Plasma Relaxation Dynamics Moderated by Current Sheets

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Abstract

Ideal magnetohydrodynamics (IMHD) is strongly constrained by an infinite number of microscopic constraints expressing mass, entropy and magnetic flux conservation in each infinitesimal fluid element, the latter preventing magnetic reconnection. By contrast, in the Taylor-relaxed equilibrium model all these constraints are relaxed save for global magnetic flux and helicity.

A Lagrangian is presented that leads to a new variational formulation of magnetized fluid dynamics, *relaxed MHD* (RxMHD), all static solutions of which are Taylor equilibrium states. By postulating that some long-lived macroscopic current sheets can act as barriers to relaxation [1], separating the plasma into multiple relaxation regions, a further generalization, *multi-relaxed MHD* (MRxMHD), is developed.

These concepts are illustrated using a simple two-region slab model similar to that proposed by Hahm and Kulsrud — the formation of an initial shielding current sheet after perturbation by boundary rippling is calculated using MRxMHD and the final island state, after the current sheet has relaxed through a reconnection sequence [2], is calculated using RxMHD.

[1] *Helical bifurcation and tearing mode in a plasma — a description based on Casimir foliation*
Z Yoshida & RL Dewar, J Phys A **45** 365502 (2012);

[2] *Plasmoid solutions of the Hahm–Kulsrud–Taylor equilibrium model*
RL Dewar, A Bhattacharjee, RM Kulsrud and AM Wright, Phys Plasmas **20**, 082103 (2013)

Generalizations of Taylor Relaxation

This presentation

- Shows there is a reduced magneto-hydro-*dynamics* that leads to Taylor's relaxed *equilibrium* states in the static limit by using Hamilton's Principle to derive self-consistent dynamics from a *relaxed MHD* (RxMHD) Lagrangian.
- Calculates the modulated current sheet driven by a resonant perturbation at a rational surface by treating the plasma as *two* relaxation regions – 2-region example of *multi-relaxed MHD* (MRxMHD)

Hamilton's Action Principle in domain Ω: $\delta S = 0$

$S = \int dt \int_\Omega \mathcal{L}\, d^3x$ denotes the *action*. Its *first variation* is:

$$\delta S = \int dt \int_\Omega \delta\mathcal{L}\, d^3x + \epsilon \int dt \int_{\partial\Omega} \mathcal{L}\, \boldsymbol{\xi}\cdot\mathbf{n} dS$$

$\delta\mathcal{L}$ is $O(\epsilon)$ *Eulerian* variation of action density $\mathcal{L}$,
$\epsilon\boldsymbol{\xi}$ is *Lagrangian* displacement of fluid element positions $\mathbf{r}$ on boundary $\partial\Omega$

MHD Lagrangian density is

$$\mathcal{L}_{\mathrm{MHD}} = \frac{\rho\mathbf{v}\cdot\mathbf{v}}{2} - \frac{p}{\gamma - 1} - \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}$$

where $\mathbf{v} = d\mathbf{r}/dt$ is velocity, ρ is mass density, p is pressure and $\mathbf{B}$ is magnetic field

Holonomic constraints

- IMHD = Ideal MHD ($\rho, \mathbf{B}$ and p *holonomically* constrained, i.e. *locally* "frozen in" to fluid elements):
$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi}),\ \delta p = -\epsilon(\boldsymbol{\xi}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{\xi}),\ \delta\mathbf{B} = \nabla\times\delta\mathbf{A}$$
$$\delta\mathbf{A} = \epsilon\boldsymbol{\xi}\times\mathbf{B} + \nabla\delta\chi$$
- RxMHD = Relaxed MHD (only ρ holonomically constrained — no effect on static equilibrium — magnetic helicity and entropy constrained only *globally*):
$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi})$$
- MRxMHD = Multi-Relaxed MHD (multiple RxMHD regions Ω_i separated by current sheet transport barriers $\partial\Omega_i$, with holonomic constraints on either side, ±, of $\partial\Omega_i$ to keep $\mathbf{B}$ tangential to the current sheets):
$$\delta\rho = -\epsilon\nabla\cdot(\rho\boldsymbol{\xi}) \text{ in } \Omega_i,\ \delta\mathbf{A}_{\rm tgt} = (\epsilon\boldsymbol{\xi}\times\mathbf{B} + \nabla\delta\chi)_{\rm tgt} \text{ on } \partial\Omega_i^{\pm}$$

Global constraints

- IMHD = Ideal MHD (none — mass, entropy and magnetic flux and helicity within Ω all automatically conserved as a consequence of the holonomic constraints):

- RxMHD = Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within Ω using Lagrange multipliers τ and μ respectively):

$$\mathcal{L} = \mathcal{L}_{\mathrm{MHD}} + \tau \frac{\rho \ln(Cp/\rho^{\gamma})}{\gamma - 1} + \mu \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0}$$

  where γ and C are thermodynamic gas constants.

- MRxMHD = Multi-Relaxed MHD (mass and flux automatic, entropy and magnetic helicity are constrained globally within the multiple RxMHD regions Ω_i using Lagrange multipliers τ_i and μ_i giving p and q profile control).

