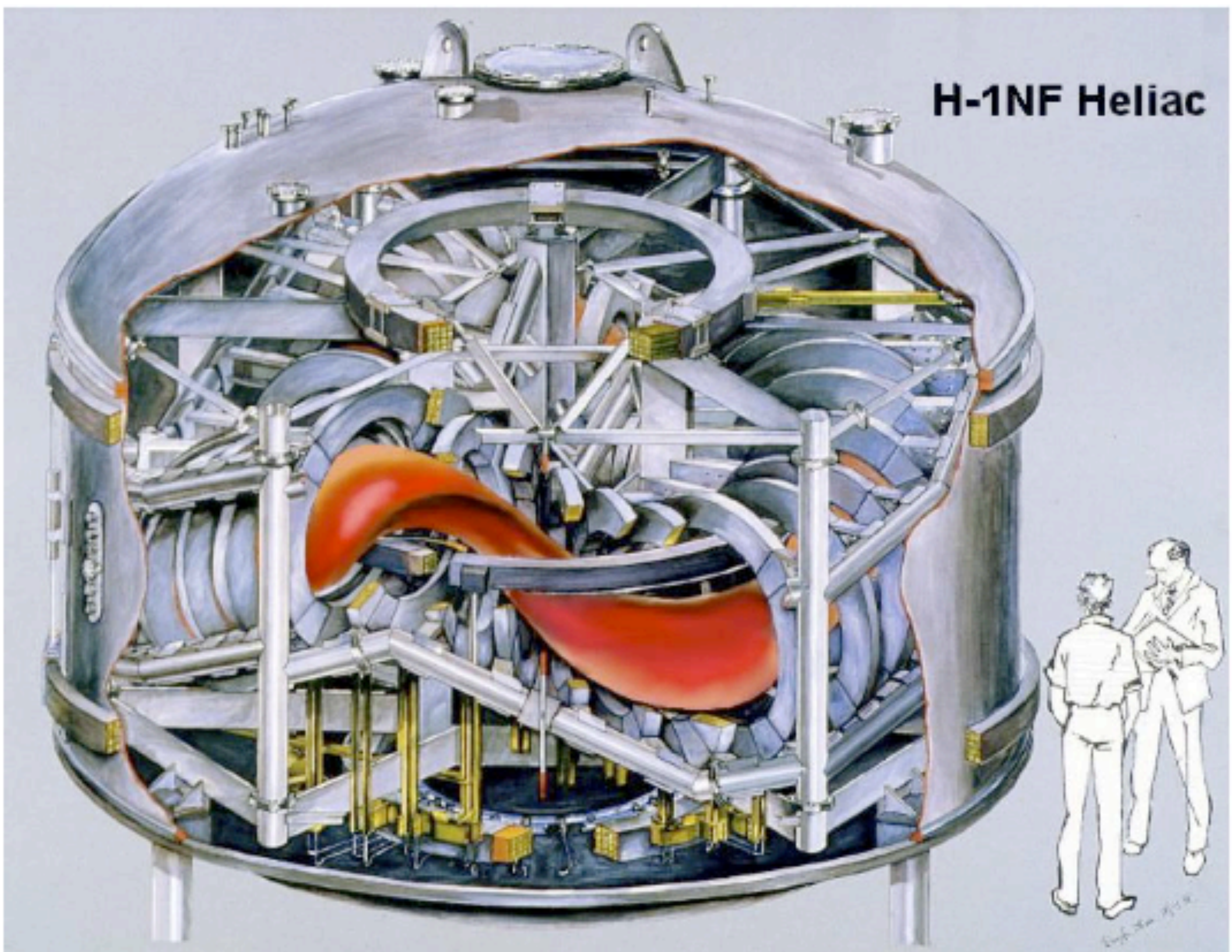
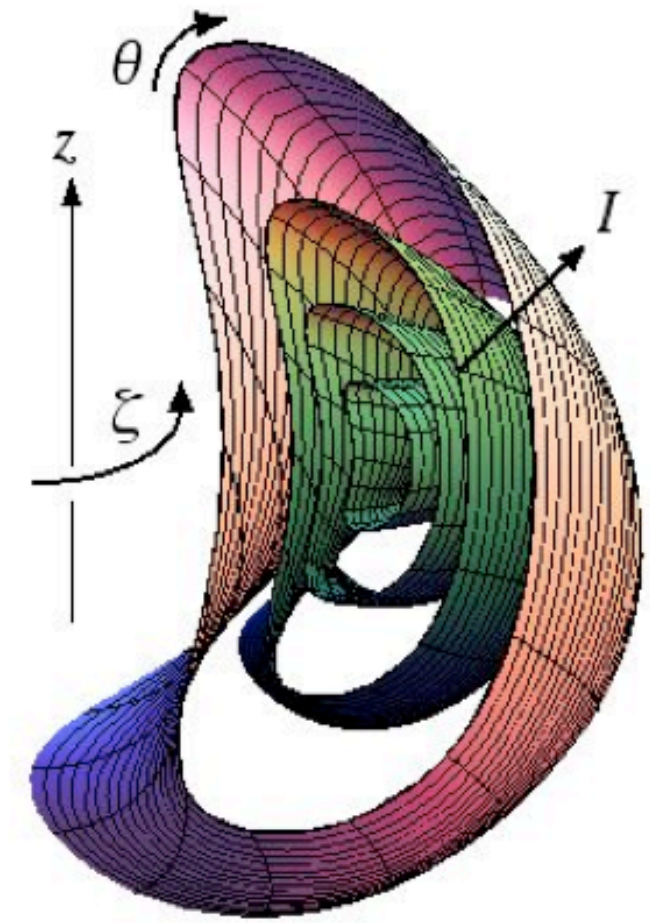
AIP Congress Melbourne
9th December 2010

