Almost-Invariant Tori in the Hamiltonian Dynamics of 3-D Magnetic Fields

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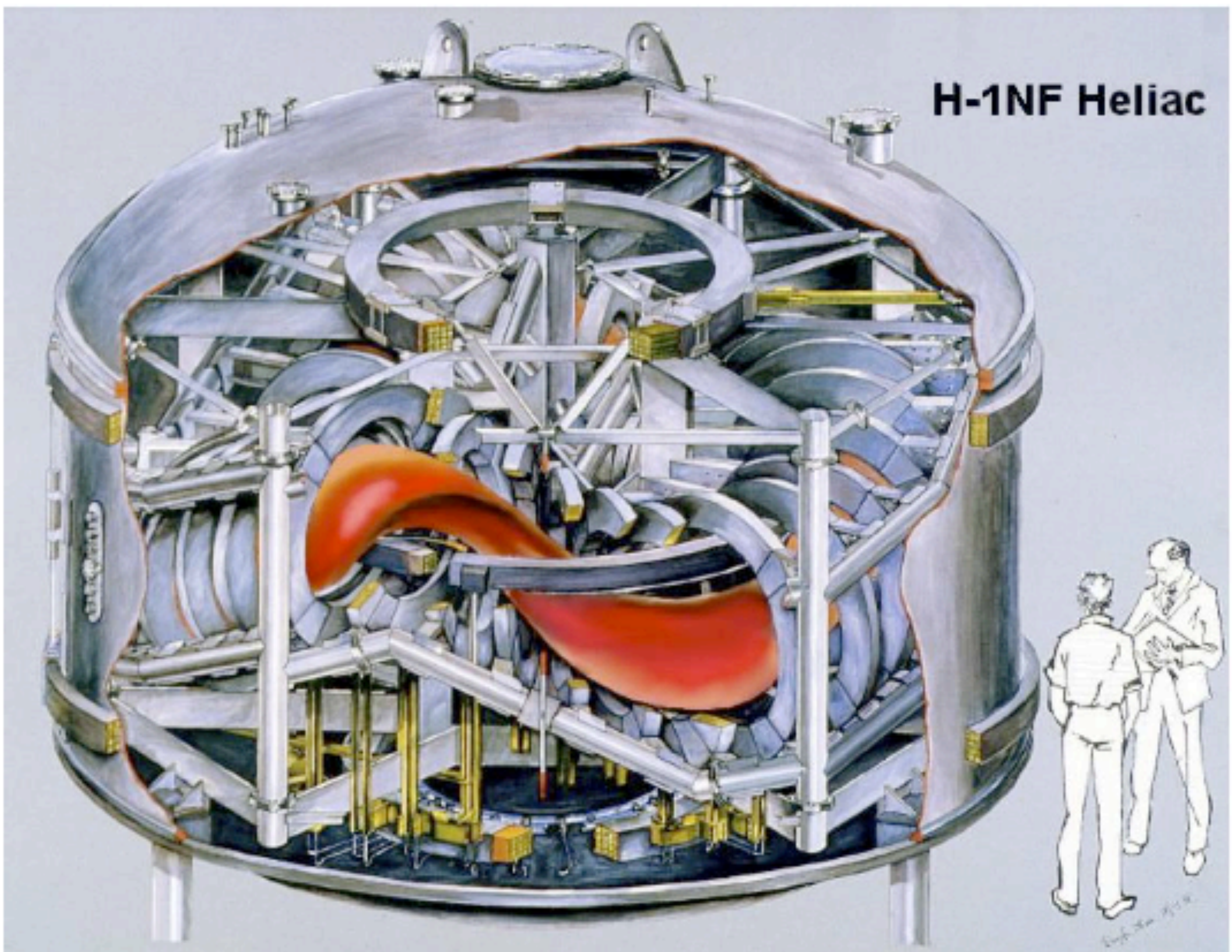
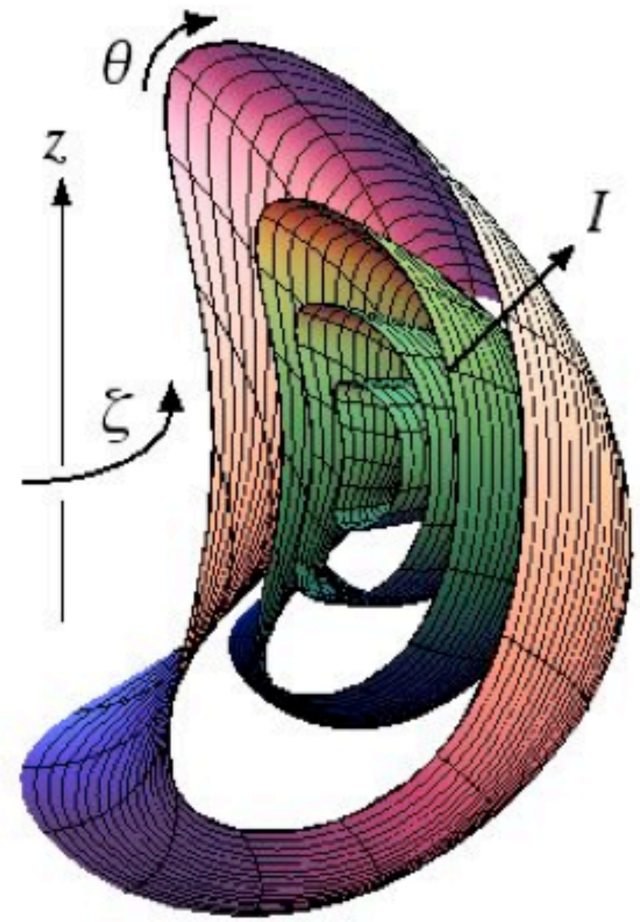
International Conference on Plasma Physics
ICPP-LAWPP Santiago, Chile August 2010

Content of talk

- Motivations:
  - magnetic coordinates when magnetic surfaces break
  - electron heat transport in chaotic magnetic fields
- Close-to-integrable 1½-d.o.f. systems
- Periodic pseudo-orbits as basis of approach
- Action minimization strategies for pseudo-orbits:
  - Ghost surfaces
  - Quadratic-Flux-Minimizing (QFMin) surfaces
- QFMin theorem
- Kicked Rotor model ⇒ area-preserving map ⇒ visualizations

