Partially-relaxed, partially-constrained MHD equilibria

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→ The simplest model of approximating global, macroscopic force-balance in toroidal plasma confinement with arbitrary geometry is magnetohydrodynamics (MHD).

→ Non-axisymmetric magnetic fields generally ***do not*** have a nested family of smooth flux surfaces, ***unless*** ideal surface currents are allowed at the rational surfaces.

→ If the field is non-integrable (chaotic, fractal phase space), then any ***continuous*** pressure that satisfies $\mathbf{B}\cdot\nabla p=0$ must have an ***infinitely discontinuous gradient,*** ∇p.

→ Instead, solutions with stepped-pressure profiles are guaranteed to exist. A partially-relaxed, topologically-constrained, MHD energy principle is described.

→ Equilibrium solutions are calculated numerically. Results demonstrating convergence tests, benchmarks, and non-trivial solutions are presented.

→ The constraints of ideal MHD may be applied at the rational surfaces, in which case surface currents prevent the formation of islands. Or, these constraints may be relaxed in the vicinity of the rational surfaces, in which case magnetic islands will open if resonant perturbations are applied.

<u>An ideal equilibrium with non-integrable (*chaotic*) field and *continuous* pressure, is *infinitely discontinous*</u>

ideal MHD theory $= \nabla p = \mathbf{j}\times\mathbf{B}$, gives $\mathbf{B}\cdot\nabla p = 0$

→ transport of pressure along field is "infinitely" fast
→ no scale length in ideal MHD
→ pressure adapts to fractal structure of phase space

chaos theory = *nowhere are flux surfaces continuously nested*

*for non-symmetric systems, nested family of flux surfaces is destroyed

*islands & irregular field lines appear where transform is rational (n/m); rationals are dense in space

Poincare-Birkhoff theorem → periodic orbits, (e.g. stable and unstable) guaranteed to survive into chaos

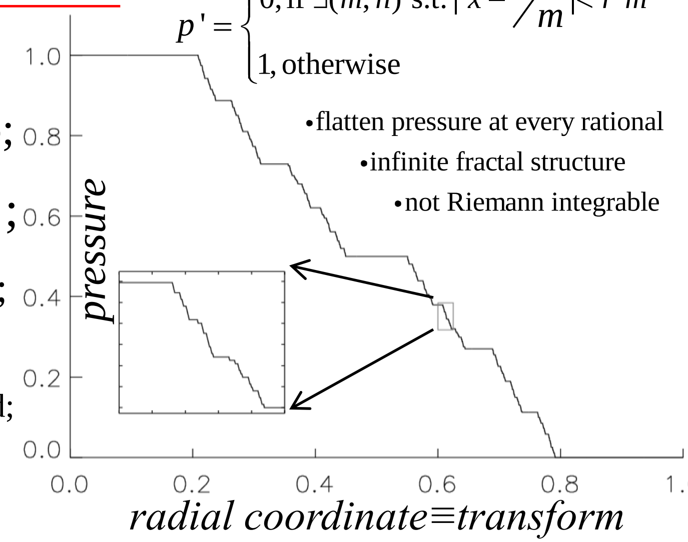
*some irrational surfaces survive if there exists an $r, k \in \Re$ s.t. for all rationals, $|\iota - n/m| > r\, m^{-k}$

i.e. rotational-transform, ι, is *poorly approximated* by rationals,

Diophantine Condition
Kolmogorov, Arnold and Moser

ideal MHD + chaos → infinitely discontinuous equilibrium

$$p' = \begin{cases} 0, \text{if } \exists(m,n) \text{ s.t.} \left| x - n/m \right| < r\, m^{-k} \\ 1, \text{otherwise} \end{cases}$$



*iterative method for calculating equilibria is ill-posed;

1) $\mathbf{B}_n \cdot \nabla p = 0$ ∇p is everywhere discontinuous, or zero;

2) $\mathbf{j}_\perp = \mathbf{B}_n \times \nabla p / B_n^2$ $\mathbf{j}_\perp$ everywhere discontinuous or zero;

3) $\mathbf{B}_n \cdot \nabla \sigma = -\nabla\cdot\mathbf{j}_\perp$ $\mathbf{B}\cdot\nabla$ is *densely and irregularly* singular;

σ is single valued if and only if $\delta\sigma = -\oint_C \nabla\cdot\mathbf{j}_\perp dl / B = 0$

pressure must be flat across every closed field line, or parallel current is not single-valued;

4) $\nabla\times\mathbf{B}_{n+1} = \mathbf{j} \equiv \sigma\mathbf{B}_n + \mathbf{j}_\perp$ solution only if $\nabla\cdot(\sigma\mathbf{B}+\mathbf{j}_\perp) = 0$

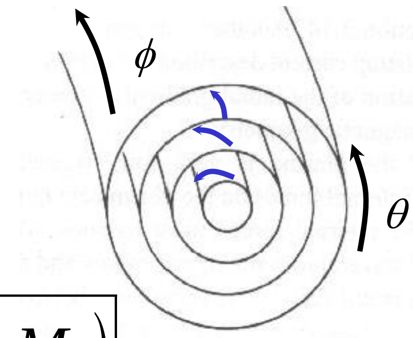
To have a well-posed equilibrium with chaotic **B** need to

→ introduce non-ideal terms, such as resistivity, η, perpendicular diffusion, $\kappa_\perp$, [*HINT*, *M3D*, *NIMROD*, ..],

→ or return to an energy principle, but relax infinity of ideal MHD constraints

Instead, a multi-region, relaxed energy principle for MHD equilibria with non-trivial pressure and chaotic fields

Energy, helicity and mass integrals (defined in nested annular volumes)



$$W_l = \underbrace{\int_{V_l} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{\text{energy}}, \quad H_l = \underbrace{\int_{V_l} (\mathbf{A} \cdot \mathbf{B})\, dv}_{\text{helicity}}, \quad M_l = \underbrace{\int_{V_l} p^{1/\gamma} dv}_{\text{mass}}$$

Seek constrained, minimum-energy state $\boxed{F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 - \nu_l M_l \right)}$

1st variation due to *unconstrained* variations δp, $\delta \mathbf{A}$, and interface geometry, ξ,

except ideal "topological" constraint $\delta \mathbf{B} = \nabla \times (\xi \times \mathbf{B})$ *imposed discretely* at interfaces

$$\delta F = \sum_{l=1}^{N} \left\{ \int_{V_l} \underbrace{\left(\frac{1}{\gamma - 1} - \frac{\nu_l p^{1/\gamma - 1}}{\gamma} \right)}_{\substack{\nu p^{1/\gamma} = \gamma p/(\gamma - 1) = const. \\ \text{in each annulus}}} \delta p \, dv + \underbrace{\int_{V_l} \delta \mathbf{A} \cdot (\nabla \times \mathbf{B} - \mu_l \mathbf{B})\, dv}_{\nabla \times \mathbf{B} = \mu_l \mathbf{B} \text{ in each annulus}} - \int_{\partial V_l} \underbrace{[[p + B^2/2]]}_{\substack{\text{continuity of total pressure} \\ \text{across interfaces}}} \xi . \mathbf{dS} \right\}$$

Equilibrium solutions when $\nabla \times \mathbf{B} = \mu_l \mathbf{B}$ in annuli, $[[p+B^2/2]]=0$ across interfaces

$\rightarrow$ partial *Taylor relaxation* allowed in each annulus; allows for topological variations/islands/chaos;

$\rightarrow$ global relaxation prevented by ideal constraints; $\rightarrow$ non-trivial *stepped – pressure* solutions;

$\rightarrow$ $\nabla \times \mathbf{B} = \mu_l \mathbf{B}$ is a linear equation for $\mathbf{B}$; depends on interface geometry; solved in parallel in each annulus;

$\rightarrow$ solving force balance $\equiv$ adjusting interface geometry to satisfy $[[p+B^2/2]]=0$;

ideal interfaces that support pressure generally have irrational rotational-transform;

standard numerical problem finding zero of multi-dimensional function; call NAG routine;

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

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PETER LAURENCE — *Universita di Roma "La Sapienza"*

We establish an existence result for the three-dimensional MHD equations

$$
\begin{aligned}
(\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p \\
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{B} \cdot n|_{\partial T} &= 0
\end{aligned}
$$

with $p \neq$ const in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

Communications on Pure and Applied Mathematics, Vol. XLIX, 717–764 (1996)

→ this was a strong motivation for pursuing the stepped-pressure equilibrium model

→ how large the "sufficiently small" departure from axisymmetry can be needs to be explored numerically

By definition, an equilibrium code must constrain topology;

Definition: Equilibrium Code (fixed boundary)

given (1) boundary (2) pressure (3) rotational-transform $\equiv$ inverse q-profile (or current profile)

$\rightarrow$ calculate **B** that is consistent with force-balance; pressure profile *is not changed*!

c.f. "*coupled equilibrium-transport*" approach, that evolves pressure while evolving field

Cannot apriori specify pressure without apriori constraining topology of the field

$\rightarrow$ the constraint $\mathbf{B}\cdot\nabla p=0$ means the structure of **B** and p are intimately connected;

if p is given and **B** that satisfies force balance is to be constructed,
then flux surfaces must coincide with pressure gradients; (e.g. if p is smooth, **B** must have nested surfaces).