Content of talk

- What are close-to-integrable 1½-d.o.f. systems?
- Periodic pseudo-orbits as basis of approach
- Action-minimization strategies for pseudo-orbits
- Reconciliation of ghost and QFMin approaches

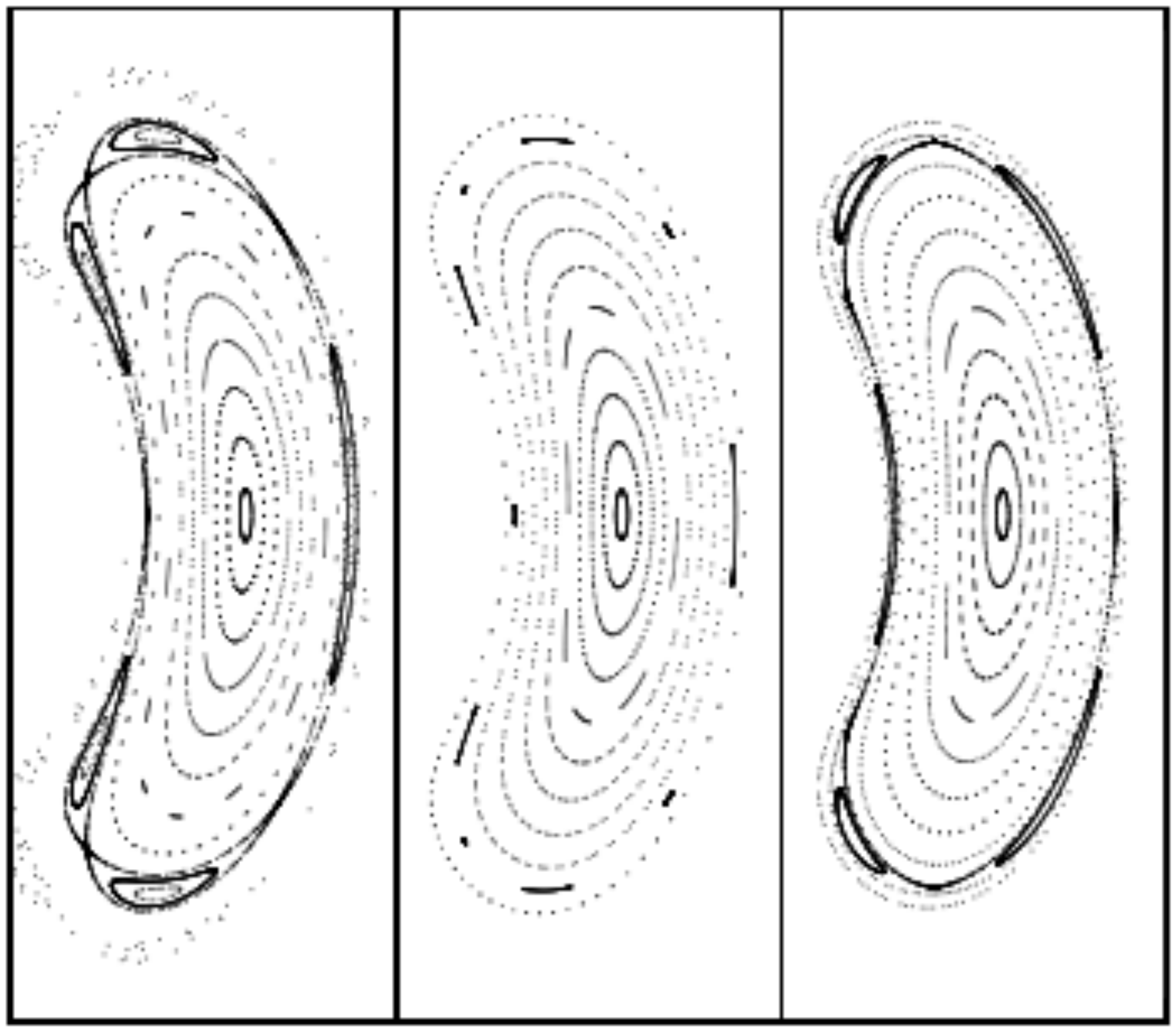
Example of 1½ d.o.f. 3-D Magnetic fields





- Field line flow is a 1-degree-of-freedom Hamiltonian system, with toroidal angle ζ playing the role of time
- Helical deformation *breaks toroidal symmetry* so ζ is not an ignorable coordinate: extra "½ degree of freedom"

