How do chaotic magnetic fields confine plasma equilibria?

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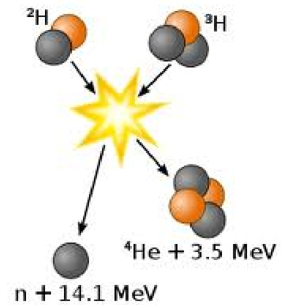
Grand Challenge of Magnetic Plasma Confinement
is to create sustainable energy

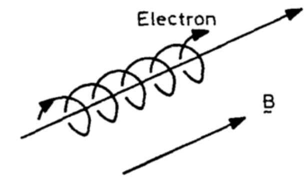


- Need to confine a **hot**, **dense** plasma (ionized gas) for a **long time**

We need to confine the plasma in a stable equilibrium.

$$\mathbf{F} = q\ (\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Macroscopic force balance = Lorentz force balances gas pressure



simplest equilibrium equation is $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$

The calculation of the equilibrium is fundamental.

Both particle transport studies, and stability calculations, depend on the equilibrium calculation

→ pointless to follow *microscopic* particle trajectories if the *macroscopic* forces are not balanced
→ to determine the stability of an equilibrium, first the equilibrium state must be known

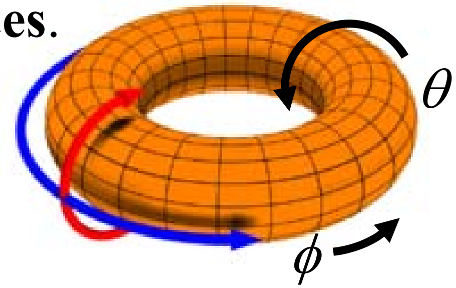
→ many plasma disruptions etc. are caused by the lack of a stable, equilibrium state
→ experimental design "begins" with an equilibrium calculation

Axisymmetric toroidal fields have "nested" flux-surfaces, and flux surfaces are good for confinement

- Because (1) there are no magnetic monopoles, (2) and the "hairy" ball theorem, (3) Noether's theorem,
  in toroidal, axisymmetric fields (i.e. an idealized tokamak),
  the field lines wrap around on nested, magnetic flux surfaces.

  each surface characterized by frequency $\omega \equiv \dfrac{\Delta\theta}{\Delta\phi}$

  if ω is **rational**, i.e. ω = n / m, the field line is "closed", i.e. "periodic"
  if ω is **irrational**, the field line will come arbitrarily close to every point on the surface



- This is great for confinement → the field lines lie on surfaces
  → the particles are tied to the field lines
  → <u>the pressure is constant on flux surfaces,</u> *p=p(s)*

- $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ is easy to solve <u>because the equilibrium is smooth</u>

  if $\dfrac{\partial}{\partial\phi} = 0,$ i.e. two-dimsional, then $\nabla p = \mathbf{j} \times \mathbf{B}$ reduces to the Grad-Shafranov equation

  (1) calculate equilibrium (2) determine stability (3) study particle transport

But, non-axisymmetric (3D) perturbations introduce *chaos*

→ the smooth, continuously-nested family of flux-surfaces is *"broken"*

→ the equilibrium is greatly complicated, becomes the equilibrium becomes <u>*fractal*</u>

With increasing non-axisymmetry, the flux surfaces become increasingly “broken”

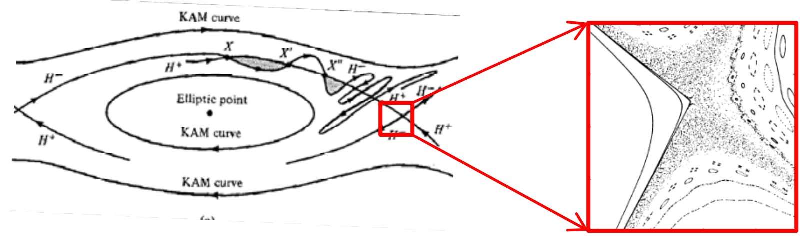
- Invariant flux surfaces are destroyed near “resonances”, ω = n / m, n, m are integers

    construction of action-angle coordinates for perturbed system fails because of “small-denominators”

- Magnetic islands (resonance zones) form

    chaotic, “irregular” field lines emerge,

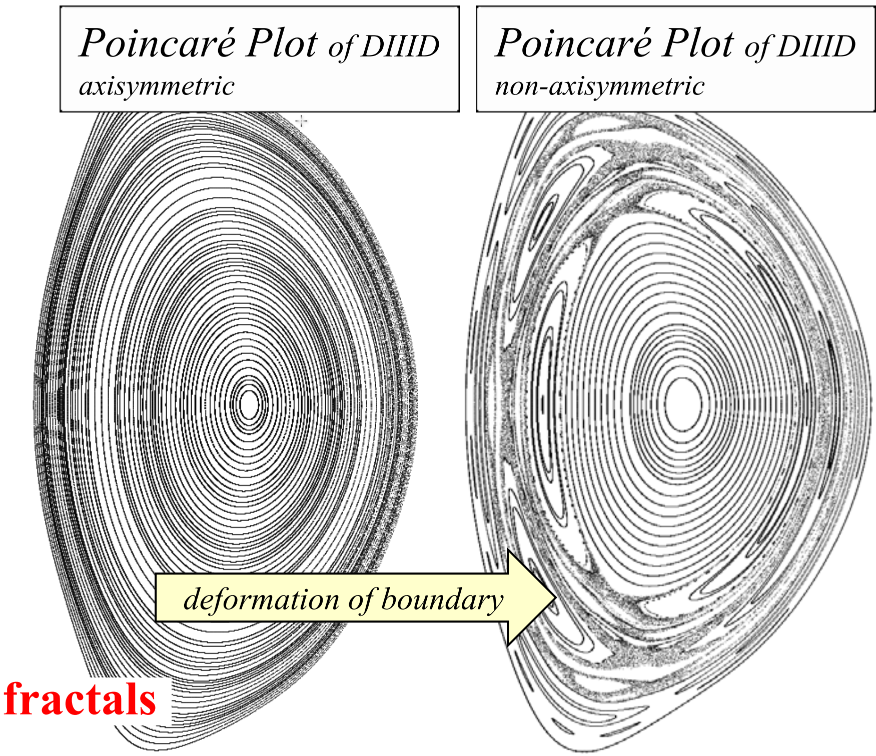
    that wander **seemingly randomly** over a volume



separatrix splitting, unstable manifold, *“chaotic tangle”*

- Confinement deteriorates,

    the pressure is flat inside islands and chaos

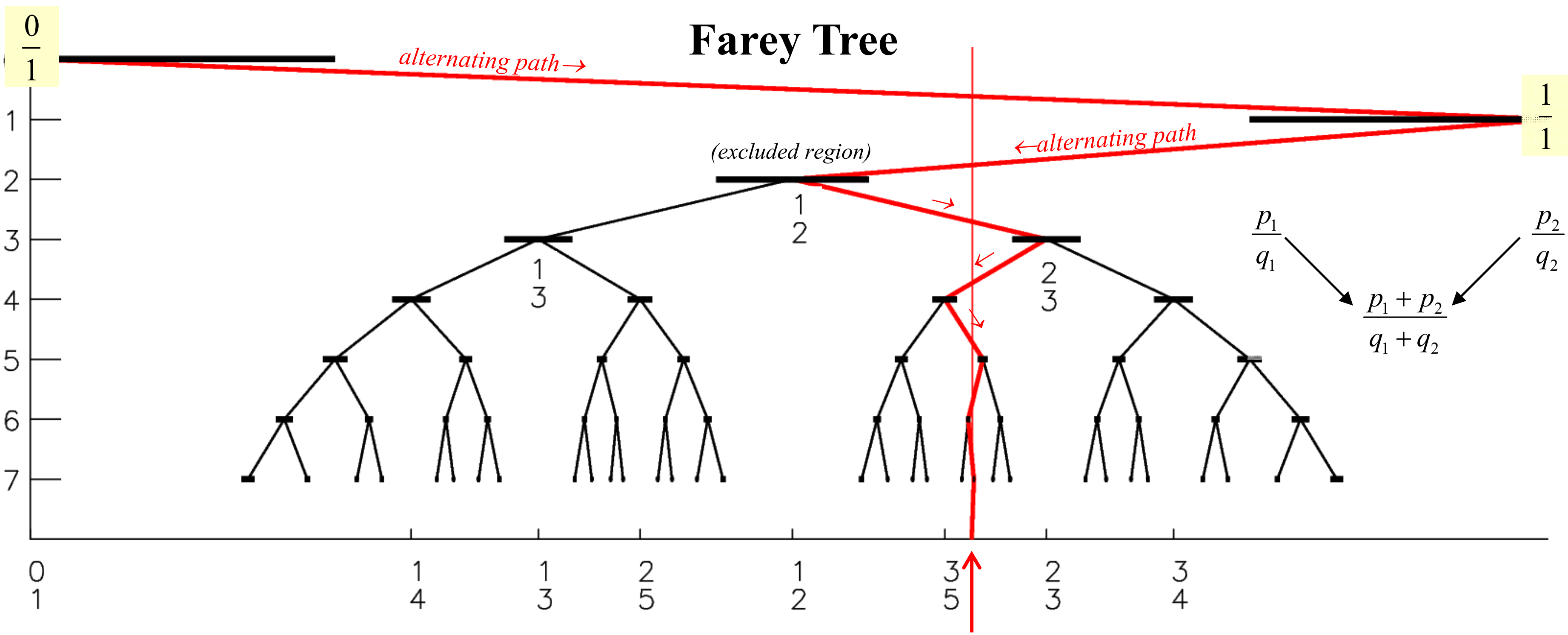


- The calculation of three-dimensional partially-chaotic equilibria must

1) Be consistent with theoretical plasma physics
2) Be consistent with experimental results
3) **Be consistent with Hamiltonian chaos theory**
4) **employ numerical methods that accommodate fractals**

WHERE TO START? START WITH CHAOS

The fractal structure of chaos is related to the structure of numbers



islands & chaos emerge at every rational $\rightarrow$ about each rational n/m, introduce excluded region, width r/m^k

KAM Theorem $\rightarrow$ flux surface can survive if $|\omega - n/m| > r/m^k$, for all n, m

(Kolmogorov, Arnold, Moser) *we say that ω is "strongly-irrational" if ω avoids all excluded regions*

Greene's residue criterion $\rightarrow$ the most robust flux surfaces are associated with alternating paths

$\rightarrow$ Fibonacci ratios $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$

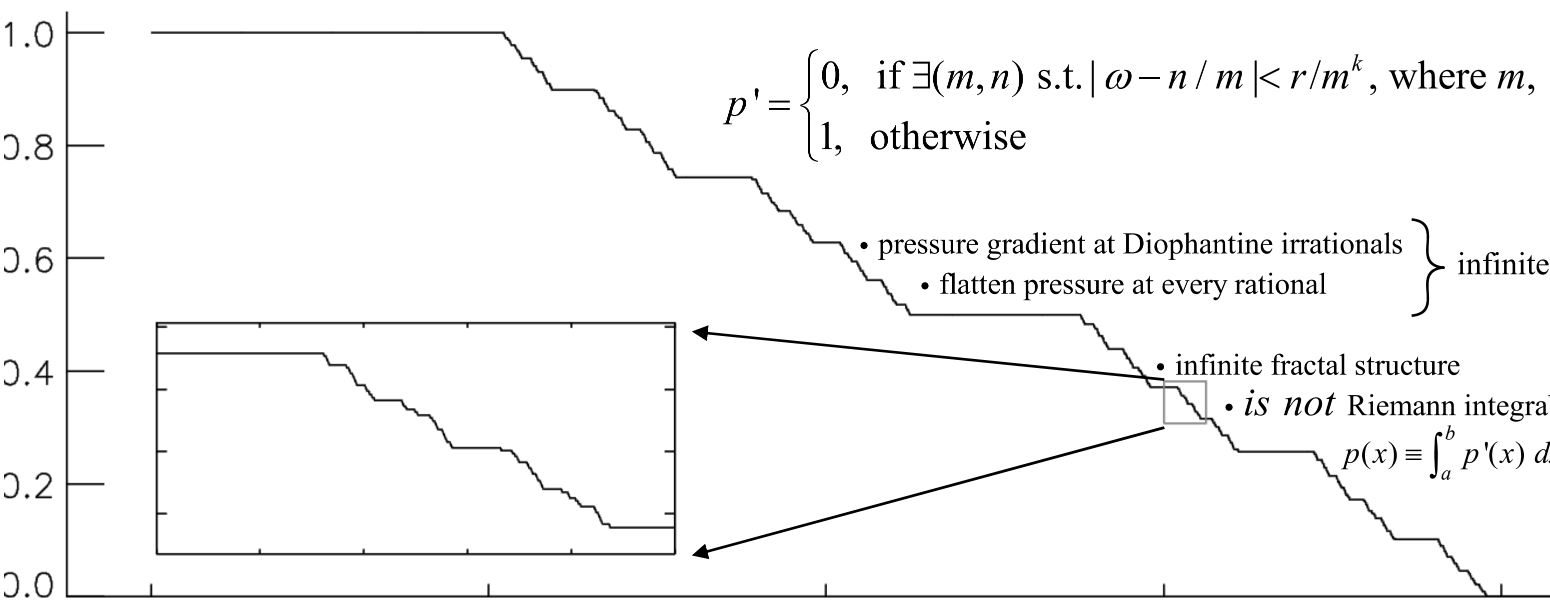
THEN, ADD PLASMA PHYSICS

Force balance means the pressure is a *“fractal staircase”*

- $\nabla p = \mathbf{j} \times \mathbf{B}$, implies that $\mathbf{B}\cdot\nabla p=0$ i.e. pressure is constant along a field line
- Pressure is flat across the rationals (assuming no “pressure” source inside the islands)
  → islands and chaos at every rational → chaotic field lines wander about over a volume
- Pressure gradients supported on the “most-irrational” irrationals
  → surviving “KAM” flux surfaces confine particles and pressure

Diophantine Pressure Profile

is it pathological?