$\rightarrow$ specifying the profiles discretely is a practical means of retaining *some* control over the profiles, whilst making minimal assumptions regarding the topology;

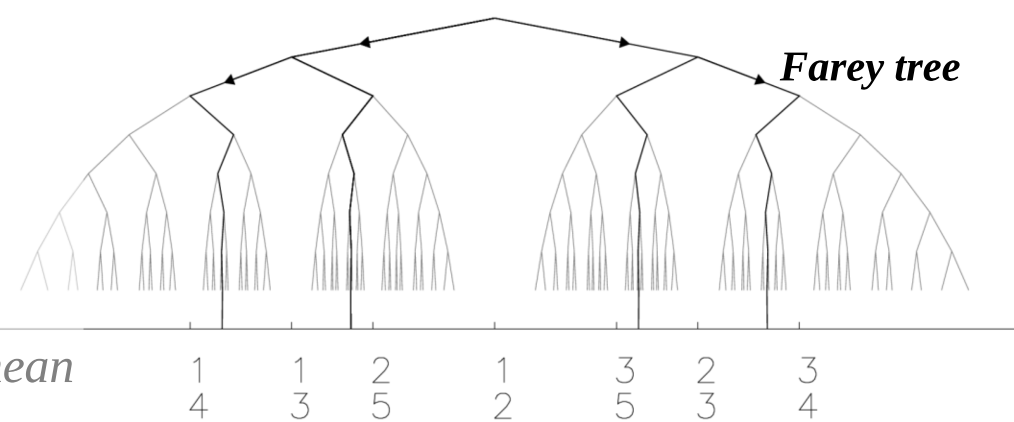
$\rightarrow$ pressure gradients are assumed to coincide with a set of strongly-irrational $\equiv$ "*noble*" flux surfaces

noble irrational

$\equiv$ limit of alternating path down Farey-tree

$\equiv$ Fibonacci sequence

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_1+p_2}{q_1+q_2}, \ldots \rightarrow \frac{p_1+\gamma p_2}{q_1+\gamma q_2}, \quad \gamma = \textit{golden mean}$$



Extrema of energy functional obtained numerically; introducing the Stepped Pressure Equilibrium Code (SPEC)

The vector-potential is discretized

* toroidal coordinates (s,ϑ,ζ), *interface geometry $R_l = \sum_{m,n} R_{l,m,n} \cos(m\vartheta - n\zeta), Z_l = \sum_{m,n} Z_{l,m,n} \sin(m\vartheta - n\zeta)$
* exploit gauge freedom $\mathbf{A} = A_\vartheta(s,\vartheta,\zeta)\nabla\vartheta + A_\zeta(s,\vartheta,\zeta)\nabla\zeta$
* Fourier $A_\vartheta = \sum_{m,n} a_\vartheta(s)\cos(m\vartheta - n\zeta)$
* Finite-element $a_\vartheta(s) = \sum_i a_{\vartheta,i}(s)\varphi(s)$ ***piecewise cubic or quintic basis polynomials***

and inserted into constrained-energy functional $F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 - \nu_l M_l\right)$

* derivatives w.r.t. vector-potential $\rightarrow$ linear equation for Beltrami field $\nabla \times \mathbf{B} = \mu\mathbf{B}$ ***solved using sparse linear solver***
* field in each annulus computed independently, distributed across multiple cpus
* field in each annulus depends on enclosed toroidal flux (boundary condition) and

$\rightarrow$ poloidal flux, ψ_P, and helicity-multiplier, μ ***adjusted so interface transform is strongly irrational***

$\rightarrow$ geometry of interfaces, $\xi \equiv \{R_{m,n}, Z_{m,n}\}$

Force balance solved using multi-dimensional Newton method.

* interface geometry is adjusted to satisfy force $\mathbf{F}[\xi] \equiv \{[[p + B^2/2]]_{m,n}\} = 0$
* angle freedom constrained by spectral-condensation, adjust angle freedom to minimize $\sum (m^2 + n^2)\left(R_{mn}^2 + Z_{mn}^2\right)$
* derivative matrix, $\nabla\mathbf{F}[\xi]$, computed in parallel using finite-differences ***minimal spectral width [Hirshman, VMEC]***
* call NAG routine: quadratic-convergence w.r.t. Newton iterations; robust convex-gradient method;

Numerical error in Beltrami field scales as expected

Scaling of numerical error with radial resolution depends on finite-element basis

$$\mathbf{A}=A_\vartheta \nabla\vartheta + A_\zeta \nabla\zeta, \quad \mathbf{B}=\nabla\times\mathbf{A}, \quad \mathbf{j}=\nabla\times\mathbf{B},$$

need to quantify **error** = **j** - μ**B**

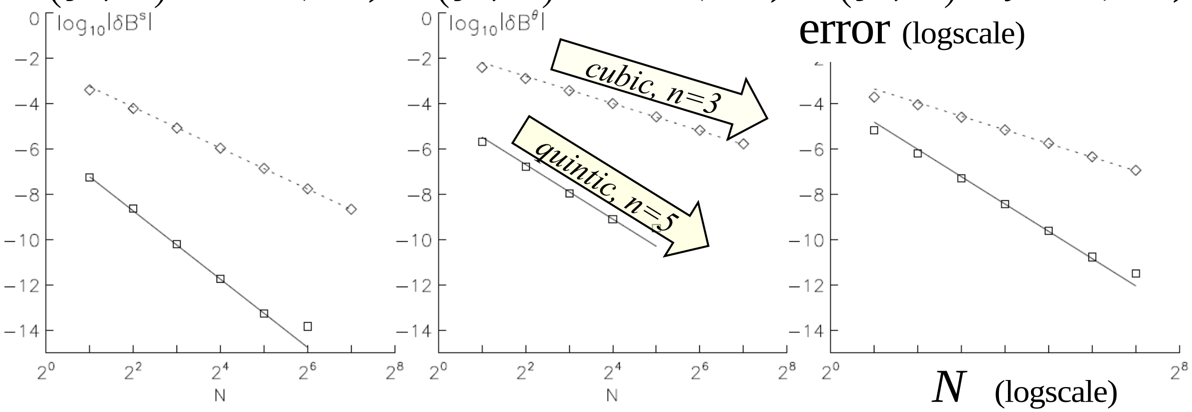
$$A_\vartheta, A_\zeta \sim O(h^n)$$

h = radial grid size = 1 / N
n = order of polynomial

$$\sqrt{g}B^s = \partial_\vartheta A_\zeta - \partial_\zeta A_\vartheta \sim O(h^n)$$
$$\sqrt{g}B^\vartheta = \qquad -\partial_s A_\zeta \sim O(h^{n-1})$$
$$\sqrt{g}B^\zeta = \partial_s A_\vartheta \qquad \sim O(h^{n-1})$$

$$\sqrt{g}\,j^s \sim O(h^{n-1})$$
$$\sqrt{g}\,j^\vartheta \sim O(h^{n-2})$$
$$\sqrt{g}\,j^\zeta \sim O(h^{n-2})$$

$(\mathbf{j}-\mu\mathbf{B})\bullet\nabla s \sim O(h^{n-1})$ $\quad(\mathbf{j}-\mu\mathbf{B})\bullet\nabla\vartheta \sim O(h^{n-2})$ $\quad(\mathbf{j}-\mu\mathbf{B})\bullet\nabla\zeta \sim O(h^{n-2})$



Example of chaotic Beltrami field in single given annulus;

$$R = 1.0 + r(\vartheta,\zeta)\cos\vartheta,$$
$$Z = \qquad r(\vartheta,\zeta)\sin\vartheta,$$

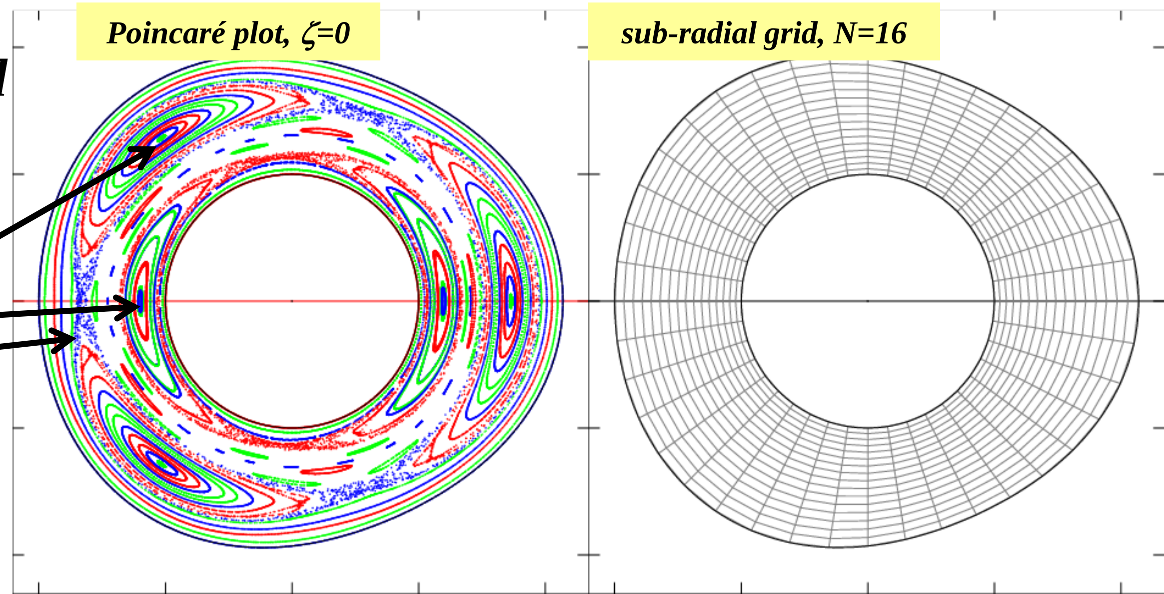
(m,n)=(3,1) island
+ (m,n)=(2,1) island
= chaos