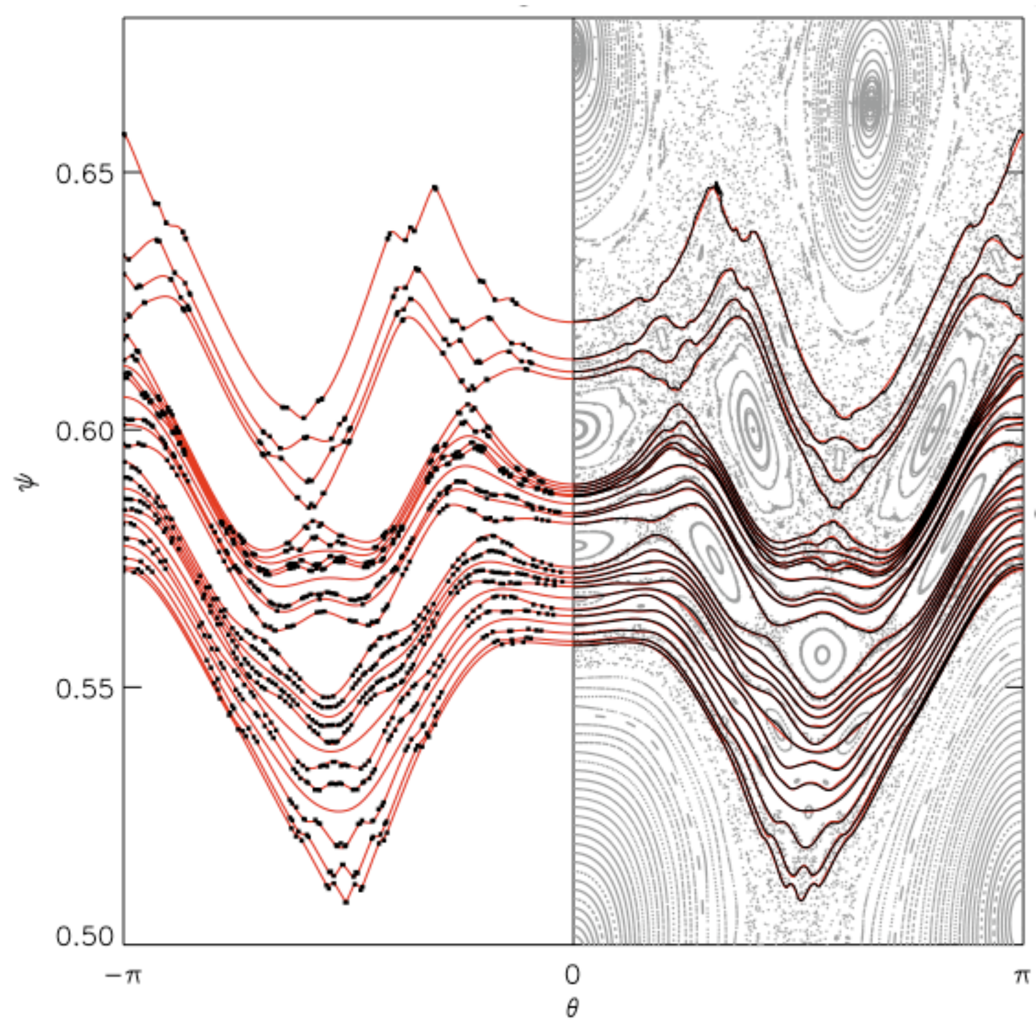
1½ d.o.f. Hamiltonian systems are not generically integrable—resonant invariant tori are destroyed



Example: island in H-1NF for 3 values of vertical field.

Our approach is to define an almost-invariant torus as a trial surface selected so as to *minimise* the quadratic flux, a weighted mean of $(\mathbf{n.B})^2$ (QFMin principle).

Heat transport in chaotic magnetic fields: Temperature contours fitted well by "ghost surfaces"



Hudson & Breslau Phys Rev Letters **100**, 095001 (2008) show that *temperature contours* for heat diffusion in fields with imperfect magnetic surfaces appear to agree very well with "ghost surfaces," an alternative to QFMin surfaces. What is the relation between these two approaches, and can they be unified?

Generalisation to 1½-d.o.f. Hamiltonian systems

- Consider non-autonomous, periodic-in-time* system with Hamiltonian approximately in action-angle form

$$H = H_0(I, \theta) + \epsilon H_1(I, \theta, t)$$

and corresponding Lagrangian

$$\begin{aligned} L &\equiv I(\theta, \dot\theta, t)\dot\theta - H(I(\theta, \dot\theta, t), \theta, t) \\ &= L_0(\theta, \dot\theta) + \epsilon L_1(\theta, \dot\theta, t) \end{aligned}$$

where $I(\theta, \dot\theta, t)$ is obtained by solving ***one*** of the Hamiltonian eqs. of motion *exactly*: $\dot\theta - H_I(I, \theta, t) \equiv 0$

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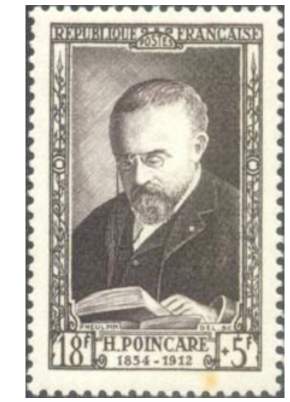
*We identify *t* with *toroidal angle* in field line application.

Periodic orbits as a key to chaos

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici reputée inabordable."