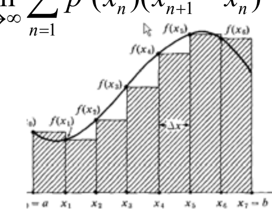


$$p' = \begin{cases} 0, & \text{if } \exists(m,n) \text{ s.t.} |\omega - n/m| < r/m^k \text{, where } m,\ n \text{ are integer} \\ 1, & \text{otherwise} \end{cases}$$

$\left.\begin{array}{l}\bullet \text{ pressure gradient at Diophantine irrationals} \\ \bullet \text{ flatten pressure at every rational}\end{array}\right\}$ infinitely discontinuous

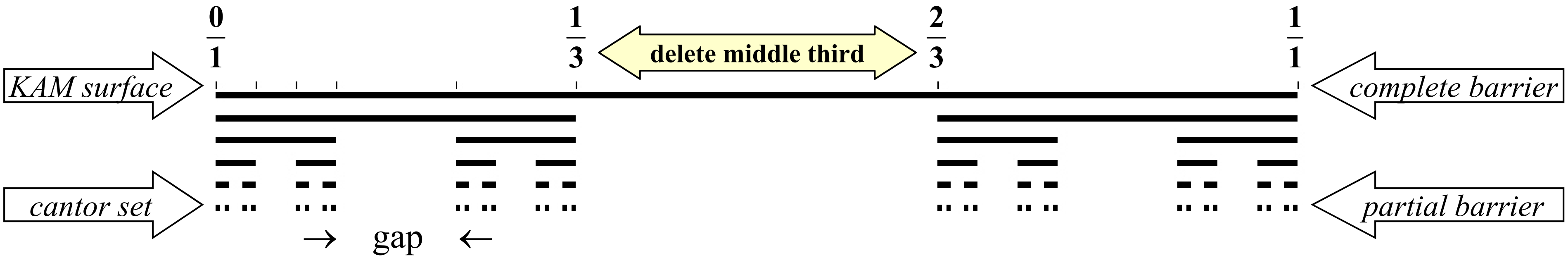
- infinite fractal structure
- *is not* Riemann integrable

$$p(x) \equiv \int_a^b p'(x)\,dx \neq \lim_{N\to\infty} \sum_{n=1}^{N} p'(x_n)(x_{n+1} - x_n)$$



Q) How do non-integrable fields confine field lines?

A) Field line transport is restricted by KAM surfaces and cantori

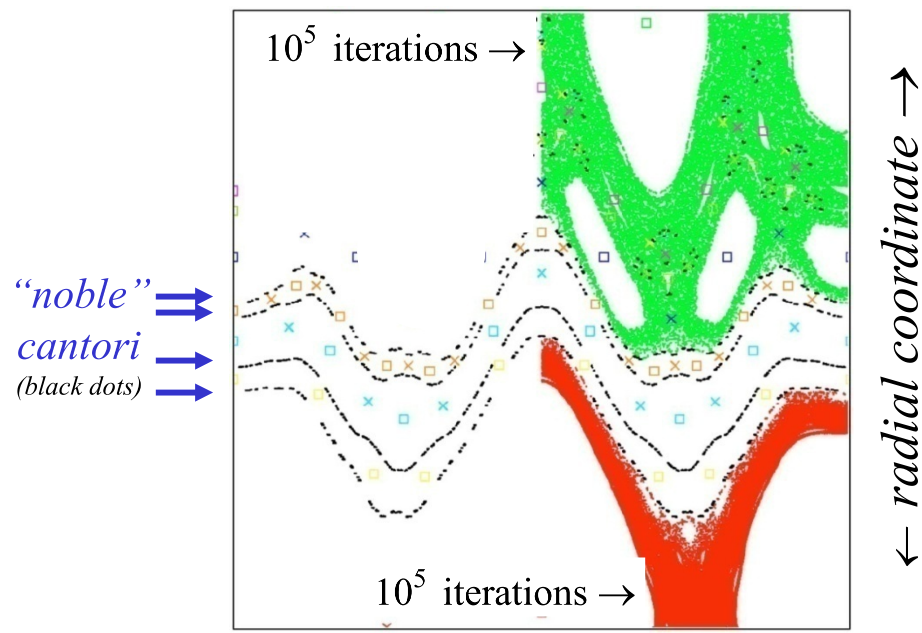


→ *KAM surfaces are closed, toroidal surfaces; and **stop** radial field line transport*

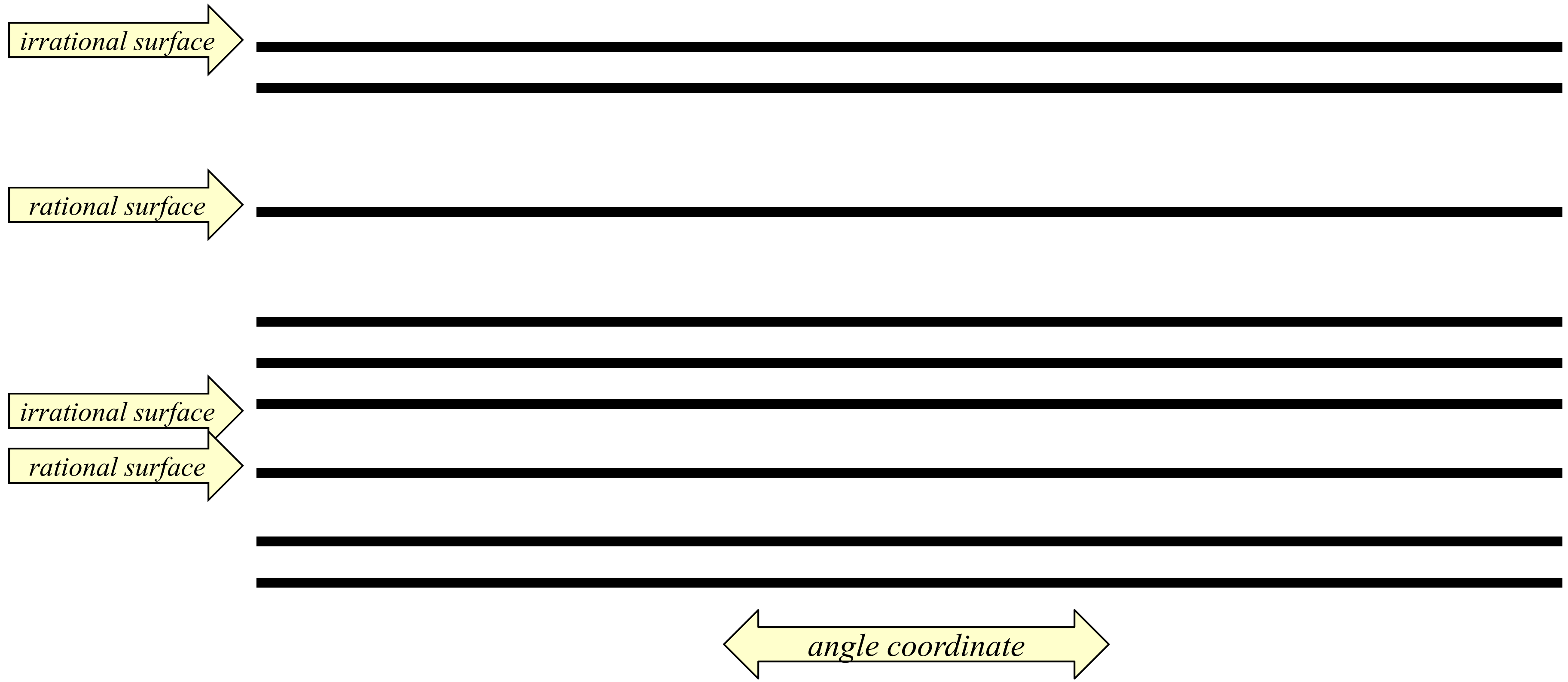
→ *Cantori have many holes, but still **cantori can severely "slow down"** radial field line transport*

→ *Example, all flux surfaces destroyed by chaos, but even after **100 000 transits** around torus the field lines **cannot get past cantori***

Calculation of cantori for Hamiltonian flows
S.R. Hudson, Physical Review E 74:056203, 2006



Simplified Diagram of the structure of integrable fields,
→showing continuous family of invariant surfaces

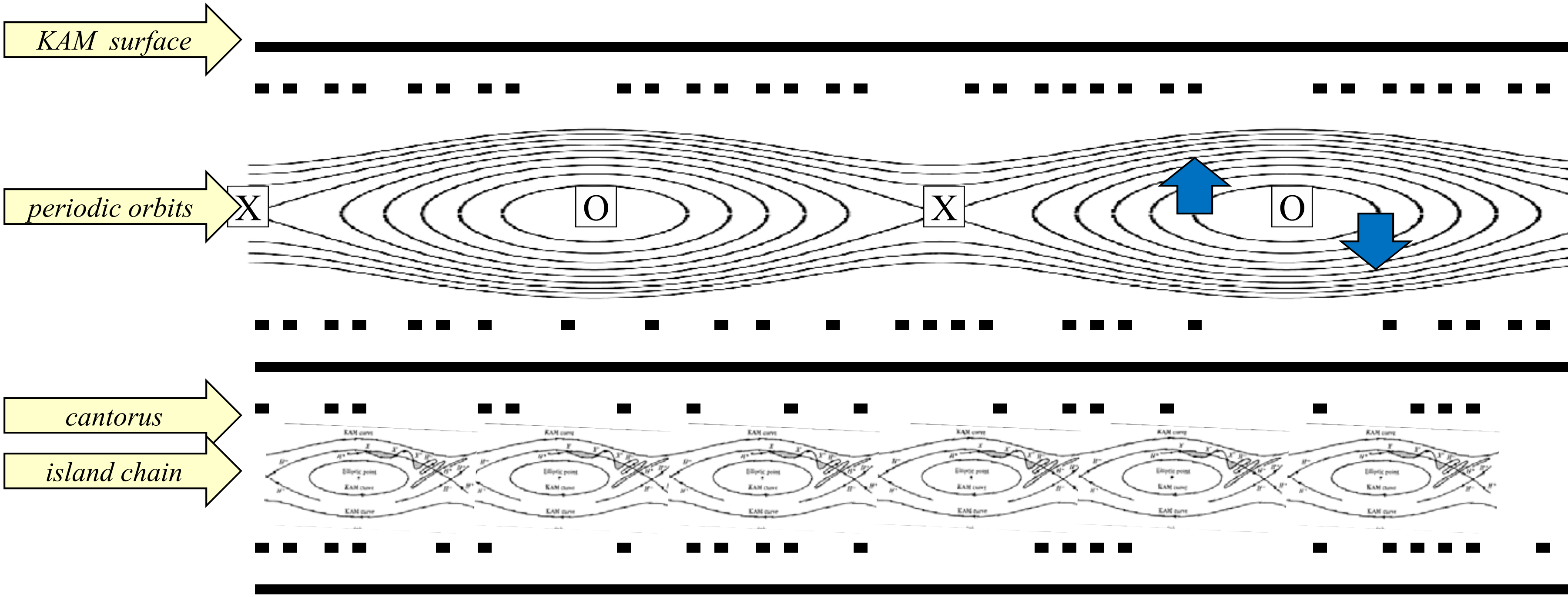


Action-angle coordinates can be constructed for “integrable” fields

- the “action” coordinate coincides with the invariant surfaces
- dynamics then appears simple