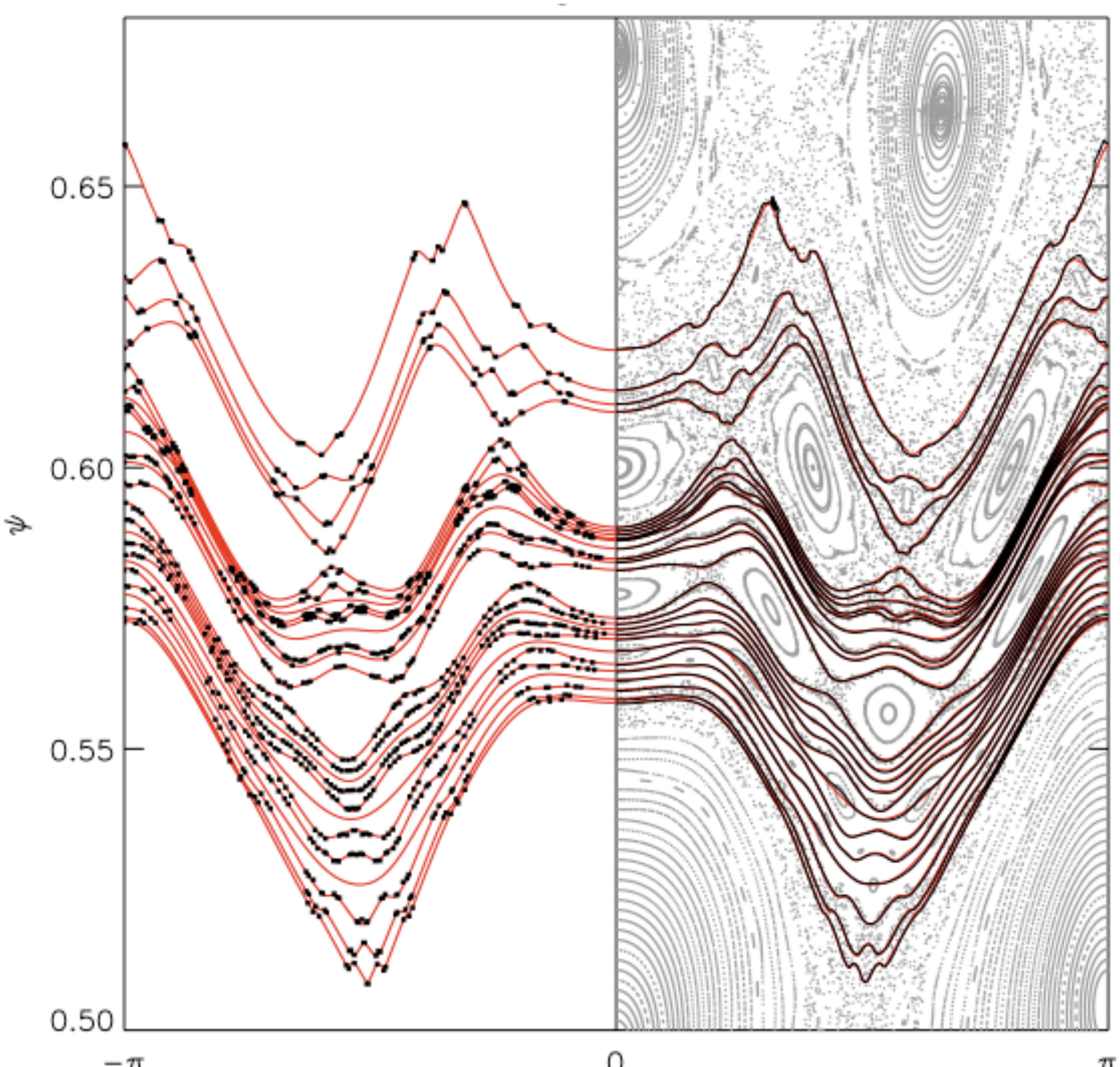
Coordinates for 3-D Magnetic fields





Equilibrium & stability (e.g. VMEC or SPEC) calculations in 3-D require magnetic coordinates. But how to define when good magnetic surfaces don't necessarily exist?

Almost-invariant tori act as barriers to heat diffusion in chaotic magnetic fields



Hudson & Breslau Phys Rev Letters **100**, 095001 (2008) show that *temperature contours* for heat diffusion in fields with imperfect magnetic surfaces appear to agree very well with "ghost surfaces"

Magnetic fields in 3D toroidal confinement systems are close-to-integrable 1½-d.o.f. Hamiltonian systems

- Consider non-autonomous, periodic-in-time system with Hamiltonian approximately in action-angle form

$$H = H_0(I,\theta) + \epsilon H_1(I,\theta,t)$$

Or corresponding Lagrangian

$$L \equiv I(\theta,\dot{\theta},t)\dot{\theta} - H(I,\theta,t)$$

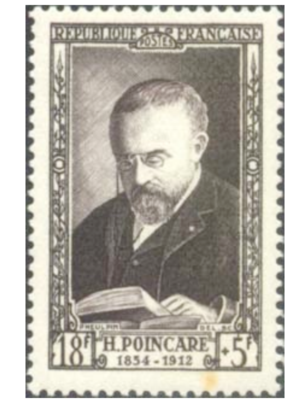
$$= L_0(\theta,\dot{\theta}) + \epsilon L_1(\theta,\dot{\theta},t)$$

where $I(\theta,\dot{\theta},t)$ is obtained by solving *one* of the Hamiltonian eqs. of motion *exactly*: $\dot{\theta} - H_I(I,\theta,t) \equiv 0$

- Define a *pseudo-orbit* as a path satisfying the other Hamiltonian eq. of motion *approximately*: $\dot{I} + H_\theta = O(\epsilon)$

Periodic orbits as a key to chaos

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissions essayer de pénétrer dans une place jusqu'ici reputée inabordable."