H. Poincaré: *Les Méthodes Nouvelles de la Mécanique Céleste* quoted by Bountis & Helleman in *Lecture Notes in Physics — Volta Memorial Conference, Como, 1977* (Springer, 1979)

- Periodic orbits are simpler to work with than invariant tori and cantori with *irrational* rotation numbers $\omega_{\rm irrat.}$
- Alternative to KAM: Periodic orbits from a rotation number *sequence* $\omega_{p,q} = p/q \rightarrow \omega_{\rm irrat.}$, $p, q \in \mathbb{Z}$ (chosen by a continued fraction construction) can be used to characterize the breaking of invariant tori [J. Greene, J. Math. Phys. **20**, 1183 (1979)].

Pierre-Louis Moreau de Maupertuis 1698–1759

Action another key

William Rowan Hamilton 1805–1865

- Consider *p,q* periodic *configuration-space* path $\theta = \vartheta(t)$,
$$\vartheta(t + 2\pi q) = \vartheta(t) + 2\pi p \quad \forall\, t \in \mathbb{R}$$
Then Lagrangian action over one period is
$$S[\vartheta] = \int_0^{2\pi q} L(\vartheta, \vartheta', t)\, dt$$
- Hamiltonian action on *phase-space* path $\theta = \vartheta(t), I = \mathcal{I}(t)$
$\vartheta(t + 2\pi q) = \vartheta(t) + 2\pi p, \quad \mathcal{I}(t + 2\pi q) = \mathcal{I}(t) \quad \forall\, t \in \mathbb{R}$ is
$$S_{\mathrm{ph}}[\vartheta, \mathcal{I}] = \int_0^{2\pi q} [\mathcal{I}\vartheta' - H(\mathcal{I}, \vartheta, t)]\, dt$$
- Hamilton's principle for a true periodic *orbit* is $\delta S = 0 \,\forall\, \delta\vartheta$, or $\delta S_{\mathrm{ph}} = 0 \,\forall\, \delta\vartheta, \delta\mathcal{I}$. Euler–Lagrange equations give Lagrangian or Hamiltonian eqs. of motion

Action gradient

- Define functional inner product over periodic orbit:

$$\langle f, g \rangle \equiv \int_0^{2\pi q} f g \, dt$$

- Define gradients in path space as functional derivatives:

$$\delta S = \left\langle \delta\vartheta, \frac{\delta S}{\delta\theta} \right\rangle \qquad \delta S_{\mathrm{ph}} = \left\langle \delta\vartheta, \frac{\delta S_{\mathrm{ph}}}{\delta\theta} \right\rangle + \left\langle \delta\mathcal{I}, \frac{\delta S_{\mathrm{ph}}}{\delta I} \right\rangle$$

$$\frac{\delta S}{\delta\theta} = L_\theta - \frac{d}{dt} L_{\dot\theta} \qquad \frac{\delta S_{\mathrm{ph}}}{\delta\theta} = -\dot I - H_\theta, \quad \frac{\delta S_{\mathrm{ph}}}{\delta I} = \dot\theta - H_I$$

- On a *pseudo-orbit* we constrain: $\dot\theta - H_I(I, \theta, t) \equiv 0$,

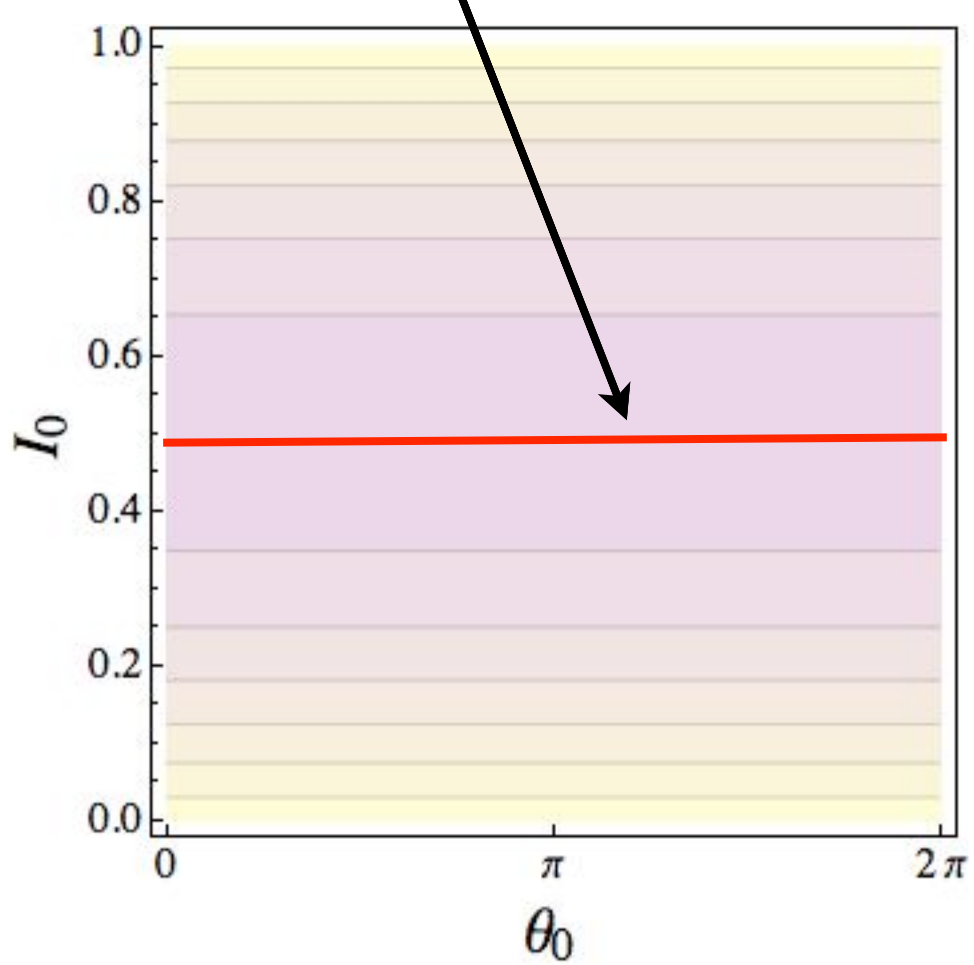
i.e. $\dfrac{\delta S_{\mathrm{ph}}}{\delta I} \equiv 0$, $\quad \Rightarrow \boxed{\dfrac{\delta S_{\mathrm{ph}}}{\delta\theta} = \dfrac{\delta S}{\delta\theta}}$ $= O(\epsilon)$