Simplified Diagram of the structure of non-integrable fields,

→showing the fractal hierarchy of invariant sets



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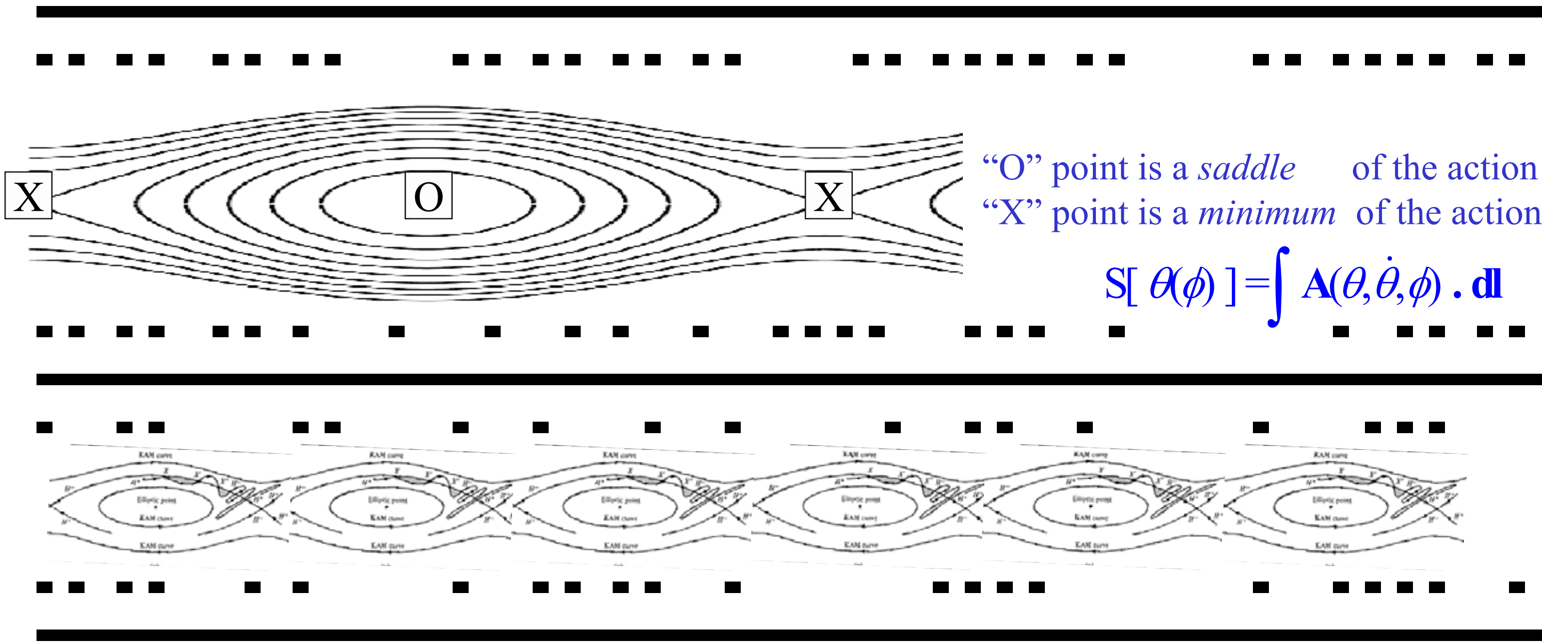
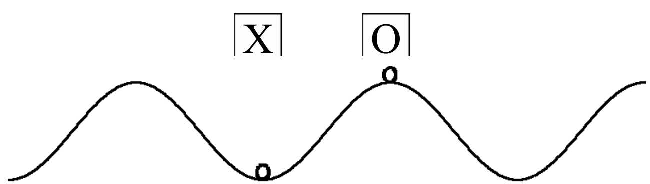


"O" point is a *saddle* of the action

"X" point is a *minimum* of the action

$$S[\ \theta(\phi)\ ] = \int \mathbf{A}(\theta,\dot{\theta},\phi)\ .\ \mathbf{dl}$$



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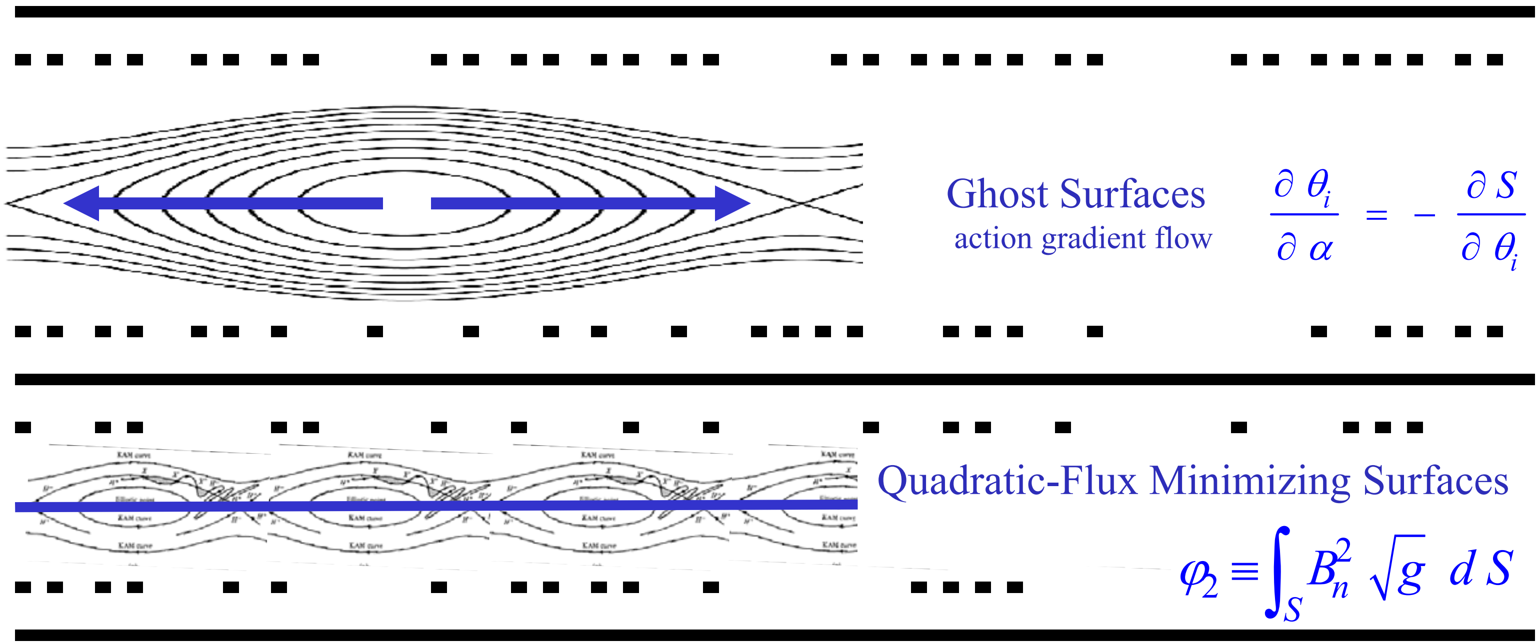

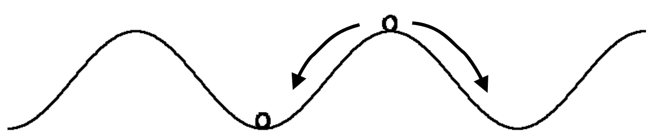

Are ghost-surfaces quadratic-flux minimizing?
S.R. Hudson & R.L. Dewar, Physics Letters A 373:4409, 2009

Unified theory of Ghost and Quadratic-Flux-Minimizing Surfaces
R.L. Dewar, S.R. Hudson & A.M. Gibson
Journal of Plasma and Fusion Research SERIES, 9:487, 2010

Simplified Diagram of the structure of non-integrable fields,
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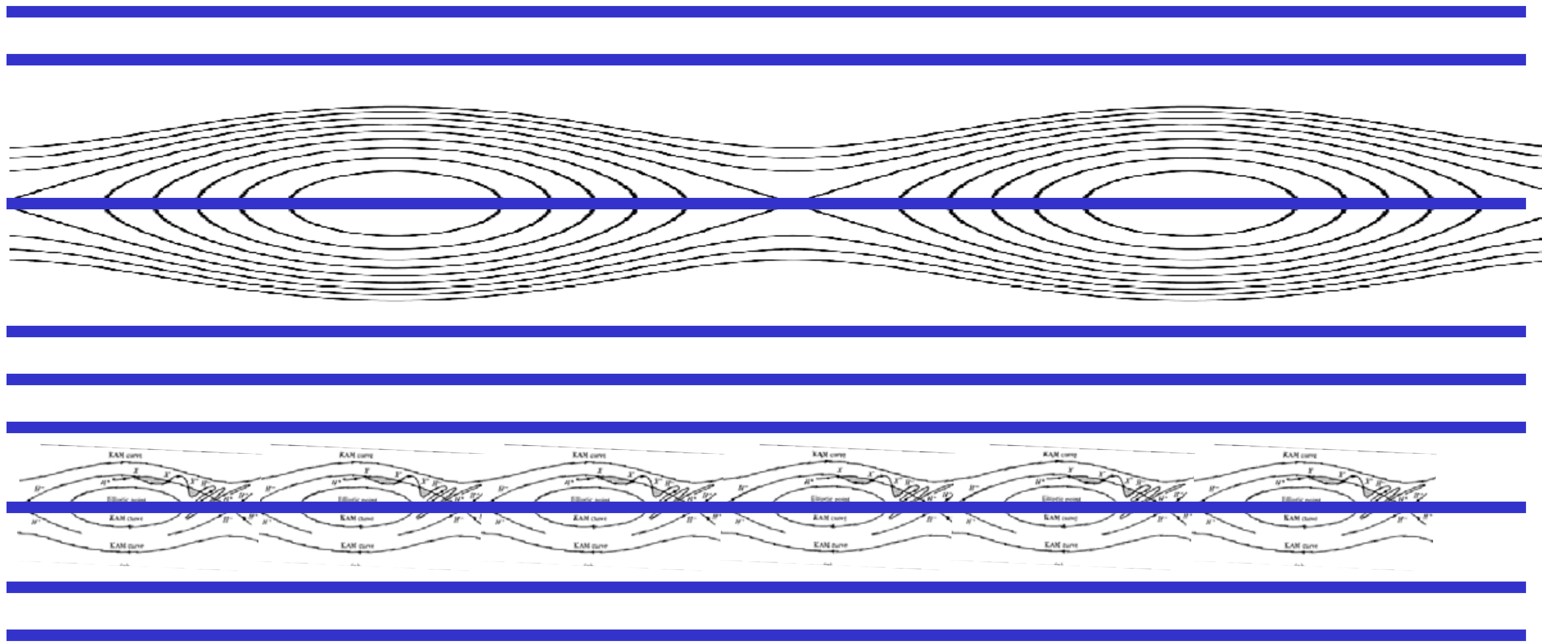