H. Poincaré: *Les Méthodes Nouvelles de la Mécanique Céleste* quoted by Bountis & Helleman in *Lecture Notes in Physics — Volta Memorial Conference, Como, 1977* (Springer, 1979)

- Periodic orbits are simpler to work with than KAM tori and cantori with irrational rotation numbers $\omega_{\rm irrat.}$
- Per. orbits with rot. no. sequence $\omega_{p,q} = p/q \to \omega_{\rm irrat.}$, $p, q \in \mathbb{Z}$ chosen by a continued fraction construction, can be used to determine the transition from invariant torus to cantorus [Greene J. Math. Phys. **20**, 1183 (1979)].

Action the other key

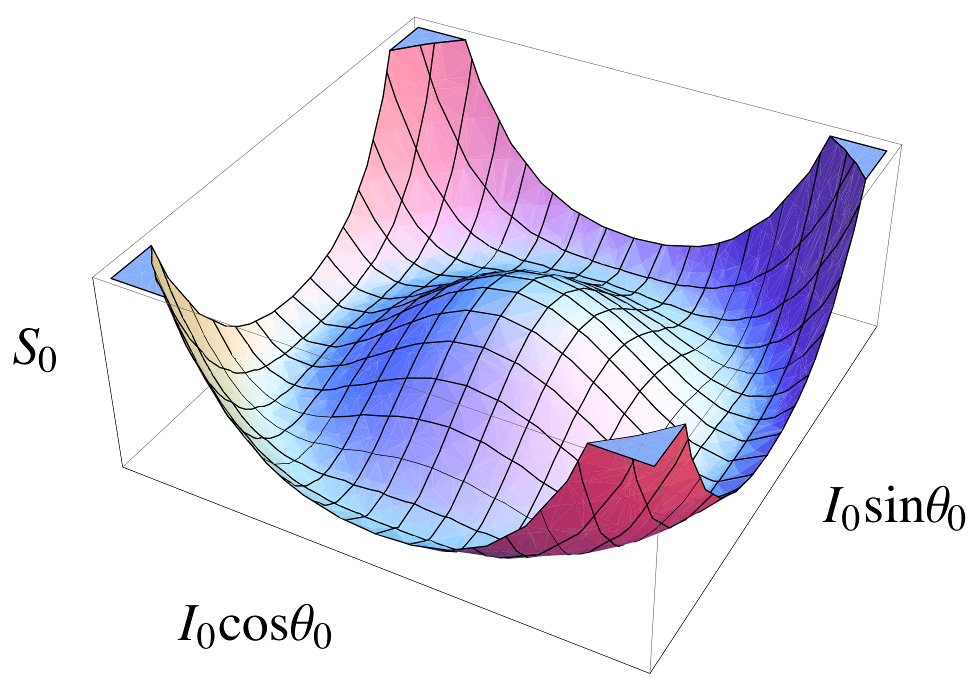
Pierre-Louis Moreau de Maupertuis 1698–1759

William Rowan Hamilton 1805–1865

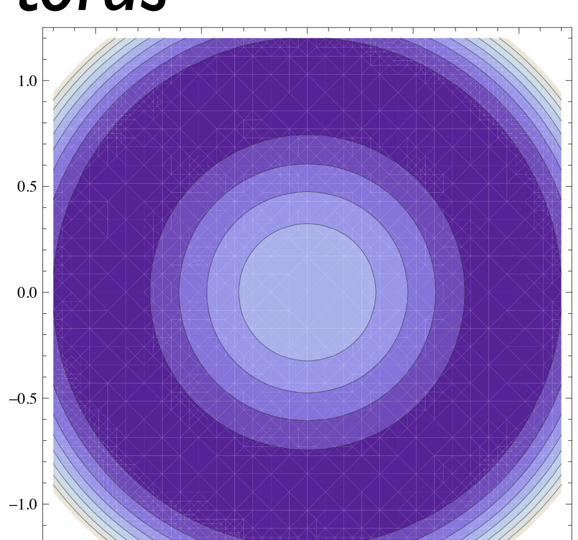
- Consider periodic pseudo-orbit $\theta = \vartheta(t)$, then Lagrangian (configuration space) action over 1 period is
$$S[\vartheta] = \int_0^{2\pi q} L(\theta, \dot{\theta}, t)\, dt$$
- Hamiltonian action on phase-space path $\theta = \vartheta(t), I = \mathcal{I}(t)$ is
$$S_{\rm ph}[\vartheta, \mathcal{I}] = \int_0^{2\pi q} [I\dot{\theta} - H(I, \theta, t)]\, dt$$
- Hamilton's principle for a true periodic *orbit* is $\delta S = 0 \,\forall\, \delta\vartheta$, or $\delta S_{\rm ph} = 0 \,\forall\, \delta\vartheta, \delta\mathcal{I}$, giving *both* Hamilton equations of motion as Euler–Lagrange equations.

Action minimizing & minimax orbits (schematic)