the *action gradient*; which can also be identified as a surface *flux density*

Hamilton's Principle (*not* "Least Action")

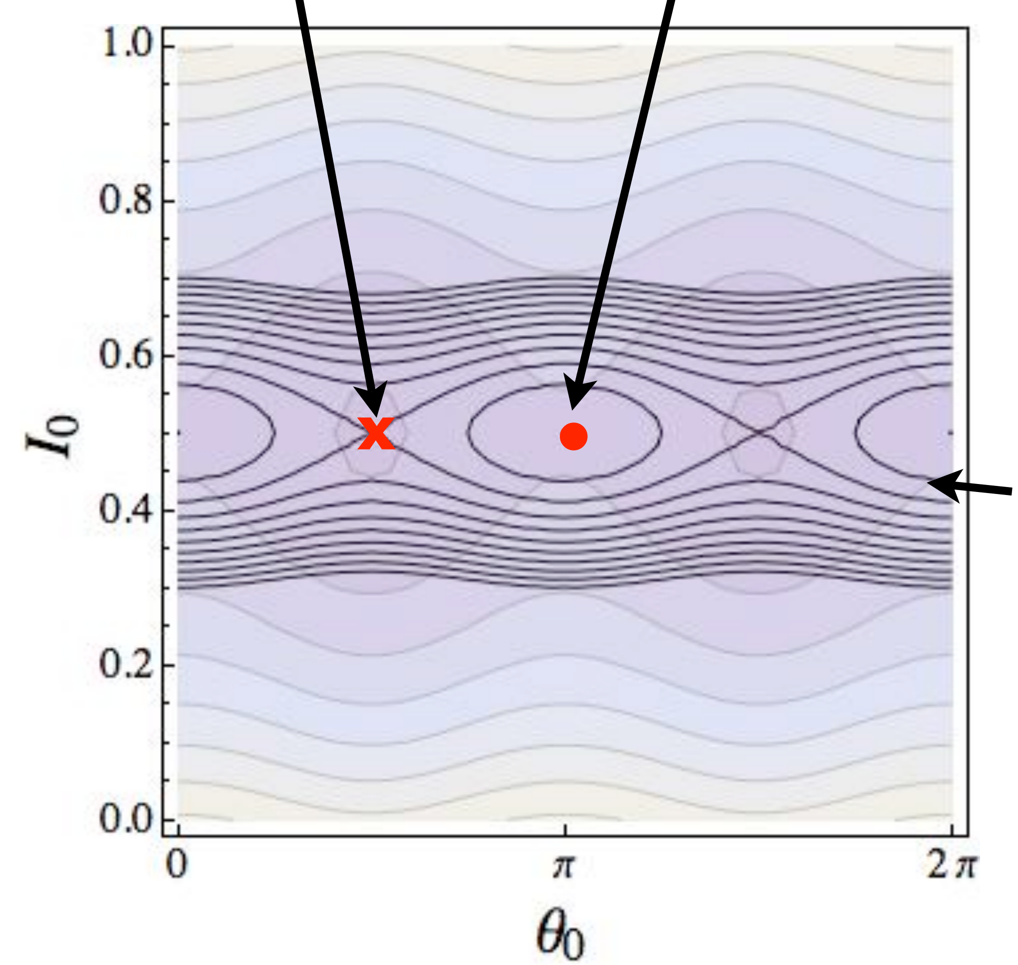
- Action contours — unperturbed case

Continuous family of 1,2-periodic orbits with *same* action, giving an *invariant torus*



- Perturbed (but integrable) case

Nearly all 1,2-periodic orbits destroyed, leaving only action-*minimizing* and *minimax* orbits



2ndry torus in island

Pseudo-orbits

- Define a configuration-space *pseudo-orbit* as a path satisfying the Lagrangian eq. of motion *approximately*:

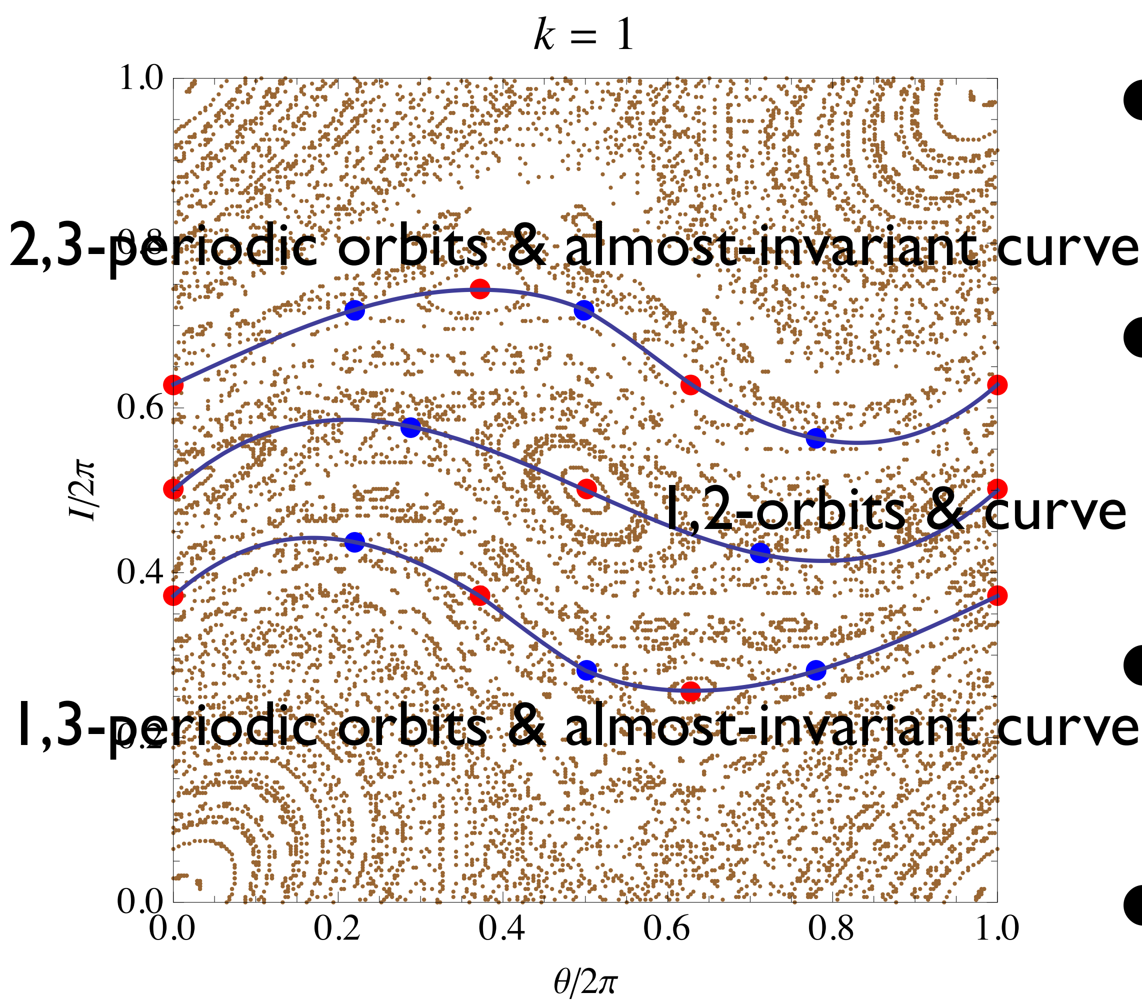
$$\frac{dL_{\dot{\theta}}}{dt} - L_\theta = O(\epsilon)$$

- For Lagrangian and Hamiltonian formulations to be equivalent we need to apply one Hamilton equation $\dot{\theta} - H_I(I, \theta, t) \equiv 0$ as a *constraint*. Thus, define a phase-space pseudo-orbit as a path satisfying the *other* Hamiltonian equation of motion approximately:

$$\dot{I} + H_\theta = O(\epsilon)$$

Minimizing and minimax orbits survive transition to chaos

by Poincaré-Birkhoff theorem [see e.g. Meiss, Rev. Mod. Phys. **64**, 795 (1992)]



Illustrated using Standard Map (see later)

- Blue dots are points on *p*,*q*-periodic orbits that *minimize* the action S
- Red dots are points on *p*,*q*-periodic orbits that are saddle (*minimax*) points of the action S
- Periodic orbits are *invariant* under the dynamics
- An *almost-invariant p,q curve* is an interpolation through the periodic orbits belonging to a *p*,*q* island chain — not unique: how to choose?

Two strategies for “joining the dots”

- *Ghost surfaces* are foliated by a *family* of pseudo-orbits constructed by action-gradient flow from minimax to minimising orbits:

$$\frac{\partial \vartheta_{\rm ghost}(t|\theta_0)}{\partial \theta_0} \propto -\frac{\delta S}{\delta \theta}$$

where we label pseudo-orbits by θ_0 s.t. $\vartheta(0|\theta_0) = \theta_0$

- *QFMin surfaces* minimise the *surface quadratic flux*:

$$\varphi_2 \equiv \frac{1}{2}\int_0^{2\pi}\!\!\int_0^{2\pi} \left(\frac{\delta S}{\delta \theta}\right)^2 d\theta dt$$

under variations of trial surface made up of a *one-parameter family* of QFMin pseudo-orbits, $\vartheta_{\rm QFMin}(t|\theta_0)$.