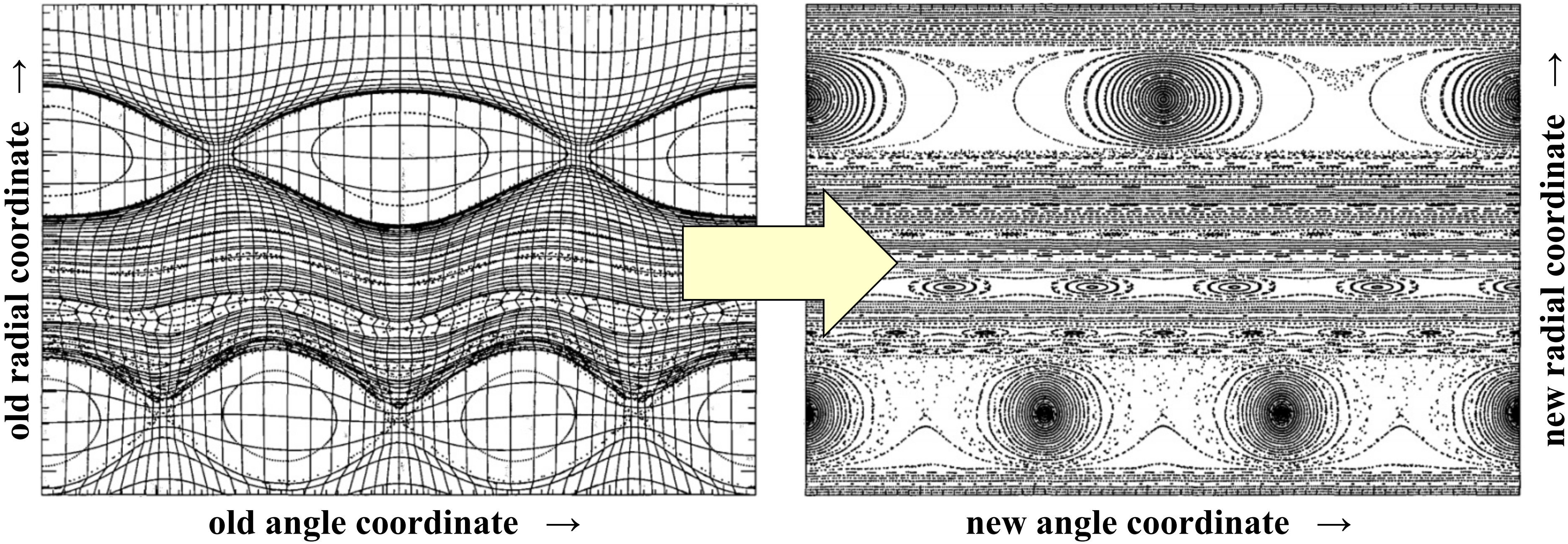
Chaotic coordinates can be constructed

- coordinate surfaces are adapted to the fractal hierarchy of remaining invariant sets
- ghost surfaces ≡ quadratic-flux minimizing surfaces are “almost-invariant”
- dynamics appears “almost-simple”

Chaotic coordinates "straighten out" chaos

Poincaré plot of chaotic field
(in **action-angle** coordinates of **unperturbed** field)

Poincaré plot of chaotic field
in chaotic coordinates



phase-space is partitioned into (1) regular ("irrational") regions with "good flux surfaces", temperature gradients
and (2) irregular (" rational") regions with islands and chaos, flat profiles

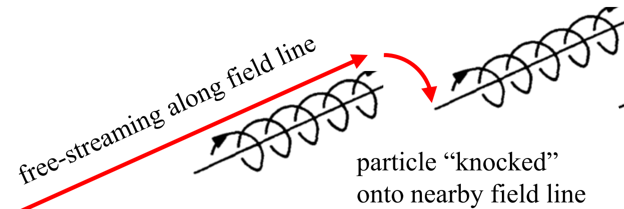
Generalized magnetic coordinates for toroidal magnetic fields
S.R. Hudson, Doctoral Thesis, The Australian National University, 1996

Chaotic coordinates simplify anisotropic transport

The temperature is constant on ghost surfaces, *T=T(s)*

1. Transport *along* the magnetic field is *unrestricted*

→ consider parallel random walk, with **long** steps≈ *collisional mean free path*



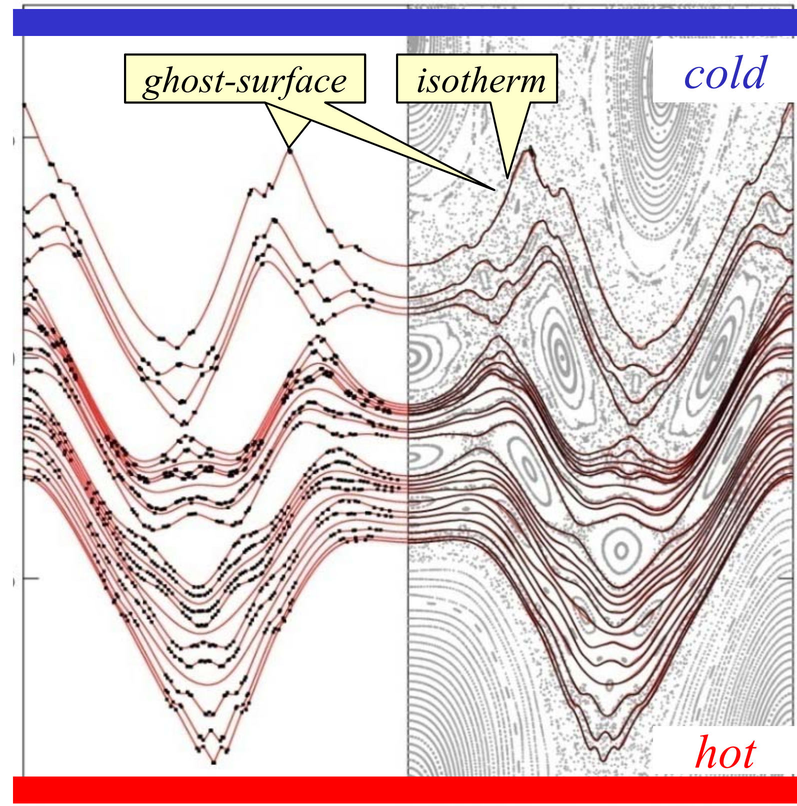
2. Transport *across* the magnetic field is *very small*

→consider perpendicular random walk with **short** steps≈ *Larmor radius*

3. Anisotropic diffusion balance $\boxed{\kappa_{\parallel} \nabla_{\parallel}^{2} T + \kappa_{\perp} \nabla_{\perp}^{2} T = 0, \quad \kappa_{\parallel} \gg \kappa_{\perp}, \quad \kappa_{\perp} / \kappa_{\parallel} \sim 10^{-10}}$

$2^{12} \times 2^{12} = 4096 \times 4096$ grid points (to resolve small structures)

4. Compare solution of numerical calculation to ghost-surfaces



5. The temperature adapts to KAM surfaces,cantori,

and ghost-surfaces!

i.e. T=T(s), where s=const. is a ghost-surface

from T=T(s,θ,ϕ) to T=T(s) is a fantastic simplification, allows analytic solution

$$\frac{dT}{ds} \propto \frac{1}{\kappa_{\parallel} \varphi_2 + \kappa_{\perp} G}$$

Temperature contours and ghost-surfaces for chaotic magnetic fields
S.R. Hudson et al., Physical Review Letters, 100:095001, 2008
Invited talk 22nd IAEA Fusion Energy Conference, 2008
Invited talk 17th International Stellarator, Heliotron Workshop, 2009

An expression for the temperature gradient in chaotic fields
S.R. Hudson, Physics of Plasmas, 16:100701, 2009

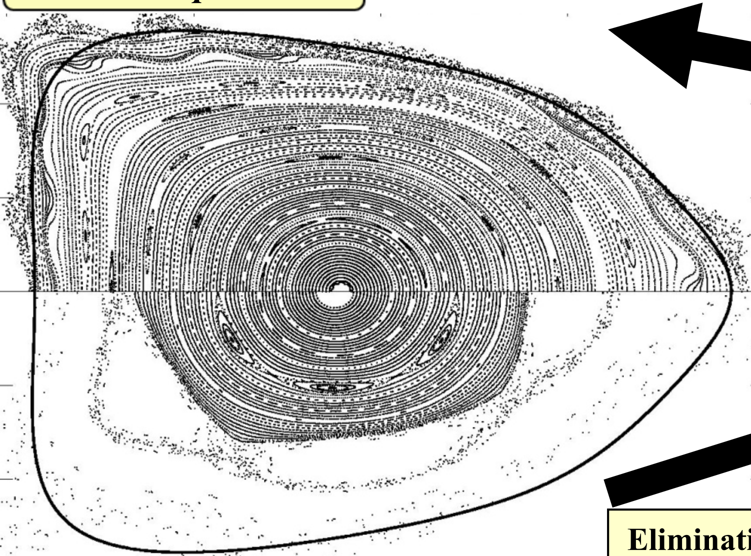
Non-axisymmetric (i.e. three-dimensional) experiments designed to have "good-flux-surfaces"

- The construction of **ghost-surfaces ≡ quadratic-flux minimizing surfaces** provides an easy-to-calculate measure of the island size

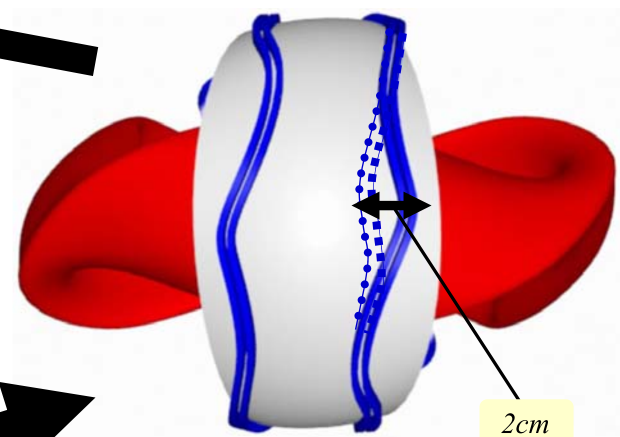
- Standard numerical optimization methods can be used to **design non-axisymmetric experiments with "good flux surfaces"**

- Example : In the design of the **National Compact Stellarator Experiment** (NCSX), small changes in the coil geometry were used to **remove resonant error fields**

"healed" equilibrium



"equilibrium" with chaos



small changes to coil geometry
→ remove resonant fields
→ stable, high-pressure plasma
→ quasi-symmetry
→ satisfy engineering constraints

Eliminating islands in high-pressure, free-boundary stellarator MHD equilibrium solutions
S.R. Hudson et al., Physical Review Letters 89:275003-1, 2002
Invited Talk 19th IAEA Fusion Energy Conference, 2002

Brief History of MHD equilibrium theory

1958 An Energy Principle for hydromagnetic stability problems

I.B. Bernstein, E.A. Freiman, M.D. Kruskal & R.M. Kulsrud

$$\mathrm{W} \equiv \int_V \left[\frac{p}{\gamma - 1} + \frac{B^2}{2} \right] dv$$

plasma displacement $\quad \mathbf{x} \rightarrow \mathbf{x} + \boldsymbol{\xi}$

ideal plasma response $\quad \dfrac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E}, \quad \mathbf{E} + \mathbf{v} \times \mathbf{B} = 0,$

$$\delta \mathrm{W} = \int_V (\nabla p \text{ - } \mathbf{j} \times \mathbf{B}) \cdot \boldsymbol{\xi} \;\; dv$$

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1954 **KAM theorem**

1962 A.N. Kolmogorov (1954), J. Moser (1962), V.I. Arnold (1963)

1967 Toroidal confinement of plasma

H. Grad *"very pathological pressure distribution"*

islands and chaos not allowed by ideal variations

dense set of singular currents

plasma variations are over constrained

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1983 3D ideal equilibrium codes

1984 BETA Garabedian et al., VMEC Hirshman et al.

(other "codes" that are ill-posed, include non-ideal effects,…)

minimize W allowing ideal variations
do not allow islands & chaos
cannot resolve singular currents, fail to converge

(If you don't get the mathematics correct, the "numerics" won't work.)

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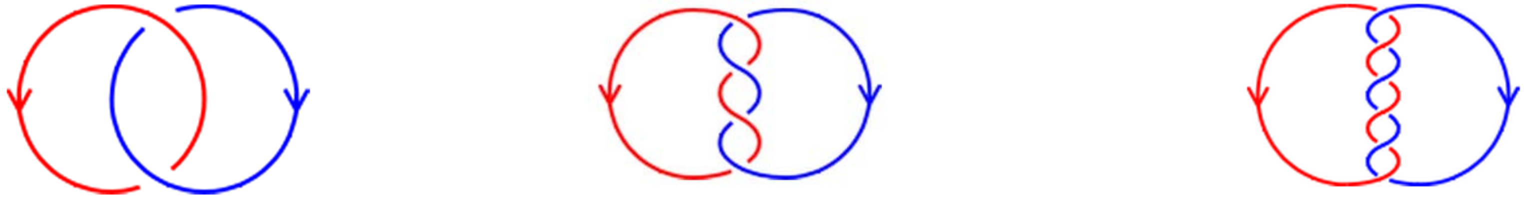
IDEAL VARIATIONS OVERLY CONSTRAIN THE TOPOLOGY, LEAD TO A DENSE SET OF SINGULARTIES

HOWEVER, IF THE VARIATION IS UNCONSTRAINED, THEN THE MINIMIZING STATE IS TRIVIAL
i.e. vacuum.

Q) What constrains a weakly resistive plasma?