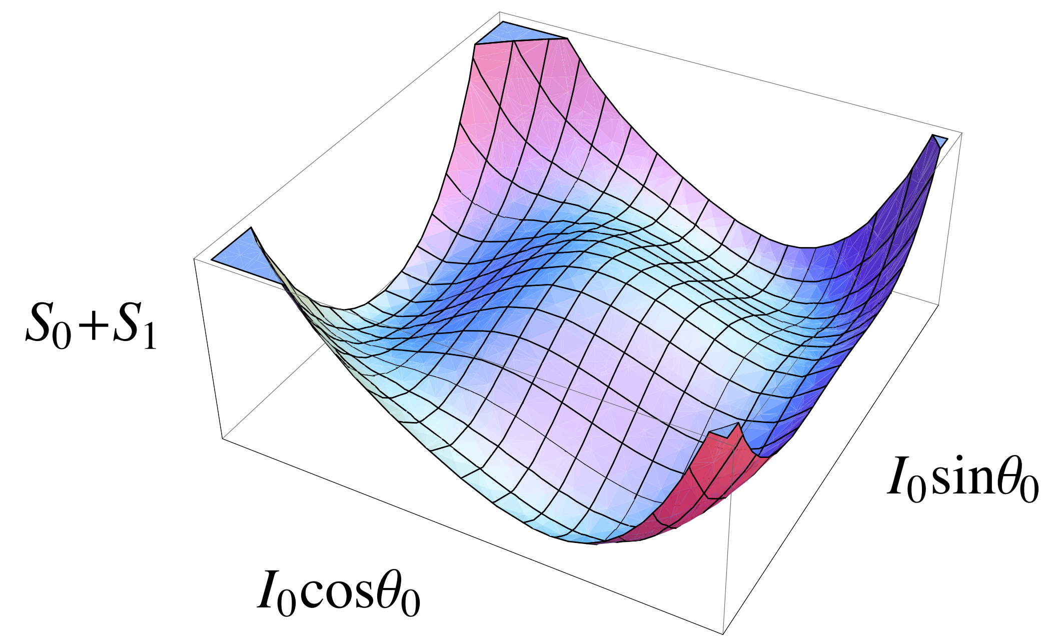
- Integrable case



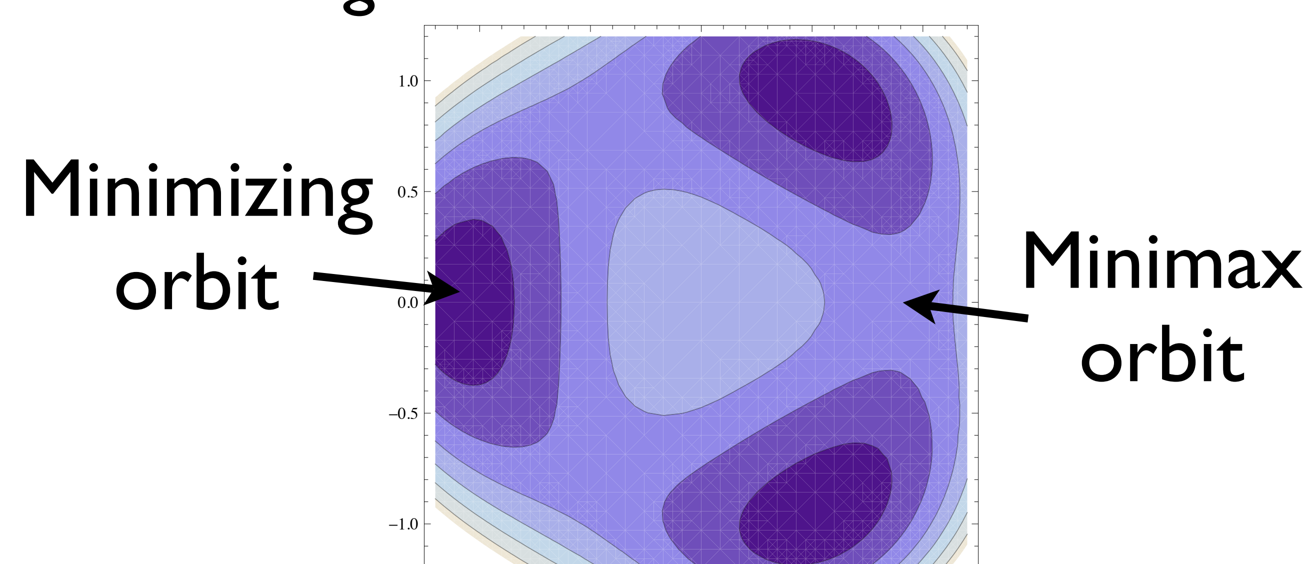
Continuous family of *p,q*-periodic orbits with *same* action, giving an *invariant torus*