inner surface

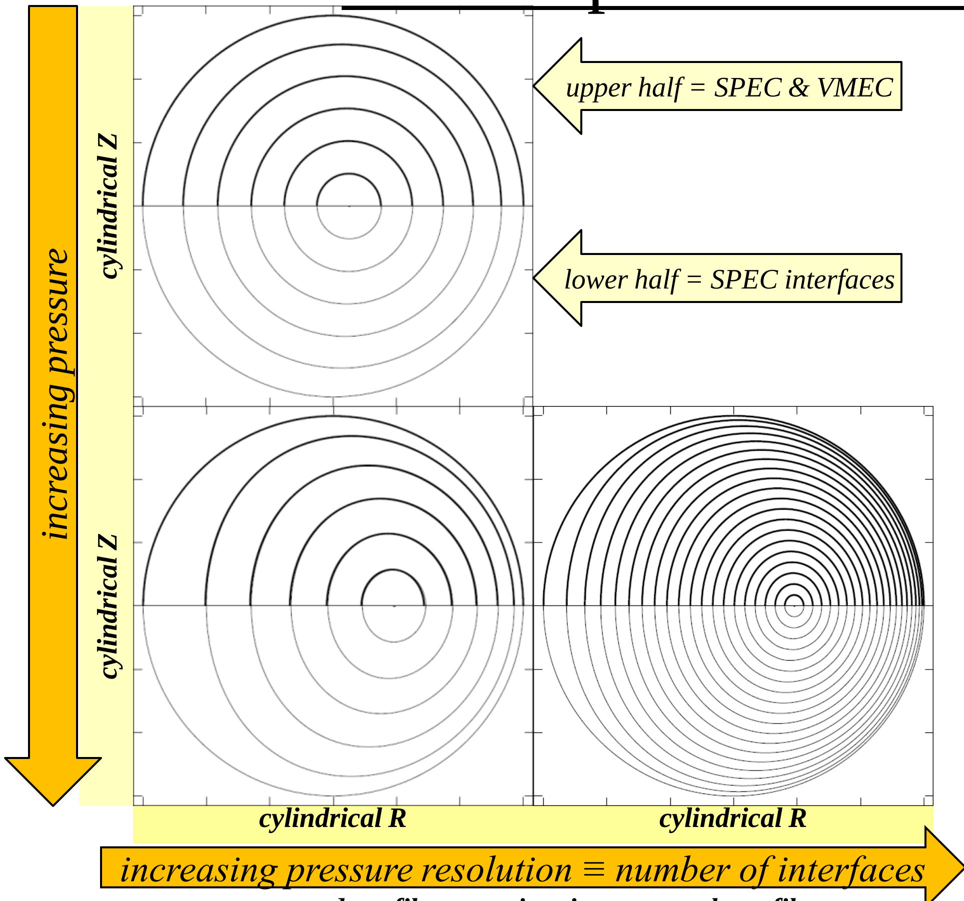
$r = 0.1$

outer interface

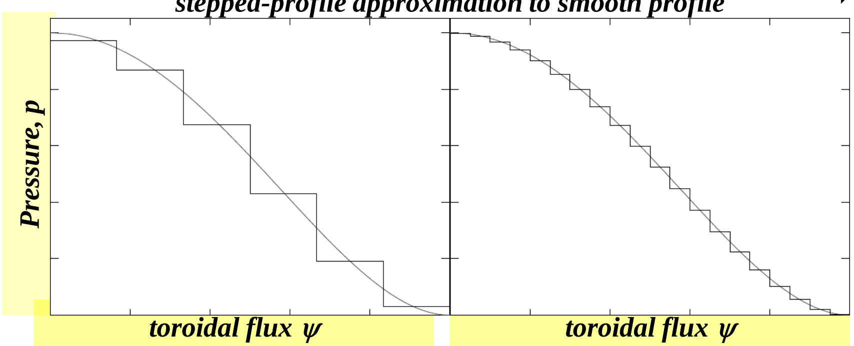
$r = 0.2 \ +\delta\left[\cos(2\vartheta-\zeta)+\cos(3\vartheta-\zeta)\right]$



Stepped-pressure equilibria accurately approximate smooth-pressure *axisymmetric* equilibria



stepped-profile approximation to smooth profile



in axisymmetric geometry . . .

→ magnetic fields have family of nested flux surfaces

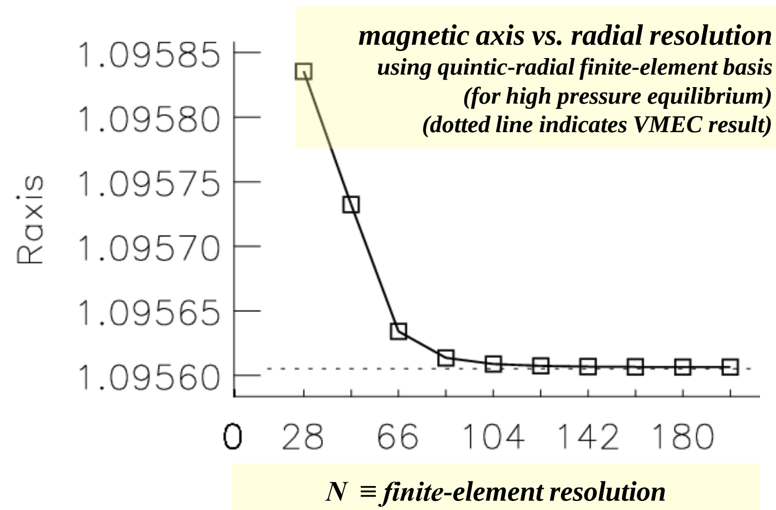
→ equilibria with smooth profiles exist,

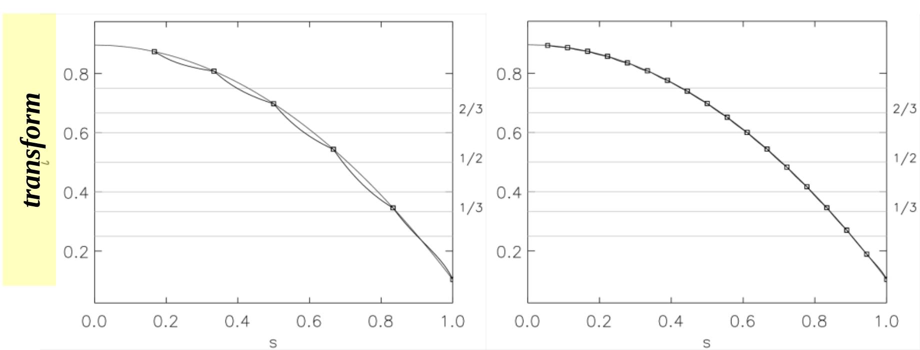
→ may perform benchmarks (e.g. with VMEC)

(arbitrarily approximate smooth-profile with stepped-profile)

→ approximation improves as number of interfaces increases

→ location of magnetic axis converges w.r.t radial resolution

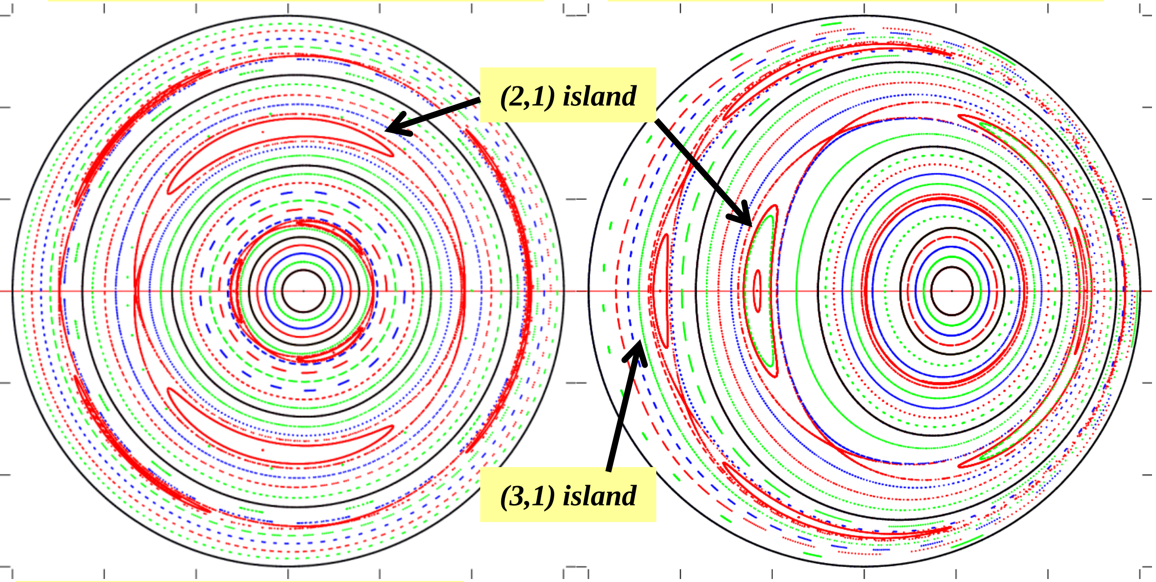




Equilibria with (i) perturbed boundary & chaotic fields, and (ii) pressure are computed.