A) Plasmas cannot easily untangle themselves



Question 1. (topology) *how many times do two closed curves loop through each other?*

$$\text{Gauss linking number} = -\frac{1}{4\pi} \oint \oint \frac{\mathbf{r}}{r^3} \times \, d\mathbf{y} \cdot \, d\mathbf{x}, \qquad \text{where } \mathbf{r} \equiv \mathbf{y} - \mathbf{x}$$

Question 2. (topology) *how 'knotted' is a magnetic field?*

$$\begin{aligned} \text{Helicity} &= -\frac{1}{4\pi} \int \int \frac{\mathbf{r}}{r^3} \times \mathbf{B}(\mathbf{y}) \cdot \mathbf{B}(\mathbf{x}) \, d^3x \, d^3y \\ &= \int \mathbf{A} \cdot \mathbf{B} \, d^3x \end{aligned}$$

$$\text{Coulomb gauge vector potential} \quad \mathbf{A}(\mathbf{x}) = -\frac{1}{4\pi} \int \frac{\mathbf{r}}{r^3} \times \mathbf{B}(\mathbf{y}) \, d^3y$$

Question 3. (physics) *are there simple principles that govern plasma confinement?*

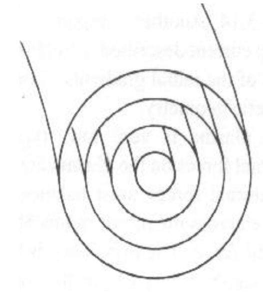
HYPOTHESIS OF TAYLOR RELAXATION

→Weakly resistive plasmas will relax to minimize the energy (and some flux surfaces may break), but the plasma cannot easily "untangle" itself i.e. constraint of conserved helicity

→Minimize $W \equiv \textit{Energy}$, subject to constraint of conserved $H \equiv \textit{Helicity} = H_0$

→Taylor relaxed state is a linear force free field, $\nabla \times \boldsymbol{B} = \mu \, \boldsymbol{B}$

Multi-Region, Relaxed MHD is a generalization of ideal MHD and Taylor relaxation



Step 1 : partition the plasma into N nested volumes (allows for non-trivial global pressure)

Step 2 : define Energy and Helicity integrals (local to each volume)

$$W_l = \underbrace{\int_{V_l} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{\text{energy}}, \qquad H_l = \underbrace{\int_{V_l} \left(\mathbf{A} \cdot \mathbf{B} \right) dv}_{\text{helicity}},$$

where $\mathbf{B} = \nabla \times \mathbf{A}$ and $pV^{\gamma} = const.$

Step 3 : construct **multi-region, relaxed MHD energy functional**, called **MRXMHD**

$$\boxed{F \equiv \sum_l \left[W_l - \frac{\mu}{2} \left(H_l - H_{l,o} \right) \right]}$$

Step 4 : The extremizing solutions satisfy the Euler-Lagrange equations $\boxed{\nabla \times \mathbf{B}_l = \mu_l \mathbf{B}_l \quad [[\mathrm{p}+\mathrm{B}^2/2]] = 0}$

relaxed Taylor state in each volume
continuity of total pressure across volume interfaces
rotational-transform on ideal interfaces is a Fibonacci irrational

Bruno & Laurence, 1996
stepped pressure equilibria are guaranteed to exist

Step 5 : Numerical implementation, **Stepped Pressure Equilibrium Code (SPEC),**

(1) uses mixed Fourier, finite-element representation for magnetic vector potential, ***A***, and geometry (2) calculation parallelized over volumes
(3) exploits spectral condensation algorithm (4) exploits sparse linear structure of $\nabla\times\boldsymbol{B}=\mu\,\boldsymbol{B}$
(5) pre-conditioned conjugate gradient methods and/or globally convergent Newton method
(6) online documentation (7) graphical user interface

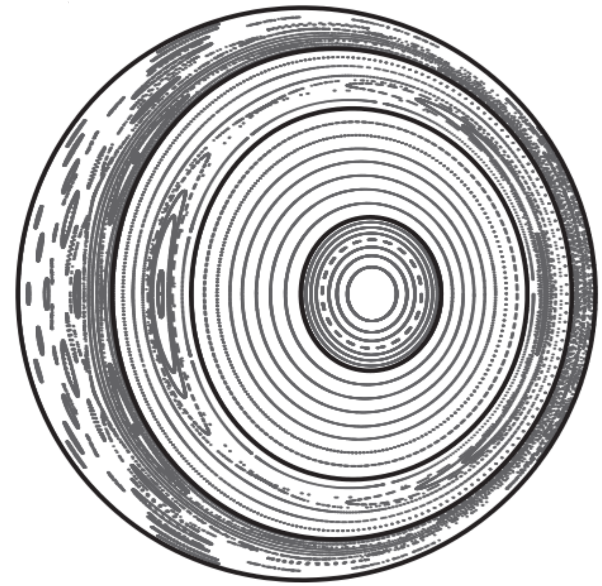
Computation of multi-region relaxed magnetohydrodynamic equilibria
S.R. Hudson, R.L. Dewar, G. Dennis, M.J. Hole, M. McGann, G.von Nessi and S. Lazerson, Physics of Plasmas, 19:112502, 2012
Invited Talk 20th International Toki Conference, 2010
Invited Talk 38th European Physical Society Conference on Plasma Physics, 2011
Invited Talk 18th International Stellarator/Heliotron Workshop, 2012

The Stepped Pressure Equilibrium Code (SPEC), has excellent convergence properties

- **First 3D equilibrium code to**

1. allow islands & chaos,
2. have a solid mathematical foundation,
3. give excellent convergence,

Poincaré Plot (of convergence calculation)
(axisymmetric equilibrium plus resonant perturbation)



approximate solution ***exact solution***

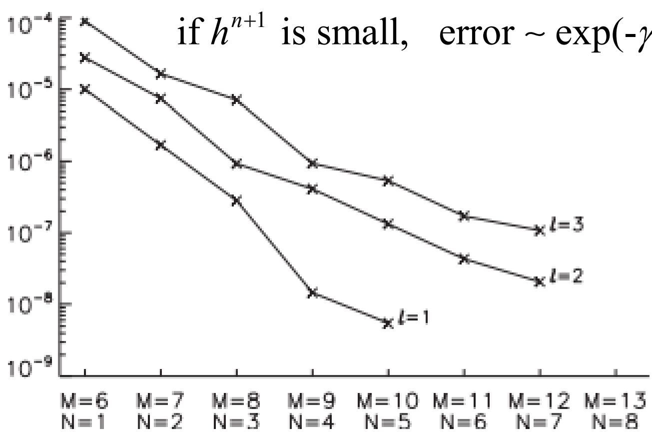
$$\text{error} \equiv f_{h,M} - f = \underbrace{\alpha\ h^{n+1} + \beta\ \exp(-\gamma M)}$$

h = 1/N radial resolution
M Fourier resolution

error decreases as N=h^{-1}, M increase

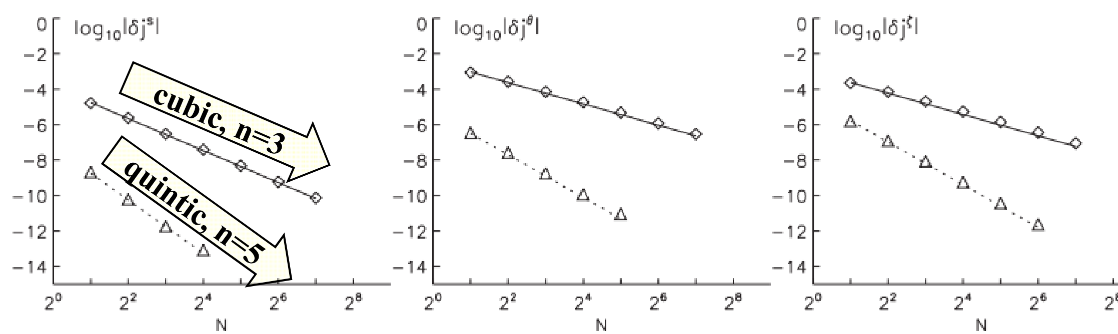
convergence with respect to Fourier resolution, M

if h^{n+1} is small, error ~ $\exp(-\gamma M)$



convergence with respect to radial resolution, N

if $\exp(-\gamma M)$ is small, error ~ h^{n+1}

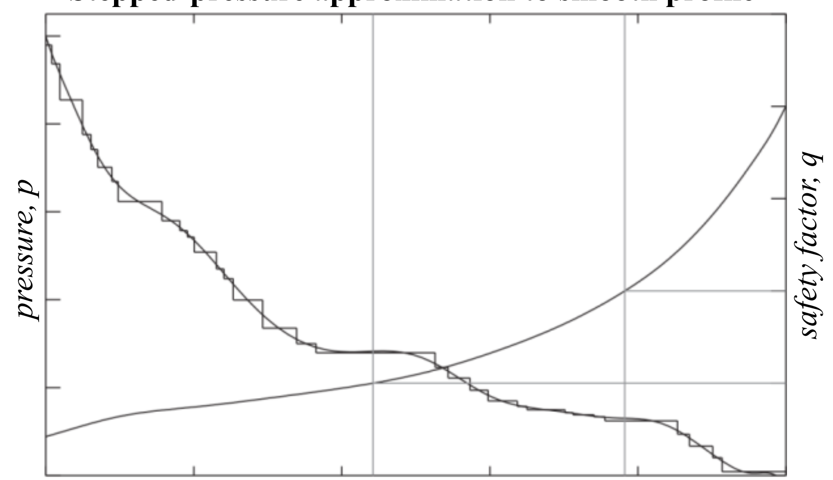


Equilibrium reconstruction of 3D plasmas is where "theory meets experiment"

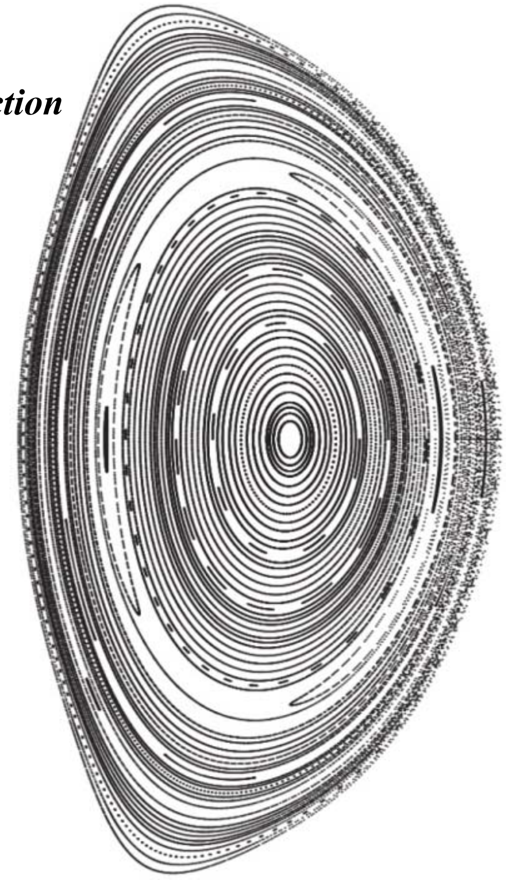
- **Consider a DIIID experimental shot, with applied three-dimensional "error fields"**
  (used to suppress edge instabilities)

- **Vary the equilibrium parameters until "numerical" diagnostics match observations**
  (i.e. plasma boundary, pressure and current profiles), (e.g. Thomson scattering, motional Stark effect polarimetry, magnetic diagnostics)

Stepped-pressure approximation to smooth profile



DIIID Poincaré Plot
experimental reconstruction using SPEC



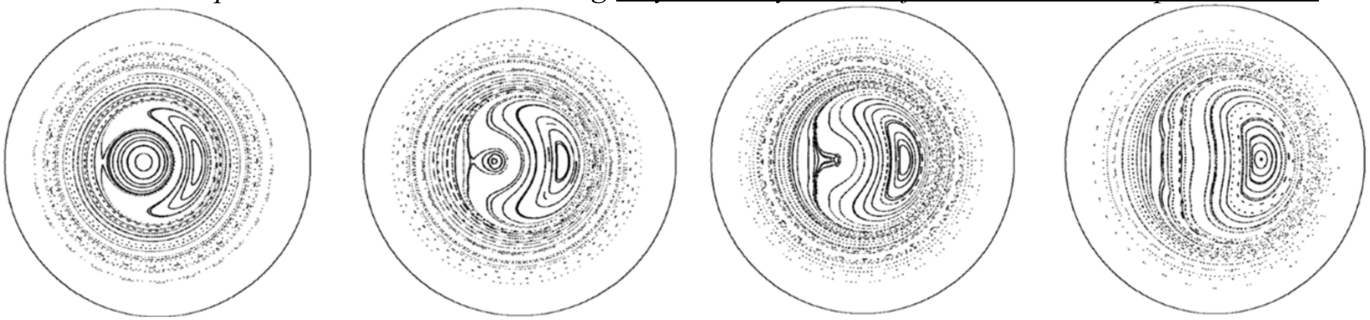
- **SPEC is being incorporated into the "STELLOPT" equilibrium reconstruction code**
  by Dr. S. Lazerson, a post-doctoral fellow at Princeton Plasma Physics Laboratory