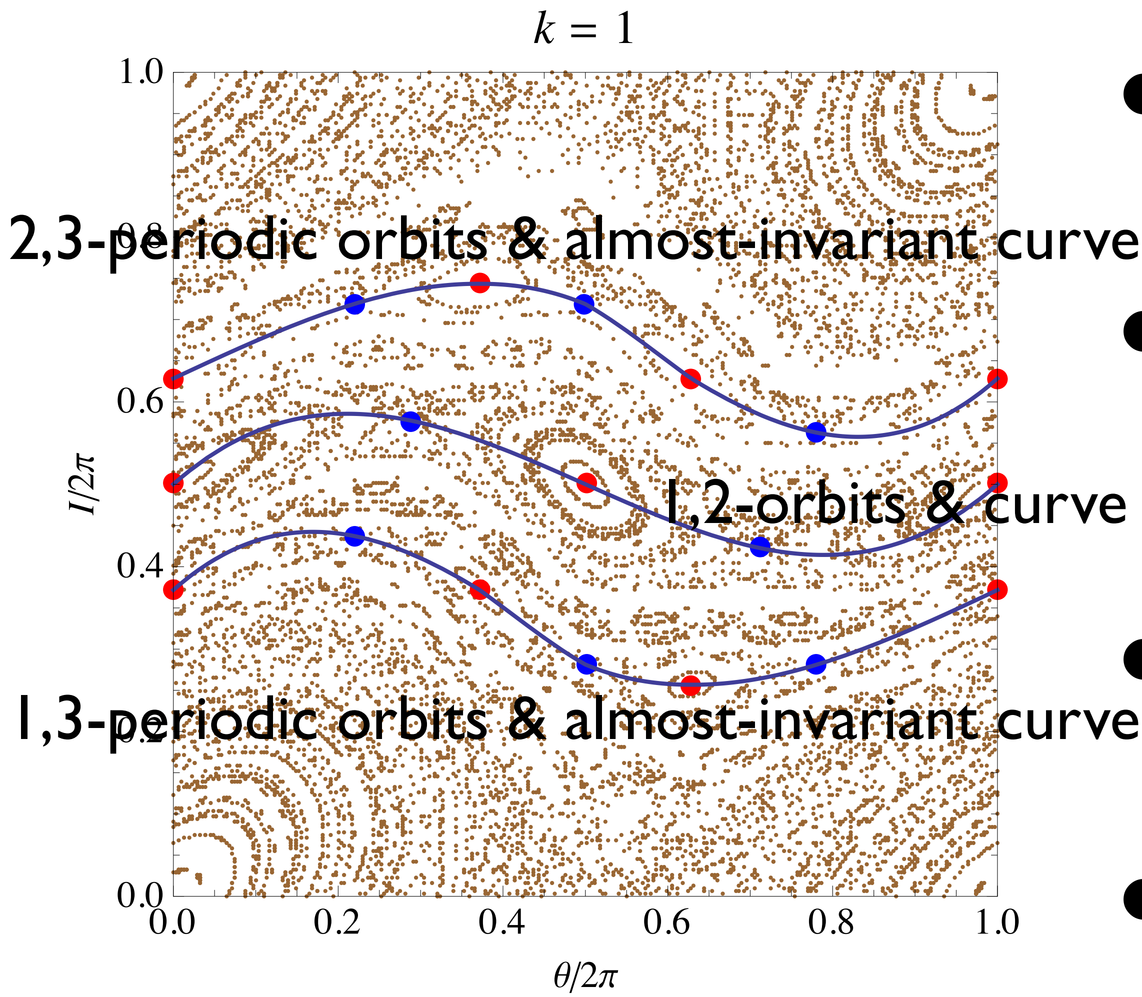
- Perturbed case



Nearly all *p,q*-periodic orbits destroyed, leaving only action-*minimizing* and *minimax* orbits



Minimizing and minimax orbits & almost-invariant surfaces



Illustrated using Standard Map (see later)

- Blue dots are *p*,*q*-periodic orbits that *minimize* the action S
- Red dots are *p*,*q*-periodic orbits that are saddle (*minimax*) points of the action S
- Periodic orbits are *invariant* under the dynamics
- An *almost-invariant p*,*q curve* is an interpolation through the periodic orbits belonging to a *p*,*q* island chain — not unique: how to choose?

Action gradients

- Define functional inner product over periodic orbit:

$$\langle f, g \rangle \equiv \int_0^{2\pi q} f g \, dt$$

- Define gradients in path space as functional derivatives:

$$\delta S = \left\langle \delta\vartheta, \frac{\delta S}{\delta \theta} \right\rangle \qquad \delta S_{\mathrm{ph}} = \left\langle \delta\vartheta, \frac{\delta S_{\mathrm{ph}}}{\delta \theta} \right\rangle + \left\langle \delta\mathcal{I}, \frac{\delta S_{\mathrm{ph}}}{\delta I} \right\rangle$$

$$\frac{\delta S}{\delta \theta} = L_\theta - \frac{d}{dt} L_{\dot{\theta}} \qquad \frac{\delta S_{\mathrm{ph}}}{\delta \theta} = -\dot{I} - H_\theta, \quad \frac{\delta S_{\mathrm{ph}}}{\delta I} = \dot{\theta} - H_I$$

- On a *pseudo-orbit* we constrain: $\dot{\theta} - H_I(I, \theta, t) \equiv 0$,

i.e. $\frac{\delta S_{\mathrm{ph}}}{\delta I} \equiv 0, \quad \Rightarrow \frac{\delta S_{\mathrm{ph}}}{\delta \theta} = \frac{\delta S}{\delta \theta}$ (Action gradient; also a surface flux density)

$= O(\epsilon)$

Strategies for "joining the dots"

- *Ghost surfaces* are foliated by a *family* of pseudo-orbits constructed by action-gradient flow from minimax to minimizing orbits:

$$\frac{\partial \vartheta_{\rm ghost}(t|\theta_0)}{\partial \theta_0} \propto -\frac{\delta S}{\delta \theta}$$

  where we label pseudo-orbits by θ_0 s.t. $\vartheta(0|\theta_0) = \theta_0$

- *QFMin surfaces* minimize the *quadratic flux*:

$$\varphi_2 \equiv \frac{1}{2}\int_0^{2\pi}\!\!\int_0^{2\pi} \left(\frac{\delta S}{\delta \theta}\right)^2 d\theta dt$$

  under variations of trial surface made up of family QFMin pseudo-orbits $\vartheta_{\rm QFMin}(t|\theta_0)$.

Action of a closed field line

Use vector potential representation $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$. *Action* is

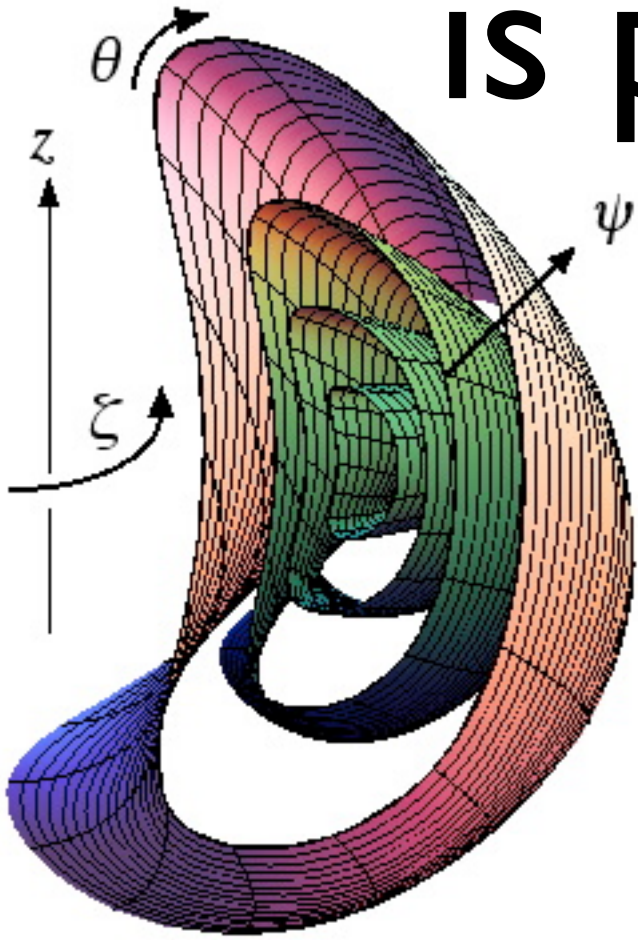
$$\mathcal{S}[\mathcal{C}] \equiv \int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{l} \equiv \int_0^{2\pi q} \mathbf{A}\cdot\dot{\mathbf{r}}\, d\zeta \ , \text{where}\ \dot{\mathbf{r}} \equiv d\mathbf{r}/d\zeta$$

where C is a periodic field line (orbit), closing on itself after making *p* poloidal rotations about the magnetic axis, and *q* toroidal rotations about *z* axis.