Poincaré plot (cylindrical)
β = 0%

Poincaré plot (cylindrical)
$\beta \approx 4\%$



boundary deformation induces islands

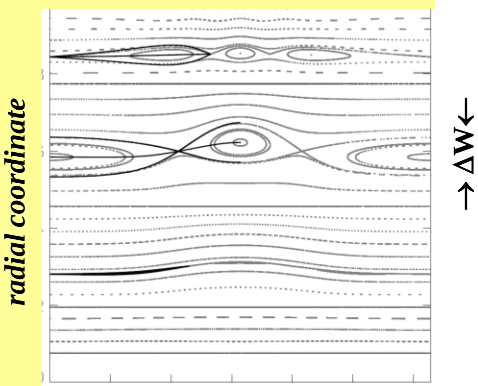
$$R = 1.0 + r\cos\vartheta,\ Z = r\sin\vartheta$$

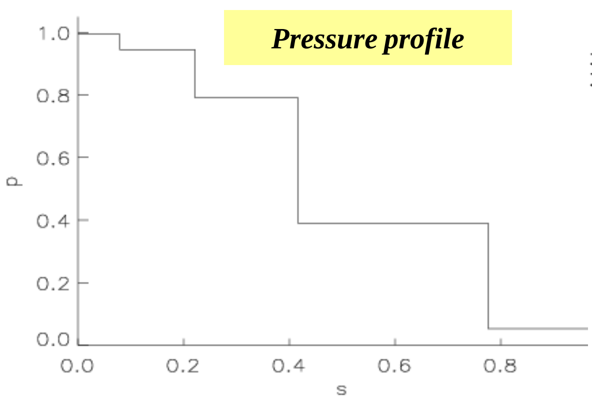
$$r = 0.3 + \delta\cos(2\vartheta - \phi) + \delta\cos(3\vartheta - \phi)$$

$$\delta = 10^{-4}$$

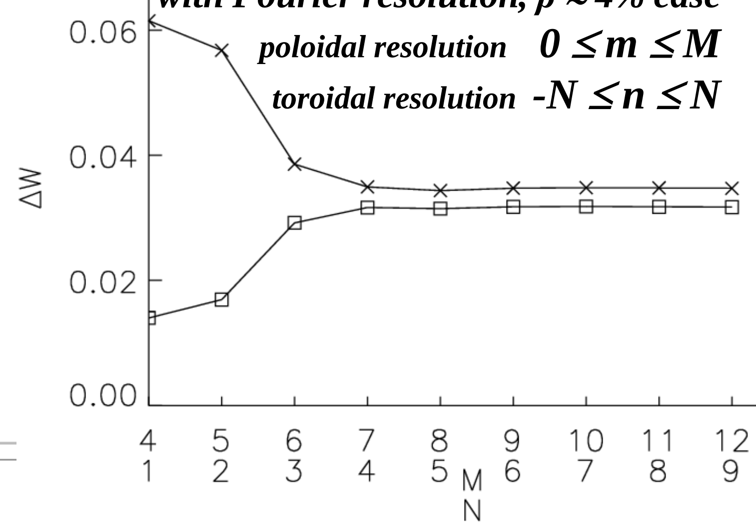
Demonstrated Convergence
of high-pressure equilibrium with islands,
with Fourier Resolution,

Poincaré plot (toroidal)
$\beta \approx 4\%$





Convergence of (2,1) & (3,1) island widths ..
with Fourier resolution, $\beta \approx 4\%$ case

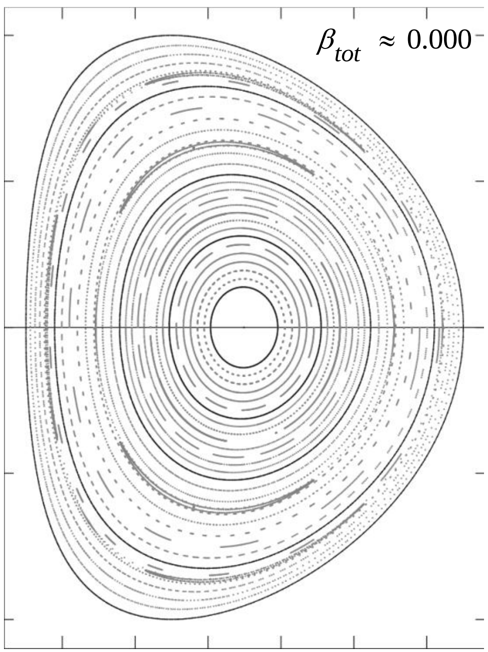


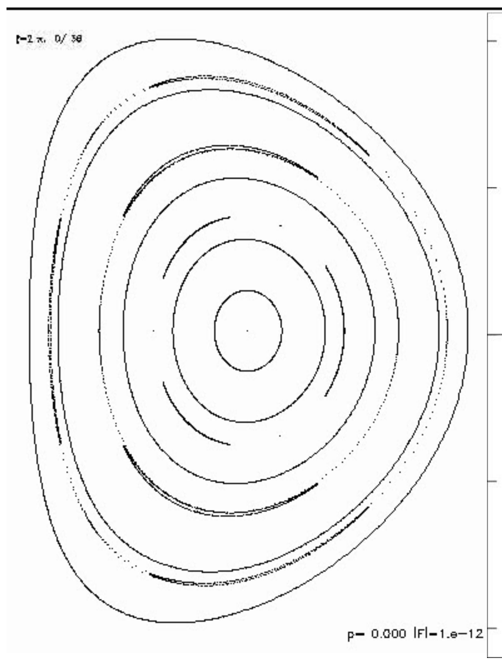
Sequence of equilibria with increasing pressure shows plasma *can* have significant response to external perturbation.

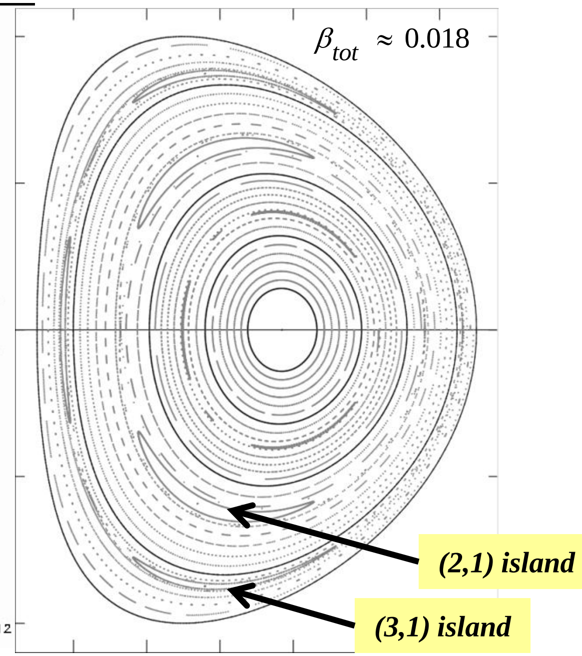
axisymmetric | plus small perturbation

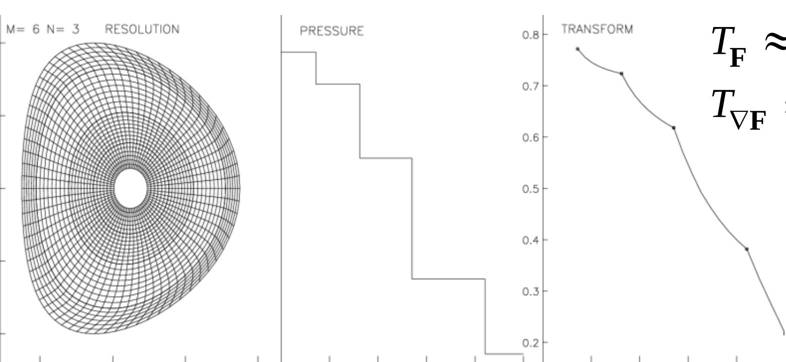
$$R = 1.00 + 0.30\cos(\vartheta) + 0.05\cos(2\vartheta) \quad + \quad [\delta_{21}\cos(2\vartheta-\zeta) + \delta_{31}\cos(3\vartheta-\zeta)]\cos(\vartheta)$$

$$Z = 1.00 + 0.40\sin(\vartheta) + \quad [\delta_{21}\cos(2\vartheta-\zeta) + \delta_{31}\cos(3\vartheta-\zeta)]\sin(\vartheta)$$