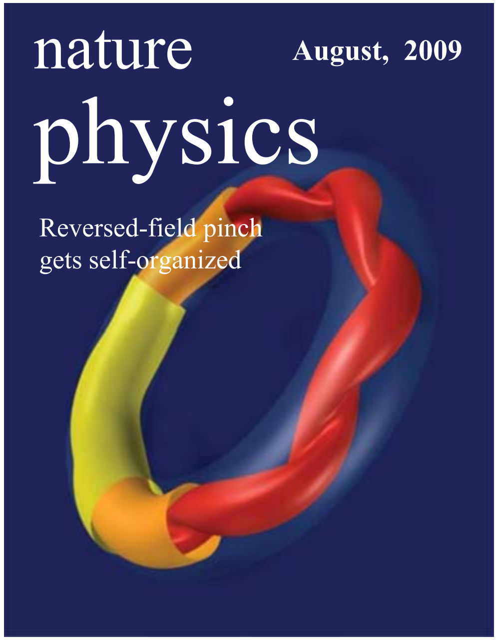
3D Equilibrium Effects due to RMP application on DIII-D
S. Lazerson, E. Lazarus, S. Hudson, N. Pablant, D. Gates
39th European Physical Society Conference on Plasma Physics, 2012

MRXMHD explains self-organization of Reversed Field Pinch into internal helical state

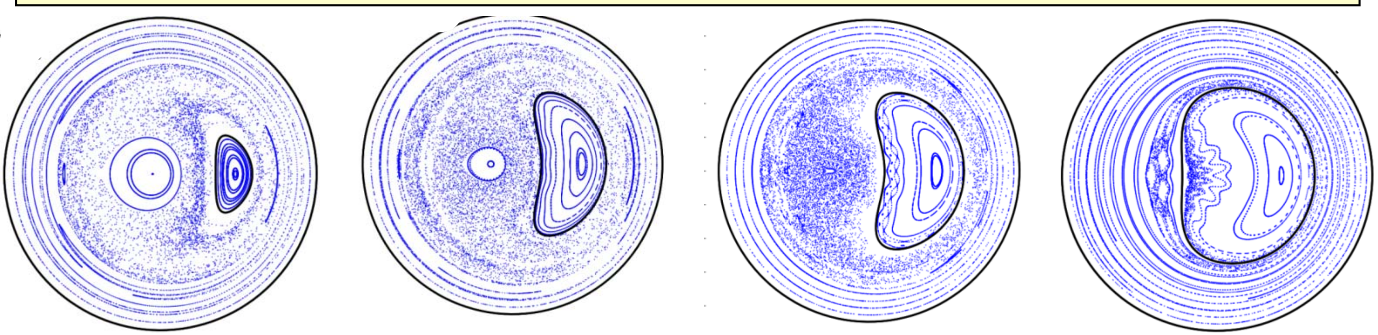
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nuclear Fusion, 49 (2009) 104019*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation"





NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
Taylor relaxation and reversed field pinches
G. Dennis, R. Dewar, S. Hudson, M. Hole, 2012 20th Australian Institute of Physics Congress



Excellent Qualitative agreement **between numerical calculation and experiment**

→ this is first (and perhaps only?) equilibrium model able to explain internal helical state with two magnetic axes
→ publication presently being prepared by Dr. G Dennis, a post-doctoral fellow at the Australian National University

Brief History of MHD equilibrium theory

1958 An Energy Principle for hydromagnetic stability problems
I.B. Bernstein, E.A. Freiman, M.D. Kruskal & R.M. Kulsrud

$$W \equiv \int_V \left[\frac{p}{\gamma - 1} + \frac{B^2}{2} \right] dv$$

1954 KAM theorem
1962 A.N. Kolmogorov (1954), J. Moser (1962), V.I. Arnold (1963)

1967 Toroidal confinement of plasma
H. Grad *"very pathological pressure distribution"*

1974 Relaxation of Toroidal Plasma and Generation of Reverse Magnetic Fields
J.B. Taylor *relax the ideal constraints, include helicity constraint*

$$H = \int_V (\mathbf{A} \cdot \mathbf{B})\, dv$$

1996 Existence of Three-Dimensional Toroidal MHD equilibria with Nonconstant Pressure
O.P. Bruno & P. Laurence *". . . our theorems insure the existence of sharp boundary solutions . . ."*
i.e. stepped pressure equilibria are well defined

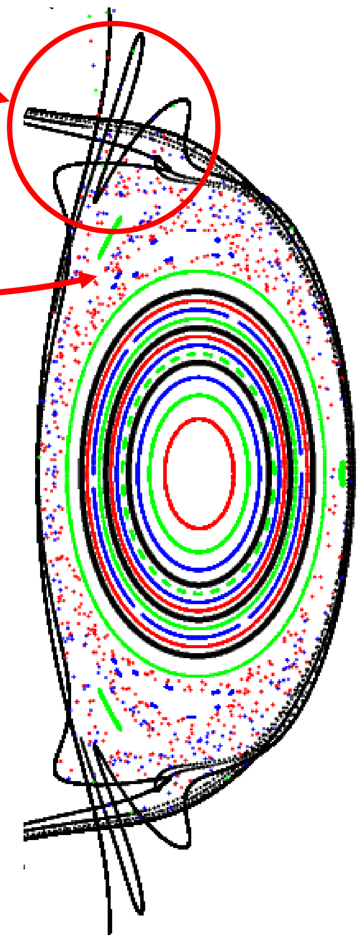
$$\nabla \times \mathbf{B}_l = \mu_l \mathbf{B}_l \qquad [[p + B^2/2]] = 0$$

2012 Computation of Multi-Region, Relaxed Magnetohydrodynamic Equilibria
S.R. Hudson, R.L. Dewar et al.
- *chaotic equilibria with arbitrary pressure*
- *combines ideal MHD and Taylor relaxation*

for $N \to \infty$, recover globally-constrained, ideal MHD
for $N=1$, recover globally-relaxed Taylor force-free state

$$F \equiv \sum_l (W_l - \mu_l H_l / 2)$$

Ongoing research activities



1. **Compute "free-boundary", partially-chaotic equilibria that are supported by vacuum fields with a chaotic-tangle**
    - investigate structure of chaotic-tangle that surrounds a high-pressure plasma
    - explore relationship between cantori and unstable manifold near plasma edge
    - explore how transport through chaotic edge is restricted by cantori

2. **Is the suppression of edge-localized modes by partial chaos a manifestation of multi-region, relaxed MHD instability phenomena?**
    - MRXMHD allows the stability of partially chaotic equilibria to be defined and calculated
    - to what extent do applied error fields ergodize the plasma edge?
    - can experimental disruptions be understood using multi-region, relaxed MHD?

3. **Equilibrium reconstruction calculations**
    - can numerical calculations predict experimental observations?
    - which equilibrium model best fits experiment?

4. **Development of transport model**
    - how do charged particles move through partially-chaotic, stepped-pressure equilibria?

5. **Compute "critical-pressure gradient", explore avalanche phenomena**
    - what is the most pressure a flux surface can support before it is destroyed?
    - do all the flux-surfaces collapse simultaneously (i.e. avalanche) when pressure exceeds a certain threshold?

For full list of publications & presentations by Dr. S.R. Hudson on multi-region, relaxed MHD

http://w3.pppl.gov/~shudson/bibliography.html

http://w3.pppl.gov/~shudson/spec.html#subroutines

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

OSCAR P. BRUNO
California Institute of Technology

PETER LAURENCE
Universita di Roma "La Sapienza"

We establish an existence result for the three-dimensional MHD equations

$$
\begin{aligned}
(\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p \\
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{B} \cdot n|_{\partial T} &= 0
\end{aligned}
$$

with $p \neq$ const in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

Communications on Pure and Applied Mathematics, Vol. XLIX, 717–764 (1996)

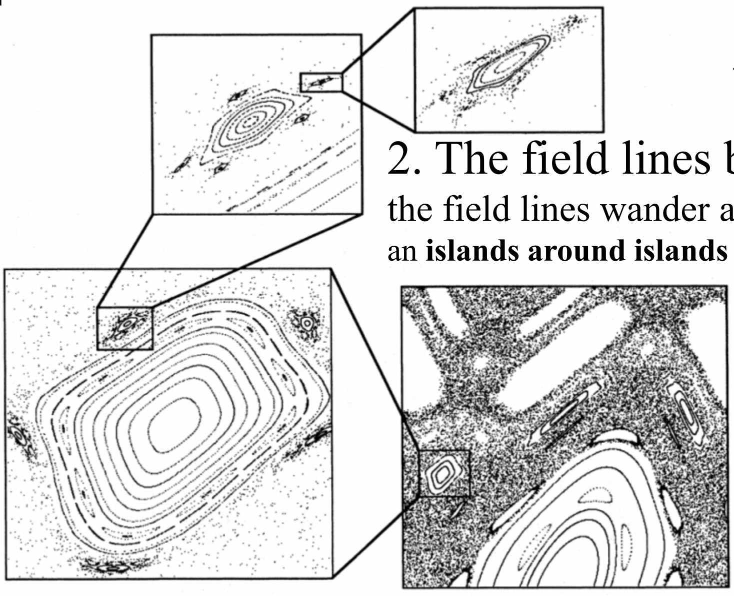
With increasing non-axisymmetry, the flux surfaces become increasingly "broken"

1.The transformation to action-angle coordinates fails where frequency $\omega = n / m$, is rational

classical problem of small denominators, resonance zones ~ magnetic islands

invariant flux surfaces are destroyed

separatrix splitting, *chaotic "tangle"*



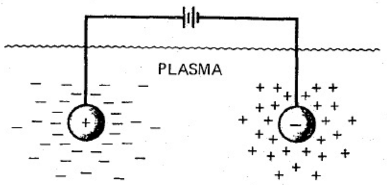
2. The field lines become "chaotic"

the field lines wander about *seemingly randomly*

an **islands around islands around islands** fractal hierarachy

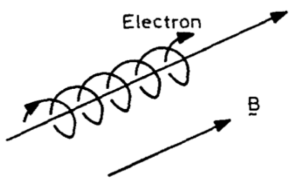
A plasma is a gas of charged particles

1. *electrically conducting, need to consider plasma currents,* $\mathbf{j}$
2. *electric fields are shielded,* $\mathbf{E} \approx 0$
3. *plasmas display "collective" behaviour, i.e. waves and oscillations*



The "physics" of magnetic confinement is based on the *Lorentz* force

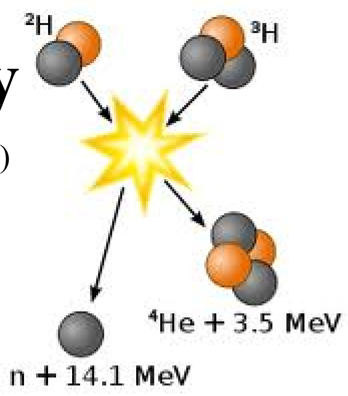
4. *the Lorentz force 'ties' the particles to the field lines,* *(but free-streaming along field lines)*



$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \qquad \text{(single particle)}$$

$$\nabla p = \mathbf{j} \times \mathbf{B} \qquad \text{Lorentz force balances gas pressure} \rightarrow \text{equilibrium}$$

Grand Challenge is to create sustainable energy



To confine a **1. hot** (fast particles can overcome Coulomb repulsion and collide)

2. dense plasma (so that many particles collide)

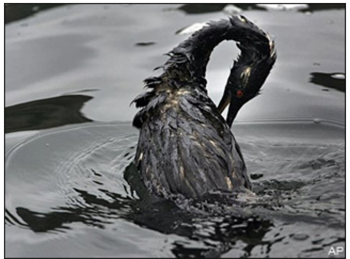
for a **3. long time** in an equilibrium state

and to have a clean, safe, 21st century

by establishing fusion as the global energy source . . . . . .

coal

oil



uranium fission

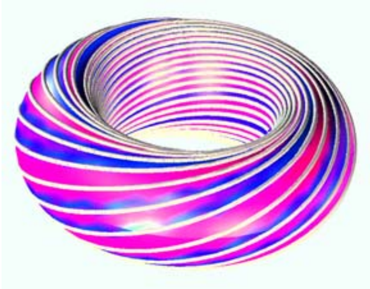
Fukushima

Fusion

1) no radioactive products

2) no uncontrollable chain reaction

Several theorems of mathematics & theoretical physics, prove that . . .