Equation of motion follows from *Hamilton's Principle* $\delta\mathcal{S}/\delta\mathbf{r} = \dot{\mathbf{r}}\times\mathbf{B} = 0 \Rightarrow \dot{\mathbf{r}} \parallel \mathbf{B}$.

Standard Hamiltonian form obtained from Clebsch representation $\mathbf{A} = \psi\boldsymbol{\nabla}\theta - \chi(\psi,\theta,\zeta)\boldsymbol{\nabla}\zeta$

In magnetic fields, action gradient is proportional to **n.B**



- Standard *linear* mag. flux through surface $\Gamma: \ \psi = \psi_\Gamma(\theta,\zeta)$ is

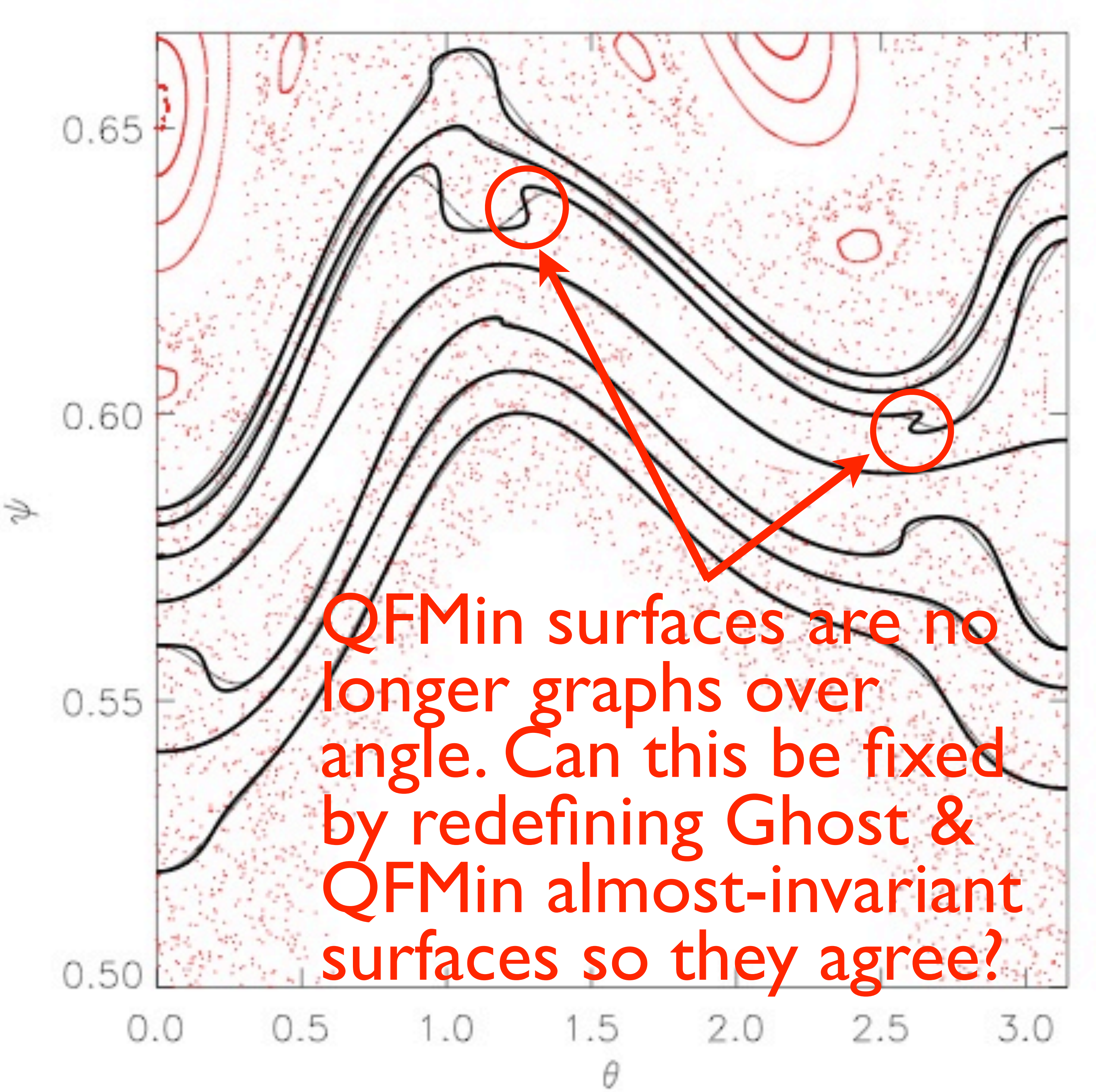
$$\varphi_1[\Gamma] \equiv \int_0^{2\pi}\int_0^{2\pi} d\theta d\zeta \, \frac{\mathbf{n \cdot B}}{\mathbf{n \cdot \nabla}\theta \times \mathbf{\nabla}\zeta} \equiv 0$$

- *Quadratic* flux through Γ is

$$\varphi_2[\Gamma] \equiv \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} d\theta d\zeta \, \frac{\mathbf{n \cdot B}}{\mathbf{n \cdot \nabla}\theta \times \mathbf{\nabla}\zeta} \frac{\mathbf{n \cdot B}}{\mathbf{n \cdot \nabla}\Theta \times \mathbf{\nabla}\zeta} \geq 0$$

Can *auxiliary poloidal angle* Θ be chosen so that quadratic-flux-minimizing (QFMin) surface Γ is also a *ghost surface*?

In strongly chaotic fields unreconciled ghost and QFMin surfaces differ



Hudson & Dewar Phys Letts A **373**, 4409 (2009) show that ghost surfaces and QFMin surfaces agree well for moderate nonlinearity.

But at strong nonlinearity they are clearly different.