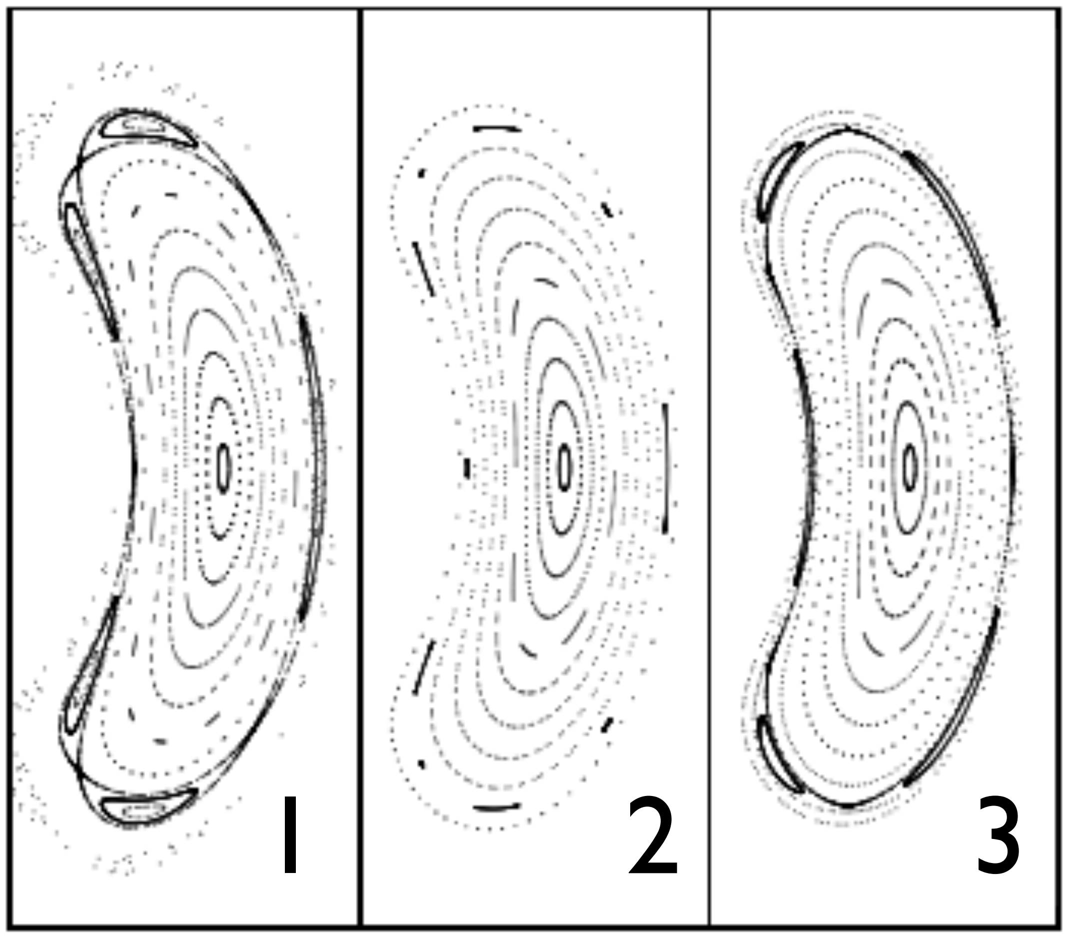
“QFMin Theorem”