1. There are no magnetic monopoles . . .

   i.e. $\nabla \bullet \mathbf{B} = 0$, *magnetic field lines have "no end"*

2. The hairy ball theorem (of algebraic topology)

   there is no nonvanishing continuous tangent vector field on a sphere, ... but there is on a torus!

3. Noether's theorem (of theoretical physics) e.g. if the system does not depend on the angle, the angular momentum is constant

   each symmetry (i.e. ignorable coordinate) of a Hamiltonian system has a integral invariant

action integral $J = \oint pdq$, "integrable" Hamiltonian, $H(J)$

$$\frac{dJ}{dt} \equiv -\frac{\partial H}{\partial \theta} = 0, \quad \text{frequency } \omega \equiv \frac{\partial H}{\partial J}$$

the action is constant and the angle increases linearly with time

4. Toroidal magnetic fields are Hamiltonian *can use all the methods of Hamiltonian and Lagrangian mechanics!*

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Action Integral $S = \int_C L(\theta, \dot{\theta}, t)dt$

$$\mathbf{B} = \nabla \times \left(\psi \nabla \theta - \chi \nabla \phi + \nabla gauge \right)$$

field line satisfies Hamilton's Equations $\boxed{\dfrac{d\theta}{d\phi} = \dfrac{\partial \chi}{\partial \psi}, \quad \dfrac{d\psi}{d\phi} = -\dfrac{\partial \chi}{\partial \theta}}$

$$S = \int_C \mathbf{A.dl}$$

$\psi \equiv$ *canonical momentum*

$\chi \equiv$ *Hamiltonian*

Trajectories are extremal curves of action-integral

Theorems from Hamiltonian chaos theory provide a solid foundation

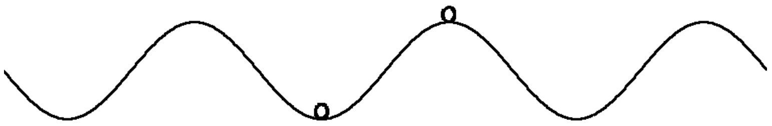
1. Poincaré-Birkhoff Theorem

magnetic field-line action $= \int_{curve} \mathbf{A} \cdot \mathbf{dl}$

curves that extremize the action integral are field lines

For every rational, $\omega = n / m$, where n, m are integers,

- a periodic field-line that is a *minimum* of the action integral will exist
- a . . . . *saddle* . . . . will exist



2. Aubry-Mather Theorem

For every $\omega \neq n / m$,

- there exists an "irrational" field-line that is a *minimum* of the action integral

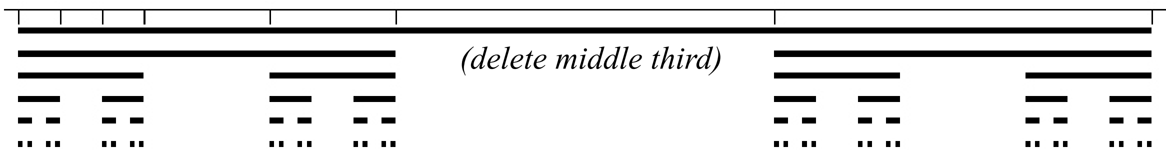
3. Kolmogorov-Arnold-Moser Theorem

- if ω is *very* irrational then the Aubry-Mather field line will cover a surface, called a KAM surface

  ω is very irrational if there exist an r, k such that $|\omega - n / m| > r\, m^{-k}$, for all integers n, m

  Diophantine condition

- if not, the Aubry-Mather field line will cover a Cantor set, called a cantorus



4. Greene's residue criterion

- *the existence of a KAM surface is related to the stability of the nearby Poincaré-Birkhoff periodic orbits*

An ideal equilibrium with non-integrable (*chaotic*) field and *continuous* pressure, is *infinitely discontinous*

ideal MHD theory = $\nabla p = \mathbf{j}\times\mathbf{B}$, gives $\mathbf{B}\cdot\nabla p = 0$

→ transport of pressure along field is "infinitely" fast
→ no scale length in ideal MHD
→ pressure adapts to fractal structure of phase space

chaos theory = *nowhere are flux surfaces continuously nested*

*for non-symmetric systems, nested family of flux surfaces is destroyed

*islands & irregular field lines appear where transform is rational (n/m); rationals are dense in space

Poincare-Birkhoff theorem → periodic orbits, (e.g. stable and unstable) guaranteed to survive into chaos

*some irrational surfaces survive if there exists an $r, k \in \Re$ s.t. for all rationals, $|\iota - n/m| > r\, m^{-k}$

i.e. rotational-transform, ι, is *poorly approximated* by rationals,

Diophantine Condition
Kolmogorov, Arnold and Moser

ideal MHD + chaos → infinitely discontinuous equilibrium

$$p' = \begin{cases} 0, \text{if } \exists (m,n) \text{ s.t. } |x - n/m| < r\, m^{-k} \\ 1, \text{otherwise} \end{cases}$$

*iterative method for calculating equilibria is ill-posed;

1) $\mathbf{B}_n \cdot \nabla p = 0$ — ∇p is everywhere discontinuous, or zero;

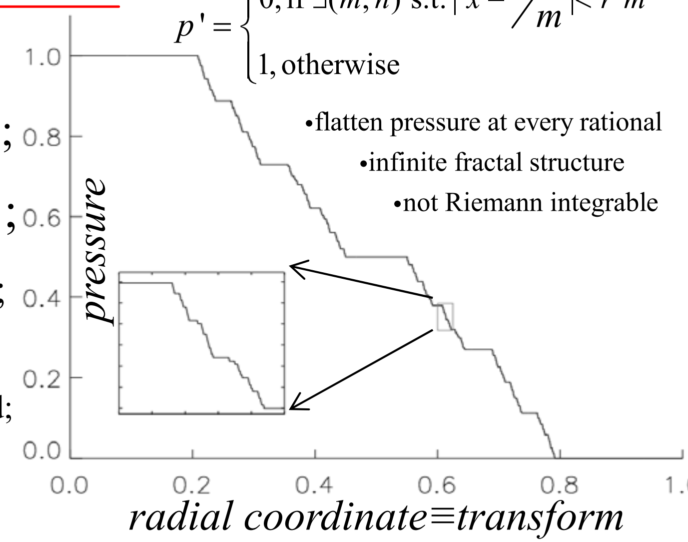
2) $\mathbf{j}_\perp = \mathbf{B}_n \times \nabla p / B_n^2$ — $\mathbf{j}_\perp$ everywhere discontinuous or zero;

3) $\mathbf{B}_n \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$ — $\mathbf{B}\cdot\nabla$ is *densely and irregularly* singular;

σ is single valued if and only if $\delta\sigma = -\oint_C \nabla\cdot\mathbf{j}_\perp dl / B = 0$

pressure must be flat across every closed field line, or parallel current is not single-valued;

4) $\nabla\times\mathbf{B}_{n+1} = \mathbf{j} \equiv \sigma\mathbf{B}_n + \mathbf{j}_\perp$ — solution only if $\nabla\cdot(\sigma\mathbf{B} + \mathbf{j}_\perp) = 0$



To have a well-posed equilibrium with chaotic **B** need to

→ introduce non-ideal terms, such as resistivity, η, perpendicular diffusion, $\kappa_\perp$, [*HINT*, *M3D*, *NIMROD*, ..],

→ or return to an energy principle, but relax infinity of ideal MHD constraints

Extrema of energy functional obtained numerically; introducing the Stepped Pressure Equilibrium Code (SPEC)

The vector-potential is discretized

* toroidal coordinates (s,ϑ,ζ), *interface geometry $R_l = \sum_{m,n} R_{l,m,n}\cos(m\vartheta - n\zeta), Z_l = \sum_{m,n} Z_{l,m,n}\sin(m\vartheta - n\zeta)$

* exploit gauge freedom $\mathbf{A} = A_\vartheta(s,\vartheta,\zeta)\nabla\vartheta + A_\zeta(s,\vartheta,\zeta)\nabla\zeta$

* Fourier $A_\vartheta = \sum_{m,n} a_\vartheta(s)\cos(m\vartheta - n\zeta)$

* Finite-element $a_\vartheta(s) = \sum_i a_{\vartheta,i}(s)\varphi(s)$ ***piecewise cubic or quintic basis polynomials***

and inserted into constrained-energy functional $F = \sum_{l=1}^{N}\left(W_l - \mu_l H_l/2 - \nu_l M_l\right)$

* derivatives w.r.t. vector-potential $\rightarrow$ linear equation for Beltrami field $\nabla\times\mathbf{B} = \mu\mathbf{B}$ ***solved using sparse linear solver***

* field in each annulus computed independently, distributed across multiple cpus

* field in each annulus depends on enclosed toroidal flux (boundary condition) and

$\rightarrow$ poloidal flux, ψ_P, and helicity-multiplier, μ ***adjusted so interface transform is strongly irrational***

$\rightarrow$ geometry of interfaces, $\xi \equiv \{R_{m,n}, Z_{m,n}\}$

Force balance solved using multi-dimensional Newton method.

* interface geometry is adjusted to satisfy force $\mathbf{F}[\xi] \equiv \left\{[[p + B^2/2]]_{m,n}\right\} = 0$

* angle freedom constrained by spectral-condensation, adjust angle freedom to minimize $\sum (m^2 + n^2)\left(R_{mn}^2 + Z_{mn}^2\right)$

minimal spectral width [Hirshman, VMEC]

* derivative matrix, $\nabla\mathbf{F}[\xi]$, computed in parallel using finite-differences

* call NAG routine: quadratic-convergence w.r.t. Newton iterations; robust convex-gradient method;

Numerical error in Beltrami field scales as expected

Scaling of numerical error with radial resolution depends on finite-element basis

$\mathbf{A} = A_\vartheta \nabla\vartheta + A_\zeta \nabla\zeta, \quad \mathbf{B} = \nabla\times\mathbf{A}, \quad \mathbf{j} = \nabla\times\mathbf{B},$ need to quantify **error** = **j** - μ**B**

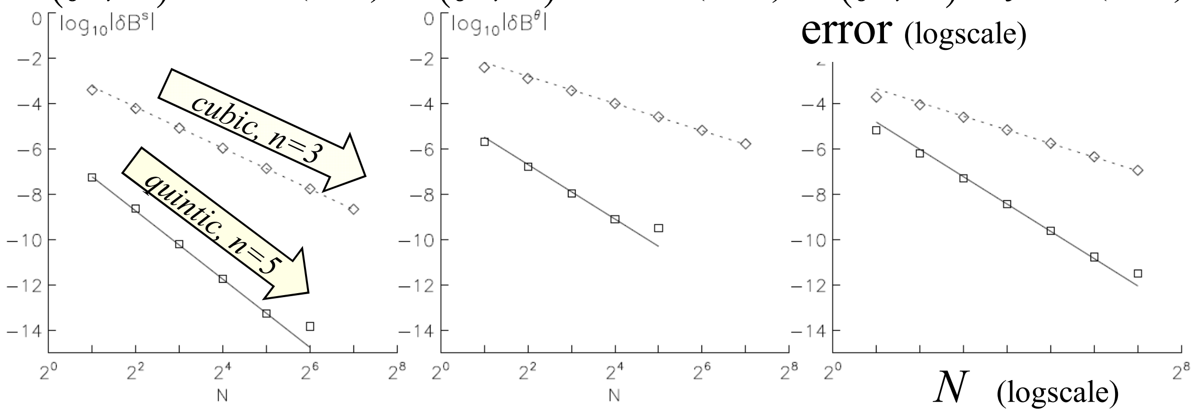
$A_\vartheta, A_\zeta \sim O(h^n)$

h = radial grid size = 1 / N
n = order of polynomial

$$\sqrt{g}B^s = \partial_\vartheta A_\zeta - \partial_\zeta A_\vartheta \sim O(h^n)$$
$$\sqrt{g}B^\vartheta = \quad -\partial_s A_\zeta \sim O(h^{n-1})$$
$$\sqrt{g}B^\zeta = \partial_s A_\vartheta \quad \sim O(h^{n-1})$$

$$\sqrt{g}j^s \sim O(h^{n-1})$$
$$\sqrt{g}j^\vartheta \sim O(h^{n-2})$$
$$\sqrt{g}j^\zeta \sim O(h^{n-2})$$