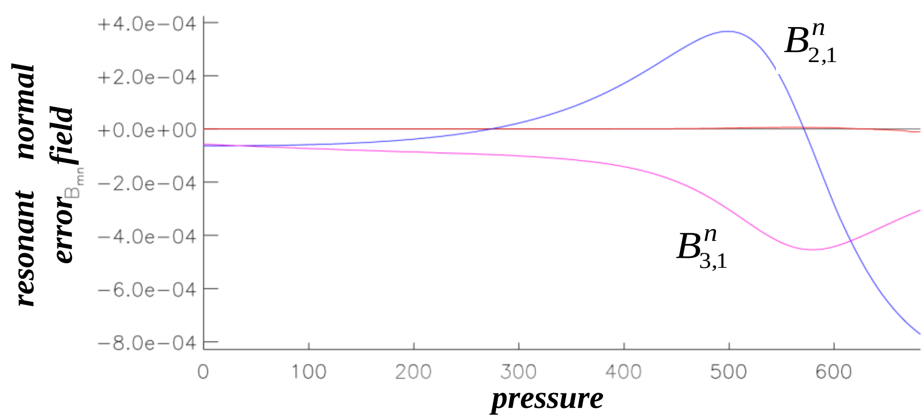




$T_{VMEC} \approx 100s$

$T_{\mathbf{F}} \approx 30s$

$T_{\nabla\mathbf{F}} \approx 1.5h$

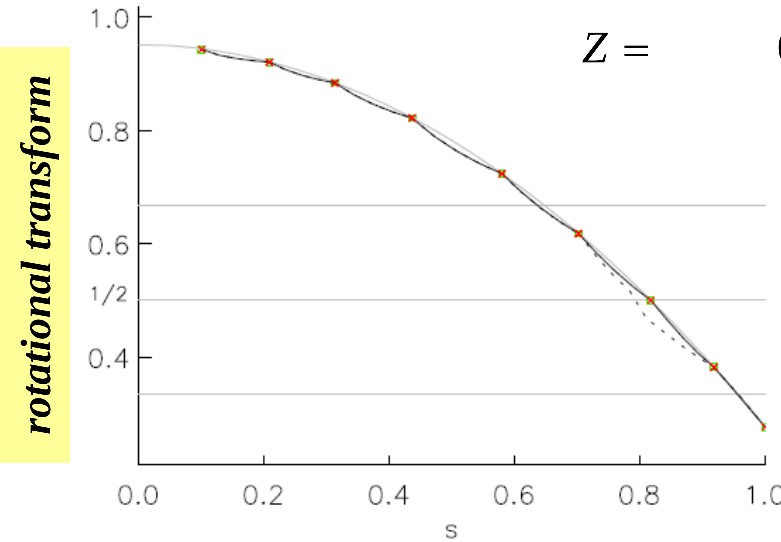


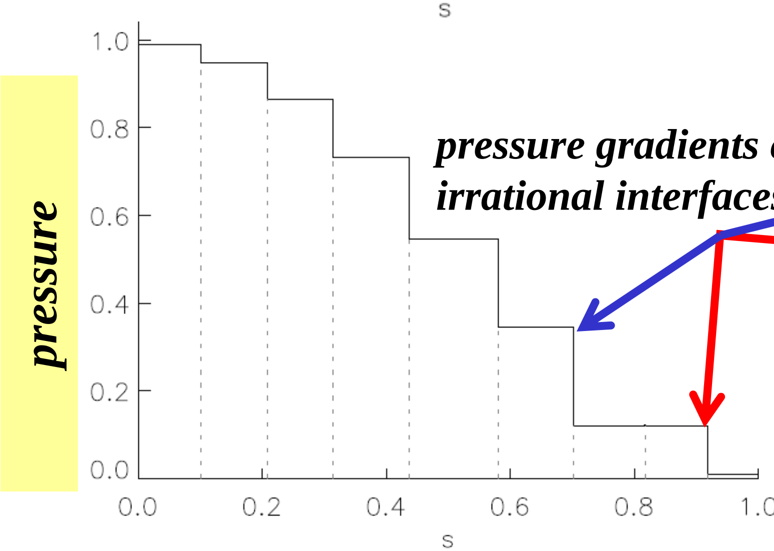
If ideal constraint applied at rational surfaces, then shielding currents prevent island formation.

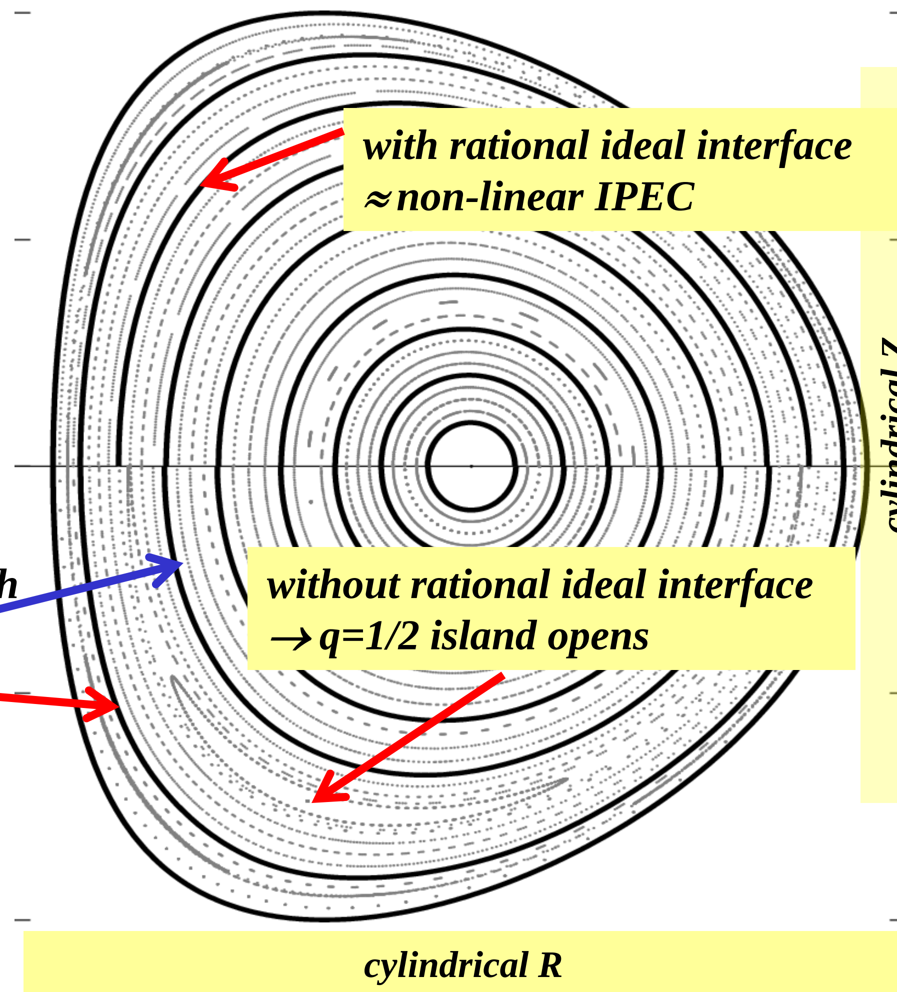
axisymmetric boundary, plus perturbation ($\delta = 10^{-4}$)

$$R = 1.0 + 0.3 \cos \vartheta + 0.05 \cos 2\vartheta, \qquad \delta R = \delta \cos(2\vartheta - \phi) \cos \vartheta$$

$$Z = 0.4 \sin \vartheta \qquad \delta Z = \delta \cos(2\vartheta - \phi) \sin \vartheta$$







Summary

→ A partially-relaxed, topologically-constrained energy principle has been described and the equilibrium solutions constructed numerically

* using a high-order (piecewise quintic) radial discretization, and a spectrally condensed Fourier representation
* workload distrubuted across multiple cpus,
* extrema located using standard numerical methods (NAG): modified Newton's method, with quadratic-convergence
* non-axisymmetric solutions with chaotic fields and non-trivial pressure guaranteed to exist (under certain conditions)

→ Specifying the profiles discretely is a practical means of retaining some control over the profiles, while making minimal assumptions regarding the topology of the field

* it is only assumed that *some* flux surfaces exist
* pressure gradients coincide with strongly irrational flux surfaces

→ Convergence studies have been performed

* expected error scaling with radial resolution confirmed
* detailed benchmark with axisymmetric equilibria (with smooth profiles)
* demonstrated convergence of island widths with Fourier resolution

→ By enforcing the ideal constraint at the rational surfaces, the formation of magnetic islands is prohibited by the formation of surface "shielding" currents

* similar to non-linear generalization of IPEC
* relaxing ideal constraint at rational surfaces allows islands to open

Force balance condition at interfaces gives rise to auxilliary pressure-jump Hamiltonian system.

$\rightarrow$ Beltrami condition, $\nabla\times\mathbf{B}=\mu\mathbf{B}$, and interface constraint, $\mathbf{B}\cdot\mathbf{n}=0$, gives $\nabla\times\mathbf{B}\cdot\nabla s=0$,

suggests surface potential, $B_\vartheta=\partial_\vartheta f$, $B_\zeta=\partial_\zeta f$, so that $\partial_\vartheta B_\zeta-\partial_\zeta B_\vartheta=0$,

$B^2=(g_{\vartheta\vartheta}f_\zeta f_\zeta-2g_{\vartheta\zeta}f_\vartheta f_\zeta+g_{\zeta\zeta}f_\vartheta f_\vartheta)/(g_{\vartheta\vartheta}g_{\zeta\zeta}-g_{\vartheta\zeta}g_{\vartheta\zeta})$, metric elements $g_{\alpha\beta}\equiv\partial_\alpha\mathbf{x}\cdot\partial_\beta\mathbf{x}$

$\rightarrow$ Force balance condition, $[[p+B^2/2]]=0$, introduce $H\equiv 2(p_1-p_2)=B_2^2-B_2^1=const.$

$\rightarrow$ Let tangential field on "inner-side" of interface be given, $B_{1\vartheta}=\partial_\vartheta f$, $B_{1\zeta}=\partial_\zeta f$,

tangential field on "outer-side", $B_{2\vartheta}=p_\vartheta$, $B_{2\zeta}=p_\zeta$, determined by characteristics

$$\dot\vartheta=\left.\frac{\partial H(\vartheta,\zeta,p_\vartheta,p_\zeta)}{\partial p_\vartheta}\right|_{\zeta,p_\vartheta,p_\zeta},\quad \dot p_\vartheta=-\frac{\partial H}{\partial\vartheta},\quad \dot\zeta=\frac{\partial H}{\partial p_\zeta},\quad \dot p_\zeta=-\frac{\partial H}{\partial\zeta}$$

$\rightarrow$ 2 d.o.f. Hamiltonian system, and invariant surfaces only exist if "frequency" is irrational

$\Rightarrow$ ideal interfaces that support pressure must have irrational transform