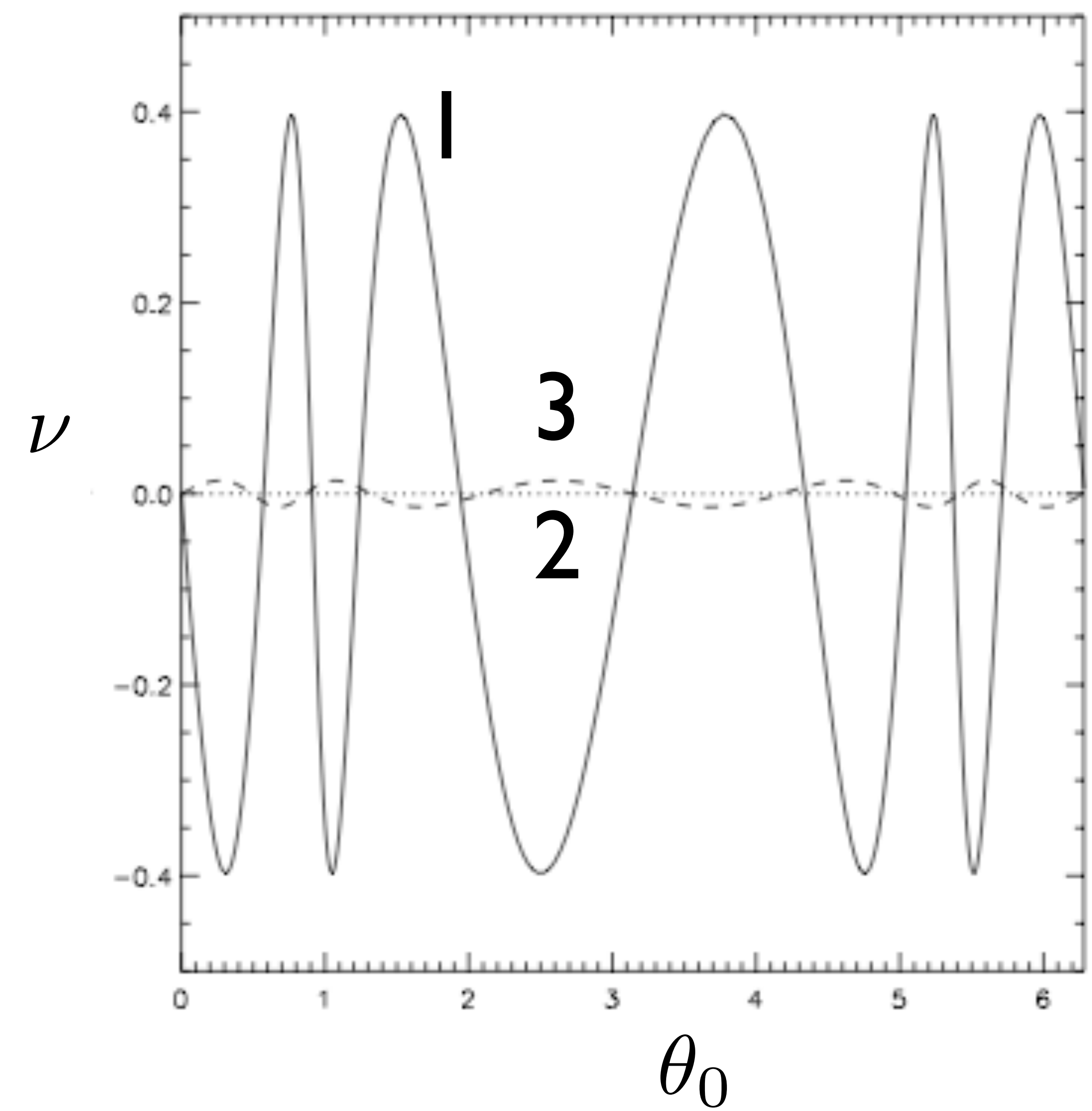
- Consider torus in 3-D phase space $\mathcal{T} :\ I = \rho(\theta, t)$
  Defines pseudo-orbit dynamics
  $$\dot{\vartheta} = H_I(\rho(\vartheta, t), \vartheta, t)$$
  $$\dot{I} = \rho_t + \dot{\vartheta}\rho_\theta$$
- Vary quadratic flux, using
  $$\delta\dot{\vartheta} = H_{II}\delta\rho$$
  $$\delta\dot{I} = \delta\rho_t + \dot{\vartheta}\delta\rho_\theta + \delta\dot{\vartheta}\rho_\theta$$
  $$\delta\frac{\delta S}{\delta\theta} = -\delta\dot{I} - H_{I\theta}\delta\rho$$
- Integrating by parts, and setting $\delta\varphi_2 = 0$ we find

$$\frac{d}{dt}\left(\frac{\delta S}{\delta\theta}\right) = 0 \quad \Rightarrow \boxed{\frac{\delta S}{\delta\theta} = \nu(\theta_0), \text{ const. on pseudo-orbit}}$$

This slight modification to Hamiltonian dynamics allows us to find a *family* of QFMin orbits defining $\mathcal{T}$.

H-1 Island-healing example





QFMin orbit constant of motion ν (action gradient) oscillates as initial poloidal angle θ_0 changes, passing through zero at the S-minimizing and minimax orbits. In case 2, ν is almost zero for all θ_0 and its oscillations are 180° out of phase in cases 1 and 3.

Action gradient is not coordinate-independent

- Change angle coordinate (for given *p*, *q* set)

$$\theta \equiv \theta(\Theta, t)$$

then *new* action gradient is

$$\frac{\delta \mathcal{S}}{\delta \Theta} = \theta_\Theta \frac{\delta \mathcal{S}}{\delta \theta}$$

- Define new ghost surfaces by the gradient flow

$$\frac{D\Theta}{DT} = -\frac{\delta \mathcal{S}}{\delta \Theta} \qquad (1)$$

where T is a parameter labelling each ghost pseudo-orbit

- Define new QFMin surfaces by

$$\frac{\delta \mathcal{S}}{\delta \Theta} = \nu(\Theta_0) \qquad (2)$$

where initial value Θ_0 labels each QFMin pseudo-orbit

Reconciliation of ghost & QFMin formulations

- Seek Θ such that the two families have a one-to-one correspondence, implying that the labels T and Θ_0 are functionally dependent: $T = T(\Theta_0)$
- Eliminating $\delta\mathcal{S}/\delta\Theta$ between (1) and (2), we find the *reconciliation condition* $\frac{D\Theta}{D\Theta_0} = -T'(\Theta_0)\nu(\Theta_0)$
- As Θ is so far completely undefined, we are free to choose it to simplify the reconciliation condition, subject to the *periodicity condition*:

$$\int_0^{2\pi} d\Theta_0 \, D\Theta/D\Theta_0 = 2\pi$$

Conclusion

- We have reviewed action-based formulations for almost-invariant tori in general Hamiltonian/Lagrangian dynamical systems
- Have unified mean-square flux minimization (QFMin) and ghost surface approaches, hopefully combining best features of each
- By defining both QFMin and ghost surfaces in terms of Θ we have found a formulation that is nicer than that of Hudson & Dewar Phys. Lett. A **373**, 4409 (2009)