$(\mathbf{j} - \mu\mathbf{B})\bullet\nabla s \sim O(h^{n-1})$ $\quad(\mathbf{j} - \mu\mathbf{B})\bullet\nabla\vartheta \sim O(h^{n-2})$ $\quad(\mathbf{j} - \mu\mathbf{B})\bullet\nabla\zeta \sim O(h^{n-2})$

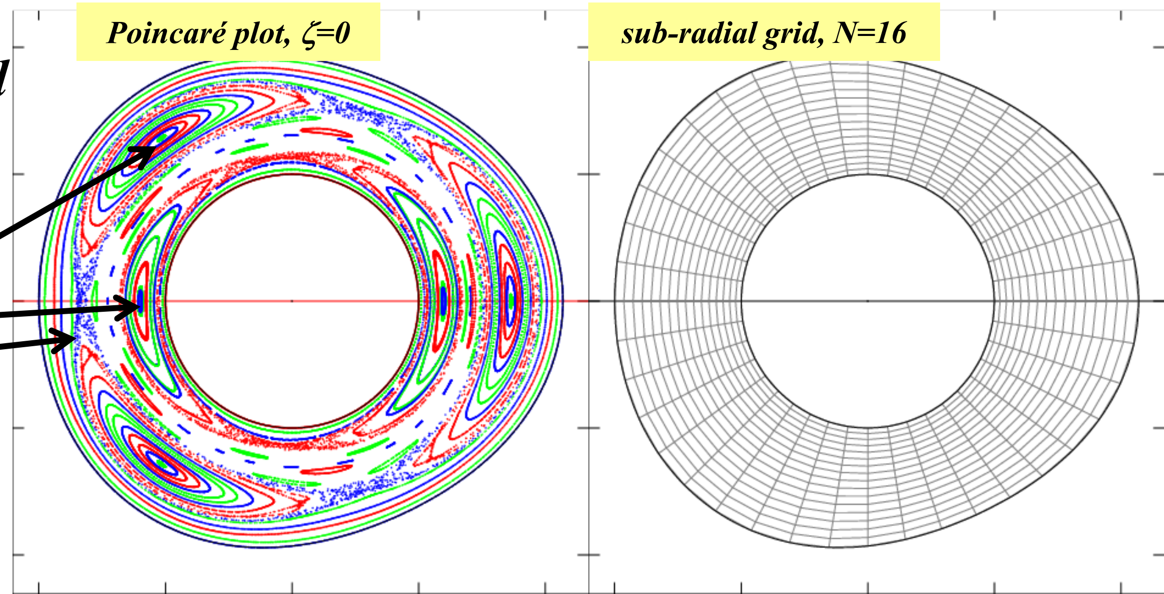


Example of chaotic Beltrami field in single given annulus;

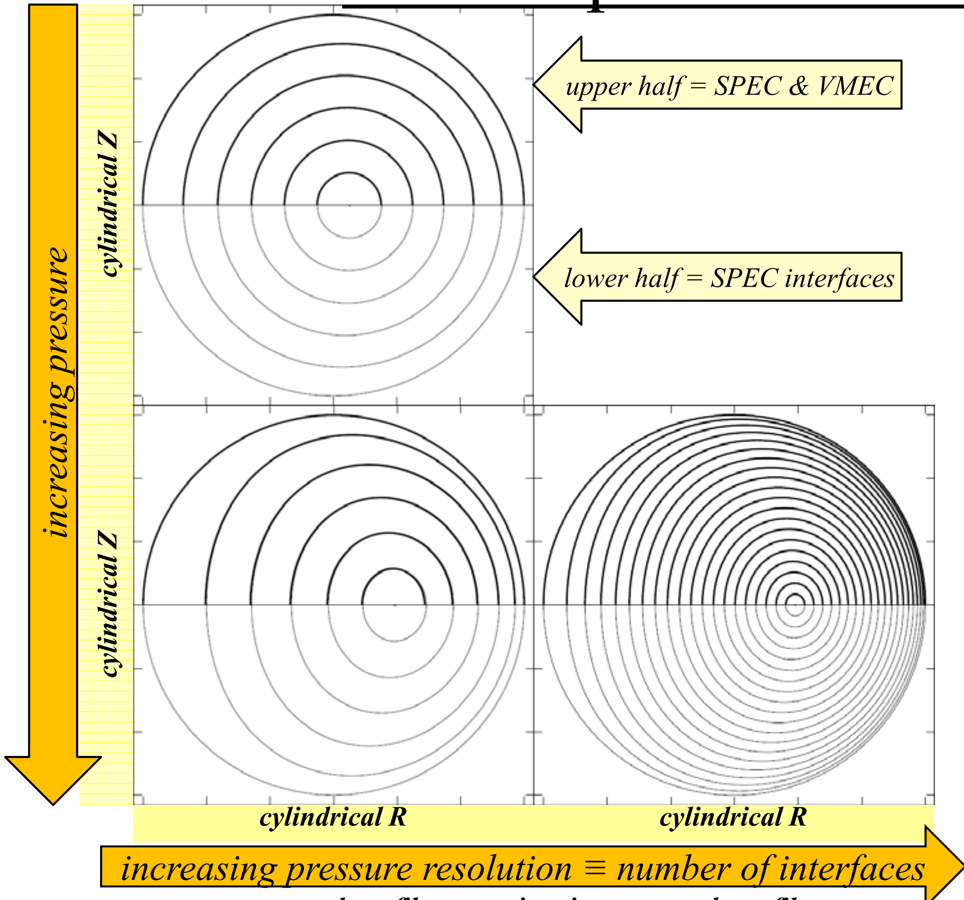
$$R = 1.0 + r(\vartheta,\zeta)\cos\vartheta,$$
$$Z = \quad r(\vartheta,\zeta)\sin\vartheta,$$

(m,n)=(3,1) island
+ (m,n)=(2,1) island
= chaos

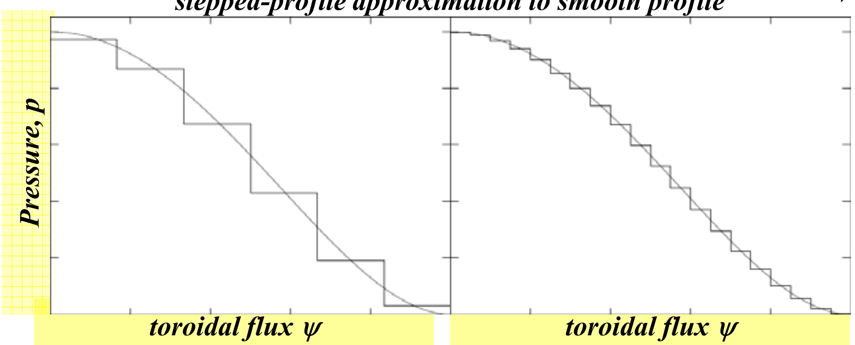
inner surface
$r = 0.1$
outer interface
$r = 0.2 \ + \delta\left[\cos(2\vartheta-\zeta) + \cos(3\vartheta-\zeta)\right]$



Stepped-pressure equilibria accurately approximate smooth-pressure *axisymmetric* equilibria



stepped-profile approximation to smooth profile



in axisymmetric geometry . . .

→ magnetic fields have family of nested flux surfaces

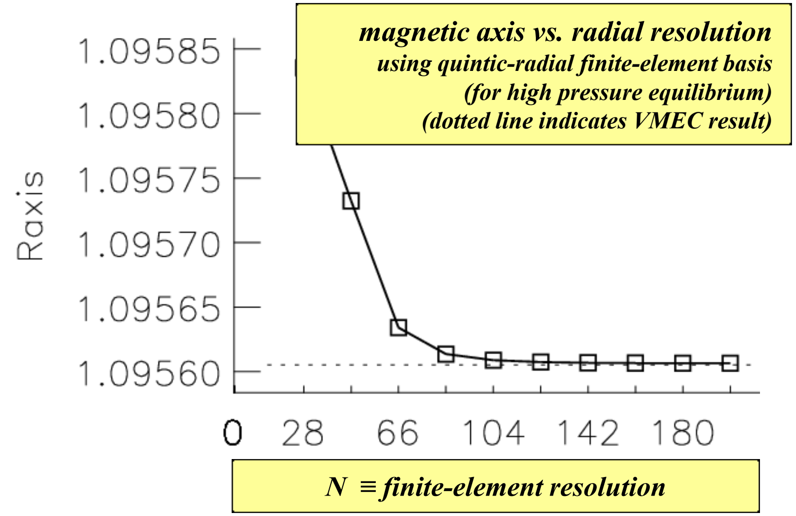
→ equilibria with smooth profiles exist,

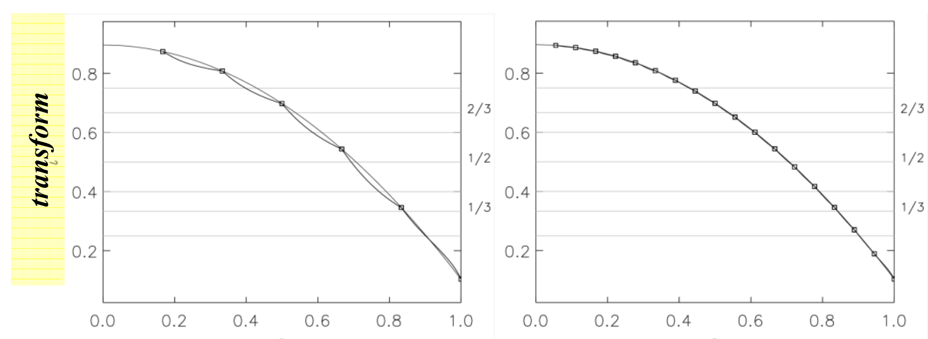
→ may perform benchmarks (e.g. with VMEC)

(arbitrarily approximate smooth-profile with stepped-profile)

→ approximation improves as number of interfaces increases

→ location of magnetic axis converges w.r.t radial resolution





Force balance condition at interfaces gives rise to auxilliary pressure-jump Hamiltonian system.

$\rightarrow$ Beltrami condition, $\nabla\times\mathbf{B}=\mu\mathbf{B}$, and interface constraint, $\mathbf{B}\cdot\mathbf{n}=0$, gives $\nabla\times\mathbf{B}\cdot\nabla s=0$,

suggests surface potential, $B_\vartheta=\partial_\vartheta f$, $B_\zeta=\partial_\zeta f$, so that $\partial_\vartheta B_\zeta-\partial_\zeta B_\vartheta=0$,

$$B^2=(g_{\vartheta\vartheta}f_\zeta f_\zeta-2g_{\vartheta\zeta}f_\vartheta f_\zeta+g_{\zeta\zeta}f_\vartheta f_\vartheta)\big/(g_{\vartheta\vartheta}g_{\zeta\zeta}-g_{\vartheta\zeta}g_{\vartheta\zeta}),$$ metric elements $g_{\alpha\beta}\equiv\partial_\alpha\mathbf{x}\cdot\partial_\beta\mathbf{x}$

$\rightarrow$ Force balance condition, $[[p+B^2/2]]=0$, introduce $H\equiv 2(p_1-p_2)=B_2^2-B_2^1=const.$

$\rightarrow$ Let tangential field on "inner-side" of interface be given, $B_{1\vartheta}=\partial_\vartheta f$, $B_{1\zeta}=\partial_\zeta f$,

tangential field on "outer-side", $B_{2\vartheta}=p_\vartheta$, $B_{2\zeta}=p_\zeta$, determined by characteristics

$$\dot{\vartheta}=\left.\frac{\partial H(\vartheta,\zeta,p_\vartheta,p_\zeta)}{\partial p_\vartheta}\right|_{\zeta,p_\vartheta,p_\zeta},\quad \dot{p}_\vartheta=-\frac{\partial H}{\partial\vartheta},\quad \dot{\zeta}=\frac{\partial H}{\partial p_\zeta},\quad \dot{p}_\zeta=-\frac{\partial H}{\partial\zeta}$$

$\rightarrow$ 2 d.o.f. Hamiltonian system, and invariant surfaces only exist if "frequency" is irrational

$\Rightarrow$ ideal interfaces that support pressure must have irrational transform

Hamilton-Jacobi theory for continuation of magnetic field across a toroidal surface supporting a plasma pressure discontinuity
M. McGann, S.R.Hudson, R.L. Dewar and G. von Nessi, Physics Letters A, 374(33):3308, 2010