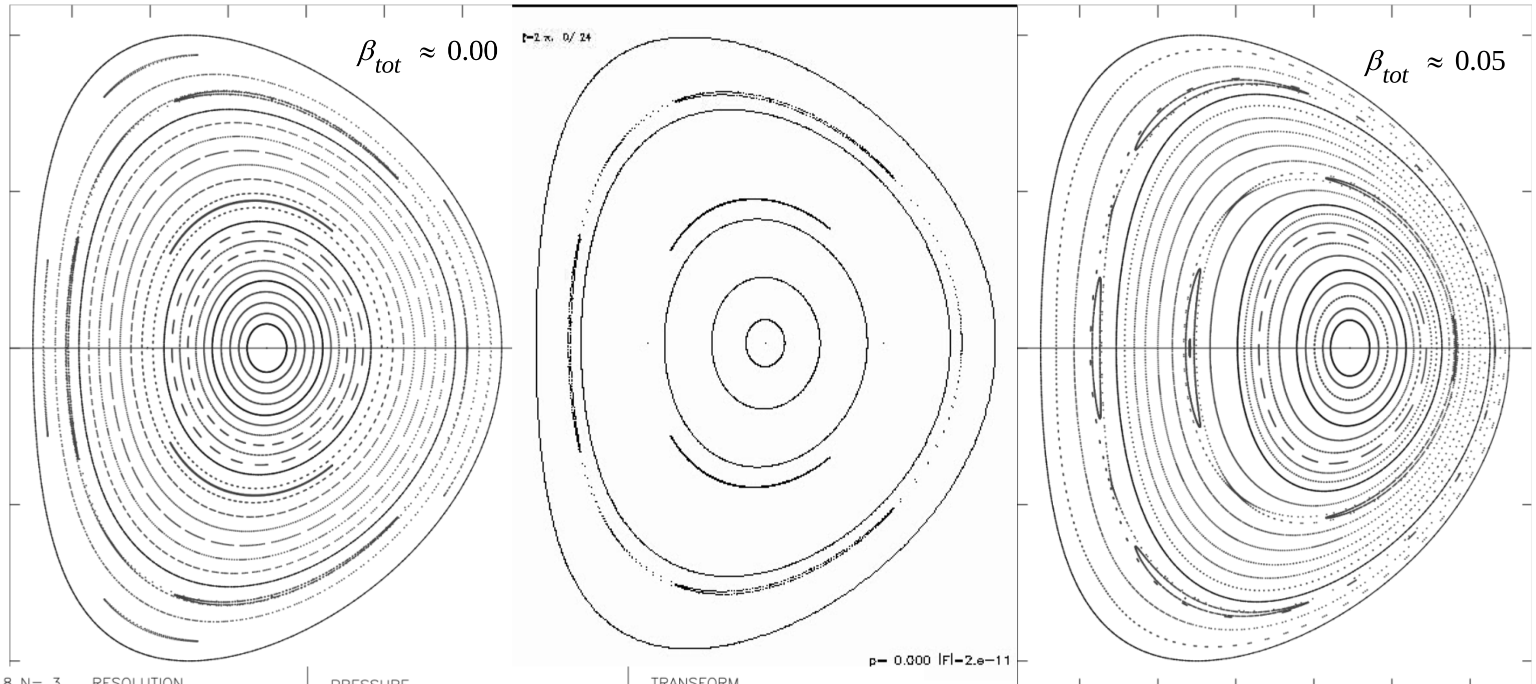
Hamilton-Jacobi theory for continuation of magnetic field across a toroidal surface supporting a plasma pressure discontinuity
M. McGann, S.R.Hudson, R.L. Dewar and G. von Nessi, Physics Letters A, 374(33):3308, 2010

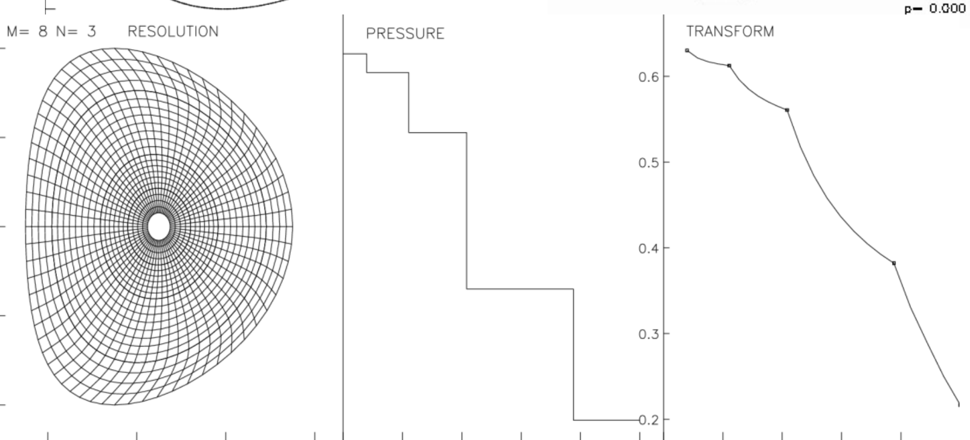
Sequence of equilibria with increasing pressure shows plasma *can* have significant response to external perturbation.

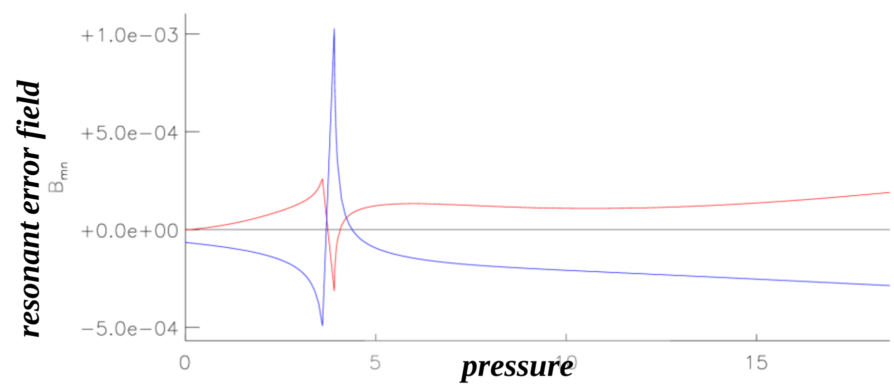
axisymmetric plus small perturbation

$$R = 1.00 + 0.30\cos(\vartheta) + 0.05\cos(2\vartheta) \quad + \quad [\delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\cos(\vartheta)$$

$$Z = 1.00 + 0.40\sin(\vartheta) + \quad [\delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\sin(\vartheta)$$





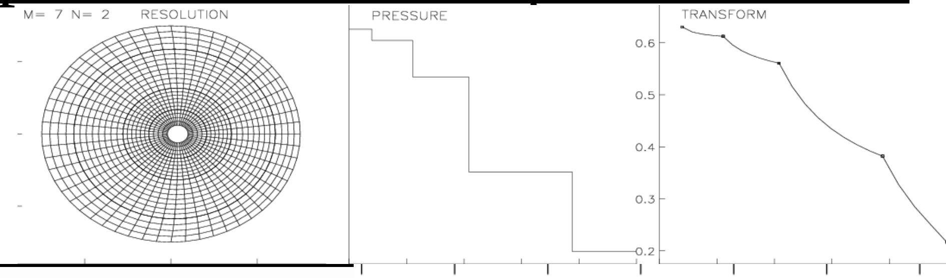


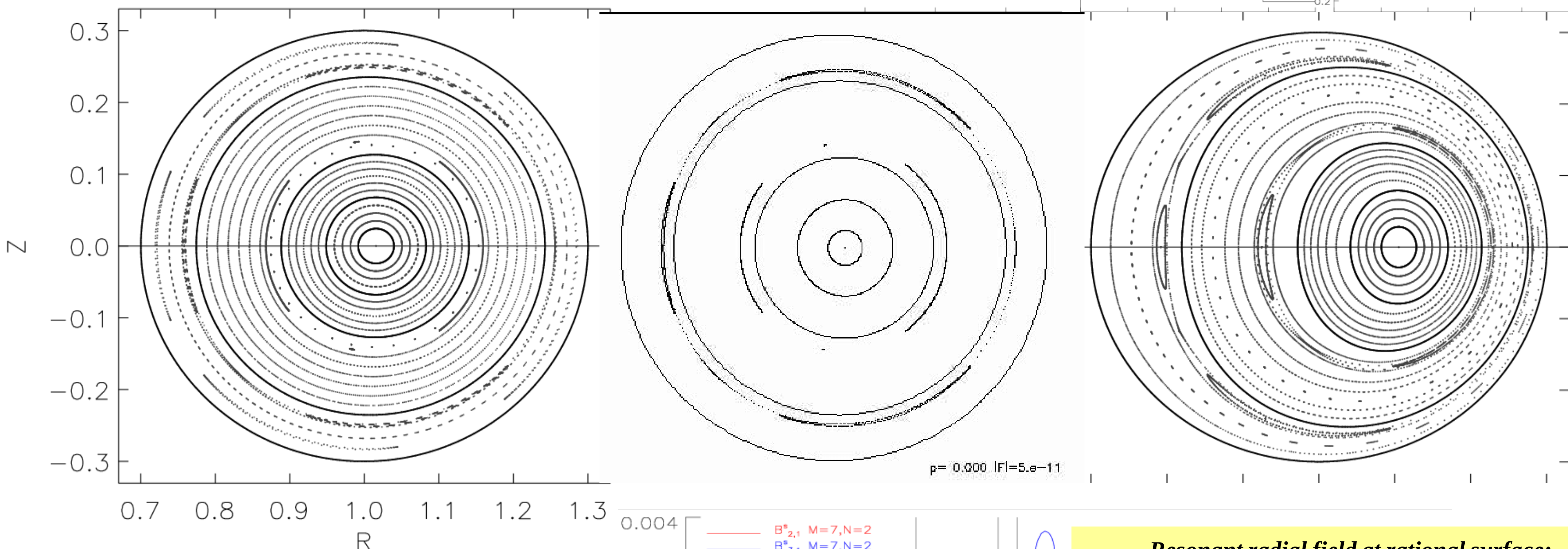
Sequence of equilibria with increasing pressure shows plasma *can* have significant response to external perturbation.