"QFMin Theorem"

- Consider torus in 3-D phase space $\mathcal{T} : \ I = \rho(\theta, t)$

  Defines pseudo-orbit dynamics
$$\dot{\vartheta} = H_I(\rho(\vartheta, t), \vartheta, t)$$
$$\dot{I} = \rho_t + \dot{\vartheta}\rho_\theta$$

- Vary quadratic flux, using
$$\delta\dot{\vartheta} = H_{II}\delta\rho$$
$$\delta\dot{I} = \delta\rho_t + \dot{\vartheta}\delta\rho_\theta + \delta\dot{\vartheta}\rho_\theta$$
$$\delta\frac{\delta S}{\delta\theta} = -\delta\dot{I} - H_{I\theta}\delta\rho$$

- Integrating by parts, and setting $\delta\varphi_2 = 0$ we find

$$\frac{d}{dt}\left(\frac{\delta S}{\delta\theta}\right) = 0 \quad \Rightarrow \boxed{\frac{\delta S}{\delta\theta} = \nu(\theta_0), \text{ const. on pseudo-orbit}}$$

This slight modification to Hamiltonian dynamics allows us to find a family of QFMin orbits defining $\mathcal{T}$

Kicked-rotor model

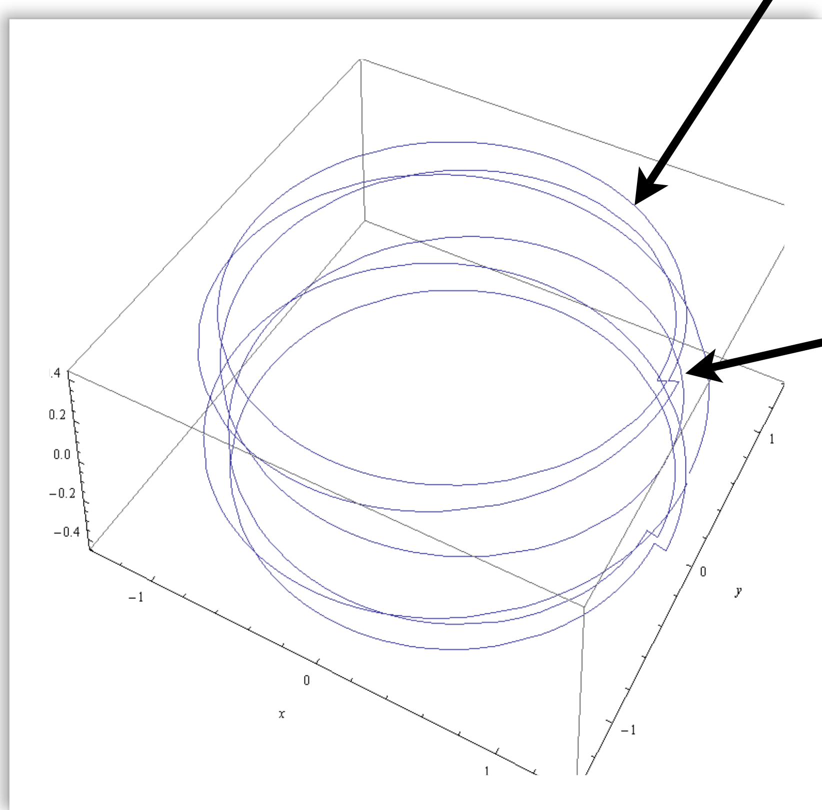
Assume $$H = \frac{1}{2}I^2 + \sum_{n=-\infty}^{\infty} \delta(t - t_n)V(\theta)$$

where $t_n \equiv 2\pi n$ are the times of the "kicks"

Solving QFMin eq. betw. kicks get piece-wise quadratic fn.

$$\vartheta\,(t) = -\frac{1}{2}\nu t^2 + \frac{1}{2\pi}\left[(t_{n+1} - t)\left(\theta_n + \frac{1}{2}\nu t_n^2\right) + (t - t_n)\left(\theta_{n+1} + \frac{1}{2}\nu t_{n+1}^2\right)\right]$$