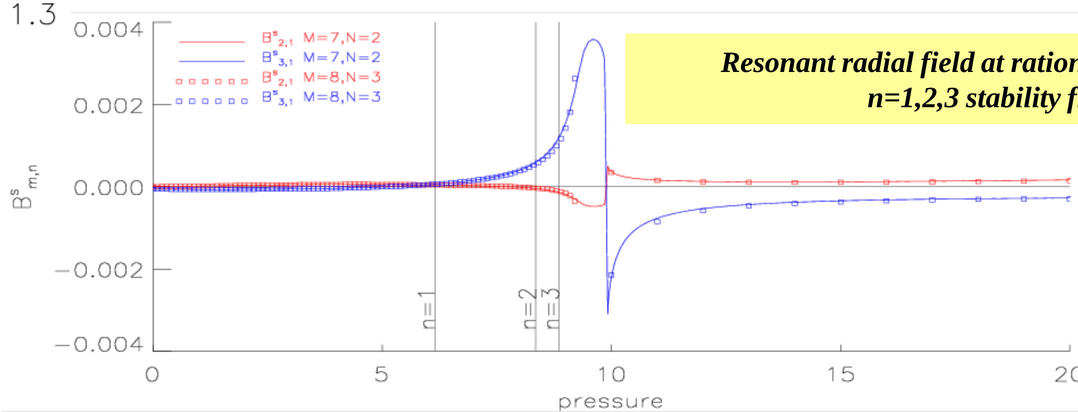
axisymmetric plus perturbation $\delta_{21} = \delta_{31} = 10^{-4}$

$$R = 1.00 + [0.30 + \delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\cos(\vartheta)$$

$$Z = 1.00 + [0.30 + \delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\sin(\vartheta)$$







Resonant radial field at rational surface; n=1,2,3 stability from PEST;

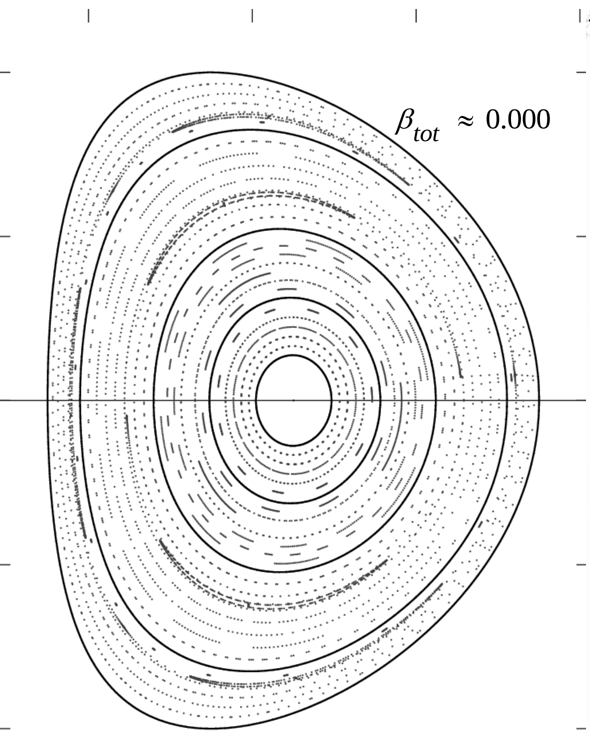
Sequence of equilibria with slowly increasing pressure

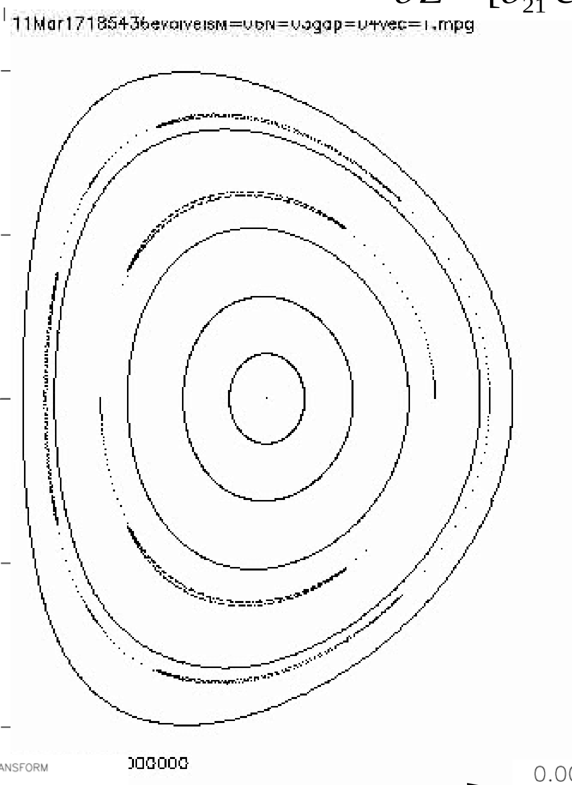
$$axisymmetric: \quad R = 1.00 + 0.30\cos(\vartheta) + 0.05\cos(2\vartheta)$$

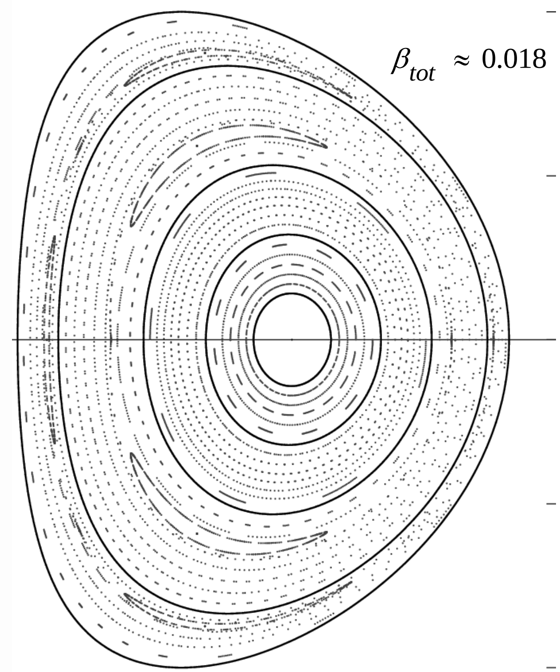
$$plus \qquad Z = 1.00 + 0.40\sin(\vartheta)$$

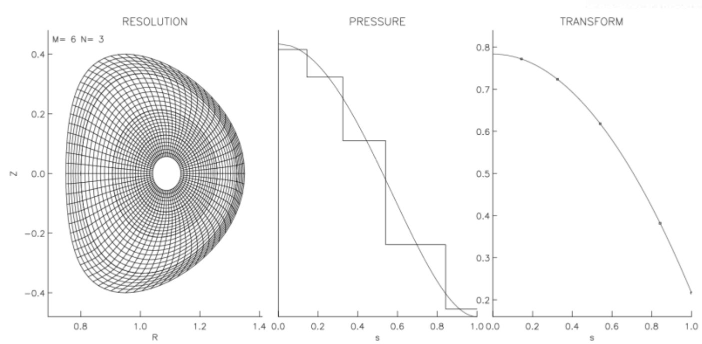
$$perturbation \quad : \delta R = [\delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\cos(\vartheta)$$

$$\delta Z = [\delta_{21}\cos(2\vartheta - \zeta) + \delta_{31}\cos(3\vartheta - \zeta)]\sin(\vartheta)$$



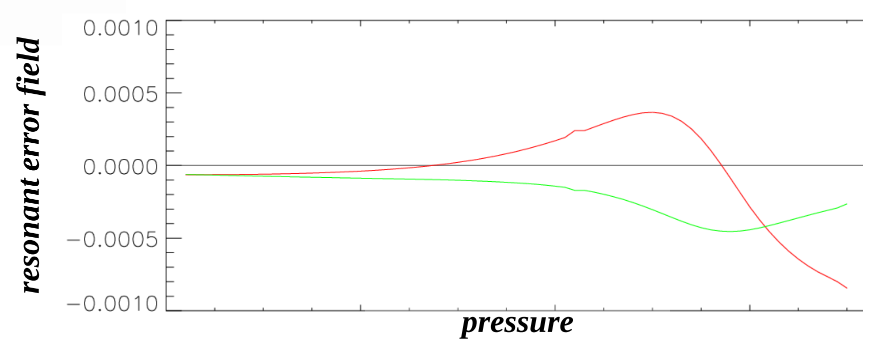






$T_F \approx 20\,\mathrm{s}$

$T_{\nabla F} \approx 60\,\mathrm{m}$



Toroidal magnetic confinement depends on flux surfaces

Transport in magnetized plasma dominately parallel to **B**

→ if the field lines are not confined (e.g. by flux surfaces), then the plasma is poorly confined

Axisymmetric magnetic fields possess a continuously nested family of flux surfaces

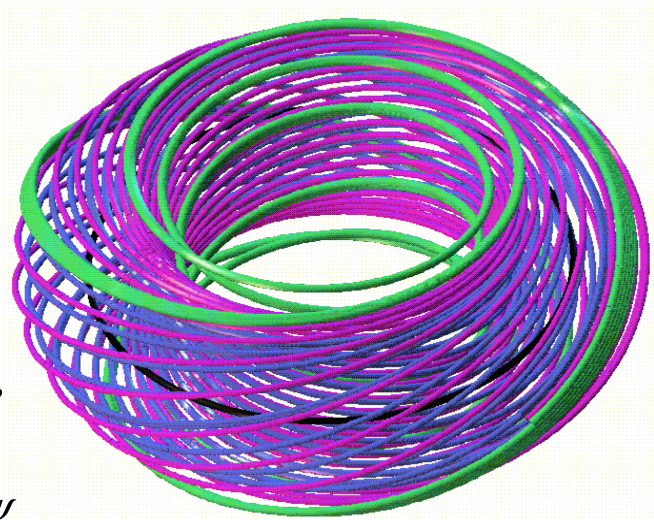
→ nested family of flux surfaces is guaranteed if the system has an ignorable coordinate