$t_n < t < t_{n+1}$

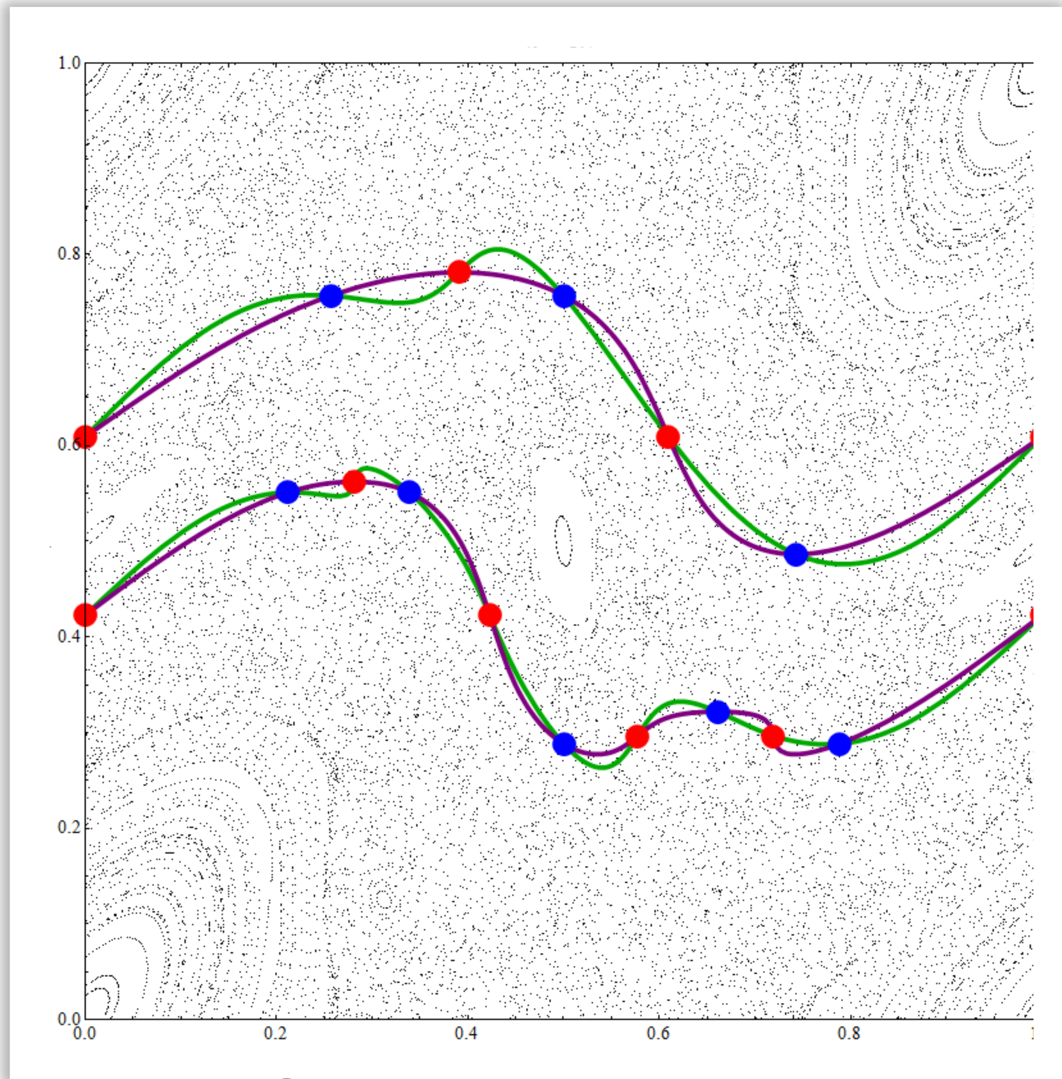


At kicks, ϑ is continuous, but $\dot{\vartheta}$ and $\mathcal{I}$ jump. Difference equation relating successive values of angles at kicks is:

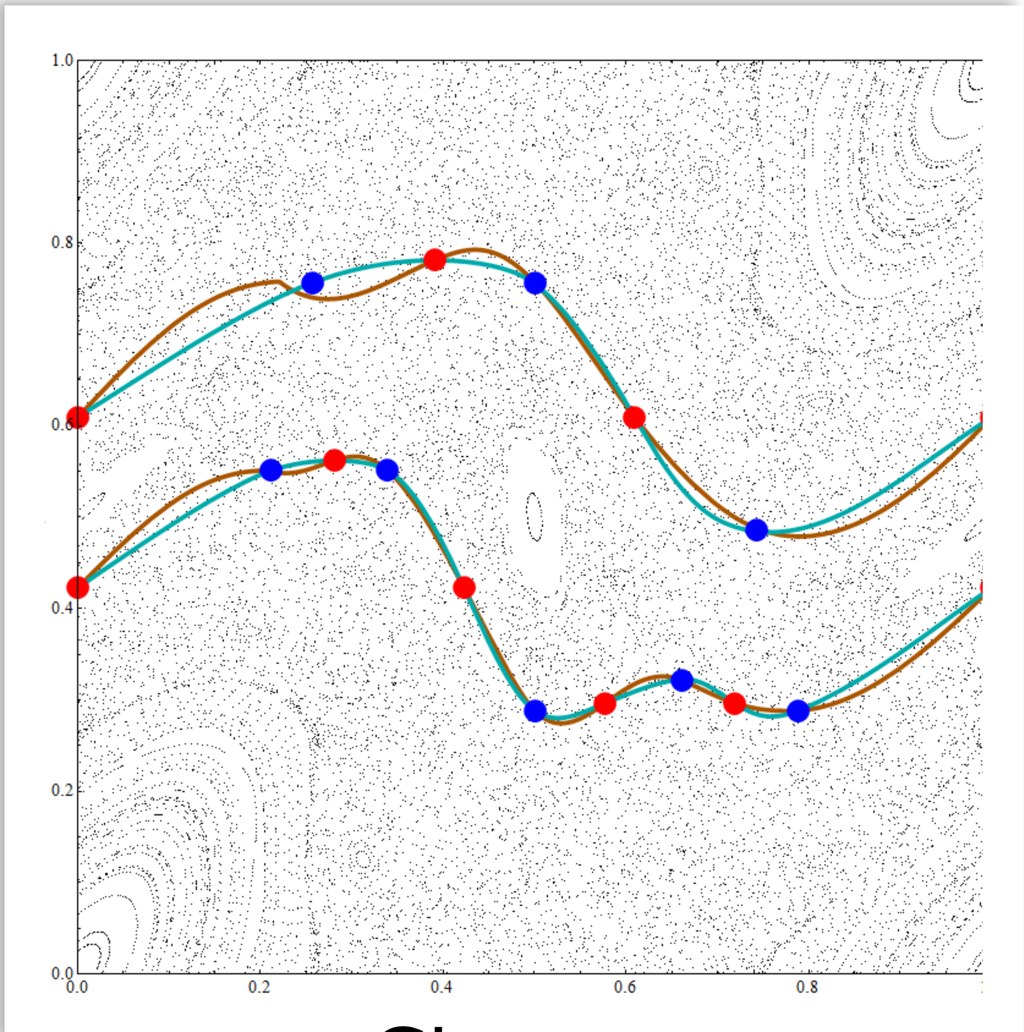
$$\theta_{n+1} - 2\theta_n + \theta_{n-1} + 2\pi V'(\theta_n) + (2\pi)^2\nu = 0$$

Ghost & QFmin curves for Standard Map

$$V(\theta) = -\frac{k}{(2\pi)^2}\cos\theta$$



QFMin curves



Ghost curves

Red/green curves images of each other — intersections invariant, periodic pts.
QFMin curves minimize vertical distance in least squares.

Conclusion

- We have given a formulation of QFMin and ghost tori for general Hamiltonian/Lagrangian dynamical systems
- Area-preserving maps appear naturally as a special case
- Mean-square flux minimization (QFMin) is a physically natural and computationally convenient way to define almost invariant tori, but until now its mathematical properties were not as good as ghost surfaces
- Currently studying unification of QFMin and ghost tori by coordinate transformation $\theta \mapsto \Theta$