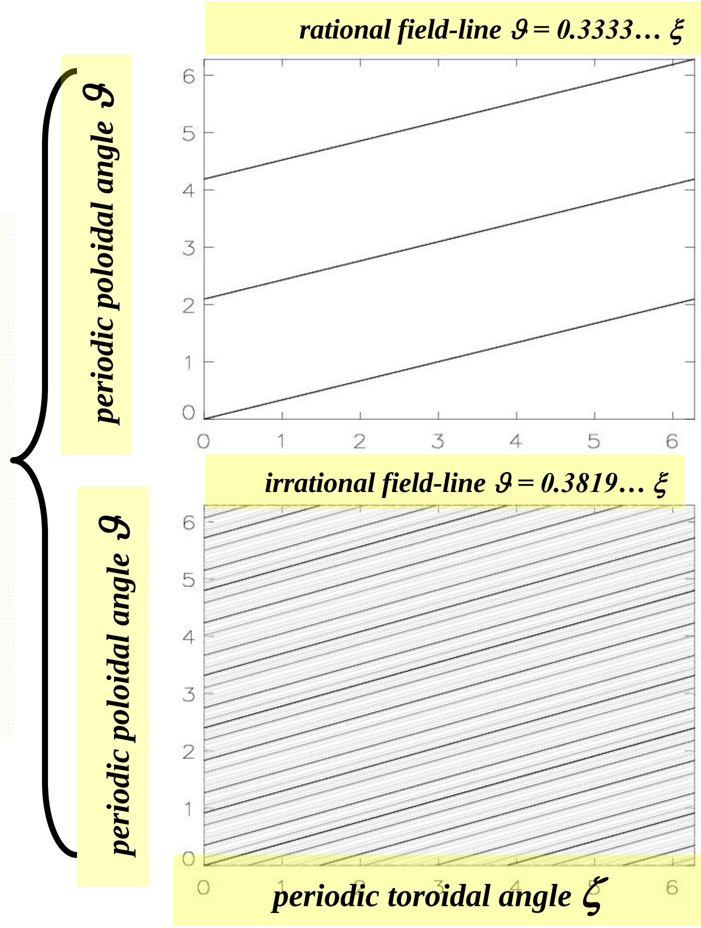
magnetic field is called integrable

→ rational field-line ≡ periodic trajectory ***family of periodic orbits ≡ rational flux surface***

→ irrational field-lines cover *irrational* flux surface

magnetic field lines wrap around toroidal "flux" surfaces





straight-field-line flux coordinates,

$$\mathbf{B}\bullet\nabla\psi = 0$$

$$\mathbf{B} = \nabla\psi\times\nabla\vartheta + \iota(\psi)\nabla\zeta\times\nabla\psi$$

$$\sqrt{g}\,\mathbf{B}\bullet\nabla \equiv \partial_\zeta + \iota\,\partial_\vartheta$$

magnetic differential equation, $\mathbf{B}\bullet\nabla\sigma = s$,

is singular at rational surfaces, $(m\,\iota - n)\sigma_{m,n} = i(\sqrt{g}\,s)_{m,n}$

Ideal MHD equilibria are extrema of energy functional

The energy functional is

$$W = \int_V (p + B^2/2)\, dv$$

$V \equiv$ global plasma volume

ideal variations

mass conservation } $\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$

state equation } $d_t (p \rho^{-\gamma}) = 0$

Faraday's law, ideal Ohm's law } $\delta \mathbf{B} = \nabla \times (\delta \xi \times \mathbf{B})$

*→ideal variations **don't** allow field topology to change "frozen-flux"*

the first variation in plasma energy is

$$\delta W = \int_V (\nabla p - \mathbf{j} \times \mathbf{B}) \cdot \delta \xi \, dv$$

Euler Lagrange equation for globally ideally-constrained variations

ideal-force-balance $\nabla p = \mathbf{j} \times \mathbf{B}$

→ two surface functions, e.g. the pressure, $p(s)$, and rotational-transform ≡ inverse-safety-factor, $\iota(s)$,

***and → a boundary surface** (. . for fixed boundary equilibria . . .),*

***constitute "boundary-conditions"** that **must** be provided to **uniquely define** an equilibrium solution*

. . . . . . The computational task is to compute the magnetic field that is consistent with the given boundary conditions . . .

nested flux surface topology maintained by singular currents at rational surfaces

from $\nabla \cdot (\sigma \mathbf{B} + \mathbf{j}_\perp) = 0$, parallel current must satisfy $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$, where $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$

$$\sigma_{m,n} = \frac{i(\sqrt{g}\ \nabla \cdot \mathbf{j}_\perp)_{m,n}}{(m\iota - n)} + \delta(m\iota - n)$$

→ magnetic differential equations are singular at rational surfaces (periodic orbits)

→ pressure-driven "Pfirsch-Schlüter currents" have 1/x type singularity

→ δ - function singular currents shield out islands

Topological constraints : pressure gradients coincide with flux surfaces

The ideal interfaces are chosen to coincide with pressure gradients

$\rightarrow$ parallel transport dominates perpendicular transport,

$\rightarrow$ simplest approximation is $\mathbf{B}\bullet\nabla p = 0$

$\rightarrow$ pressure gradients **must** coincide with KAM surfaces $\equiv$ ideal interfaces

$\rightarrow$ structure of B and structure of the pressure are intimately connected;

$\rightarrow$ cannot apriori specify pressure without apriori constraining structure of the field;

[next order of approximation, $\mathbf{B}\bullet\nabla p$ is small, e.g. $\partial_t p = \kappa_{\|}\nabla_{\|}^2 p + \kappa_{\perp}\nabla_{\perp}^2 p = 0$, with $\kappa_{\|} \gg \kappa_{\perp}$, e.g. $\kappa_{\perp}/\kappa_{\|} \sim 10^{-10}$

*pressure gradients coincide with KAM surfaces, cantori . .

*pressure flattened across islands, chaos with width > $\Delta w_C \sim (\kappa_{\perp}/\kappa_{\|})^{1/4}$

* anisotropic diffusion equation solved analytically, p' $\propto 1/\left(\kappa_{\|}\varphi_2 + \kappa_{\perp} G\right)$, φ_2 is quadratic-flux across cantori, G is metric term]

$\rightarrow$ where there are significant pressure gradients, there can be no islands or chaotic regions with width $> \Delta wc$

A fixed boundary equilibrium is defined by :

(i) given pressure, $p(\psi)$, and rotational-transform profile, $\iota(\psi)$

(ii) geometry of boundary;

(a) only stepped pressure profiles are consistent (numerically tractable) with chaos and $\mathbf{B}\bullet\nabla p = 0$

(b) the computed equilibrium magnetic field must be consistent with the input profiles

(a) + (b) = where the pressure has gradients, the magnetic field must have flux surfaces.

$\rightarrow$ non-trivial stepped pressure equilibrium solutions are *guaranteed* to exist

Taylor relaxation: a weakly resistive plasma will relax, *subject to single constraint* of conserved helicity

Taylor relaxation, [Taylor, 1974]

$$\underbrace{W = \int_V \left(p + B^2/2\right) dv,}_{\text{plasma energy}} \qquad \underbrace{H = \int_V (\mathbf{A}\cdot\mathbf{B})\, dv}_{\text{helicity, } \mathbf{B}=\nabla\times\mathbf{A}}$$

Constrained energy functional $F = W - \mu H/2$, $\mu \equiv$ Lagrange multiplier

Euler-Lagrange equation, for *unconstrained* variations in magnetic field, $\nabla \times \mathbf{B} = \mu \mathbf{B}$

linear force-free field $\equiv$ Beltrami field

But, . . .Taylor relaxed fields have no pressure gradients

Ideal MHD equilibria and Taylor-relaxed equilibria are at opposite extremes . . . .

Ideal-MHD $\rightarrow$ imposition of *infinity* of ideal MHD constraints

non-trivial pressure profiles, but structure of field is ***over-constrained***

Taylor relaxation $\rightarrow$ imposition of *single* constraint of conserved global helicity

structure of field is not-constrained, but pressure profile is trivial, i.e. ***under-constrained***

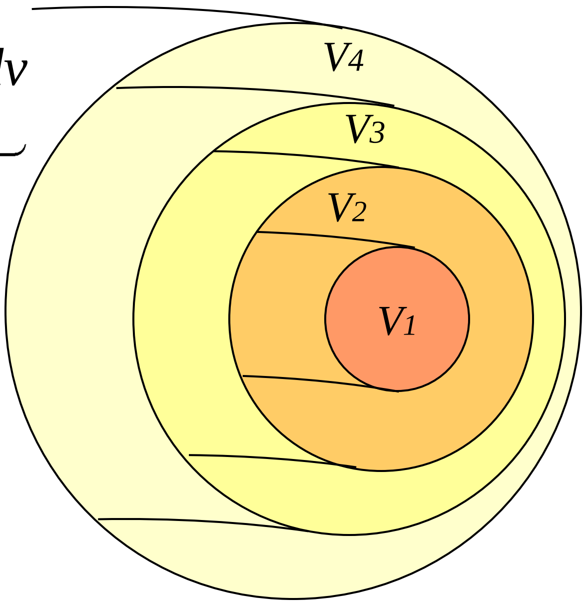
We need something in between . . .

. . . perhaps an equilibrium model with *finitely* many ideal constraints, and *partial* Taylor relaxation?

Introducing the multi-volume, partially-relaxed model of MHD equilibria with topological constraints

Energy, helicity and mass integrals

$$\underbrace{W_l = \int_{V_l} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{\text{plasma energy}}, \qquad \underbrace{H_l = \int_{V_l} (\mathbf{A} \cdot \mathbf{B}) \, dv}_{\text{helicity}}, \qquad \underbrace{M_l = \int_{V_l} p^{1/\gamma} dv}_{\text{mass}}$$



Multi-volume, partially-relaxed energy principle

* A set of N nested toroidal surfaces enclose N annulur volumes

$\rightarrow$ the interfaces are assumed to be ideal, $\delta \mathbf{B} = \nabla \times (\delta \boldsymbol{\xi} \times \mathbf{B})$

* The multi-volume energy functional is

$$\boxed{F = \sum_{l=1}^{N} \left(W_l - \mu_l H_l / 2 - \nu_l M_l \right)}$$

Euler-Lagrange equation for *unconstrained* variations in **A**

$\rightarrow$ field remains tangential to interfaces,
$\rightarrow$ a finite number of ideal constraints, imposed topologically!

In each annulus, the magnetic field satisfies $\nabla \times \mathbf{B}_l = \mu_l \mathbf{B}_l$

Euler-Lagrange equation for variations in interface geometry

Across each interface, pressure jumps allowed, but total pressure is continuous $[[p + B^2/2]] = 0$

$\rightarrow$ an analysis of the force-balance condition is that the interfaces must have strongly irrational transform

ideal interfaces coincide with KAM surfaces