MRxMHD equations

- Continuity: $\dfrac{d\rho}{dt} = -\rho \boldsymbol{\nabla}\cdot\mathbf{v}$
- Require Hamilton's Principle: $\delta S = 0$ for all independent variations of $\mathbf{r}$, p and $\mathbf{A}$, where:

$$\delta S = \sum_i \int dt \int_{\Omega_i} \delta\mathcal{L}_i \, d^3x + \epsilon \sum_i \int dt \int_{\partial\Omega_i} \mathcal{L}_i \, \boldsymbol{\xi}\cdot\mathbf{n} dS$$

- Resulting *Euler–Lagrange* equations are:

$\rho \dfrac{d\mathbf{v}}{dt} = -\boldsymbol{\nabla} p$ (momentum equation)

$p = \tau_i \rho$ (isothermal equations of state in each region)

$\boldsymbol{\nabla}\times\mathbf{B} = \mu\mathbf{B}$ (Beltrami equations)

$\left[\!\left[p + \frac{B^2}{2\mu_0} \right]\!\right]_i = 0$ (pressure jump conditions at interfaces)

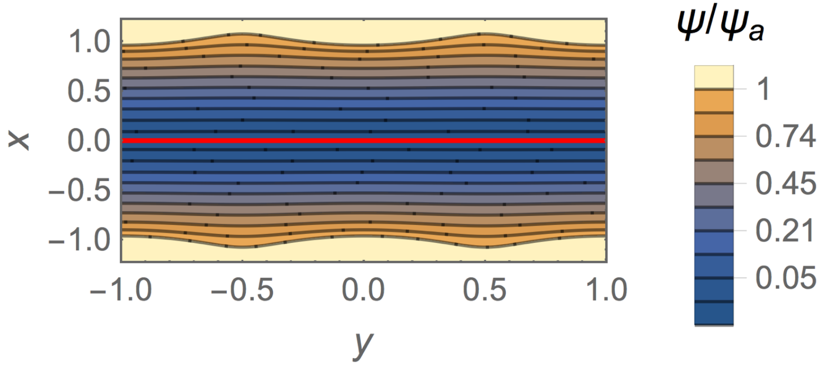
Jump and boundary conditions on a current sheet

- SPEC interfaces must be *current sheets* so a delta function $\mathbf{J}\times\mathbf{B}$ force can balance the ∇p delta function
- *Force balance* criterion is simply $\left[\!\left[p + \frac{B^2}{2} \right]\!\right] = 0$
  where $[[p]]$ denotes the *jump*, $p_+ - p_-$, between the two sides, $\pm$, of the interface
- In addition we have *tangentiality*, $\mathbf{B}\cdot\mathbf{n}$ & $\mathbf{J}\cdot\mathbf{n} = 0$, which implies the existence of two 2D *scalar* potentials $f_\pm(\theta,\zeta)$ such that $B_{\pm\theta} = \partial_\theta f_\pm$, $B_{\pm\zeta} = \partial_\zeta f_\pm$. Here ∂_i , i $=\theta,\zeta,$ are the covariant derivatives on the interface, regarded as 2D Riemannian manifold with metric $g_{i,j}$. Force balance gives *Hamilton-Jacobi equation*.

A resolution of the MRxMHD rotational transform quandary?

- A KAM argument shows 3-D toroidal *equilibrium* current sheets can in general only exist if rotational transforms on both sides of sheet are strong *irrationals*
- **But**, starting with non-equilibrium tori, relaxation of torus shape *with conserved fluxes & helicities* leads to uncontrolled *change* of rotational transforms — *no apparent relaxation mechanism to reach desired irrationals*
- In this presentation we show that a current sheet generated on a *rational* surface by a resonant perturbation causes a *jump* in rotational transform (above a small threshold in perturbation amplitude), thus *removing the resonance on the 2 sides of the current sheet* even before reconnection has occurred.

Hahm-Kulsrud Rippled Slab Model



- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$
- Hahm & Kulsrud, Phys. Fluids 1985, found 2 solutions:

- shielding current sheet on $x = 0$ (shown in red)

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

$$\psi = aB_y^a \left[\frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where B^a_y is |unperturbed poloidal field| at boundaries and $\alpha \ll 1$

2-region MRxMHD HKT model

HK-style model is natural application of MRxMHD because:

- Linearity of Beltrami equation leads to easily solvable, *linear GS equation (Poisson in small-μ limit.)*
- Symmetry about, and straightness of, current sheet at x = 0: gives most geometrically simple *2-region* geometry

Relaxation scenario:

- Switch-on: *ripple* on upper and lower boundaries slowly increased from zero (plane slab) to final amplitude
- A *shielding current* sheet at x = 0 resonance develops
- Kruskal-Kulsrud damping: evolution through *equilibria*
- Connect equilibrium sequence by *helicity conservation*

Grad-Shafranov-Beltrami equations

Grad-Shafranov equation for force-free field in slab geometry:

$\mathbf{B} = \nabla z \times \nabla\psi + F(\psi)\nabla z$ $\qquad\qquad$ $\nabla^2\psi + FF' = 0$

$\nabla \times \mathbf{B} = \mu\mathbf{B}$ (Beltrami equation) is satisfied by requiring:

$\nabla^2\psi = \mu F$ with $F(\psi) = C - \mu\psi$, giving $(\nabla^2 + \mu^2)\psi = C$

General Solution: $\psi = \overline{\psi} + \dfrac{\overline{F}}{B_0}\psi_0(x|\mu) + \widehat{\psi}(x,y)$

where $\overline{\psi}$ is cross-sectional average of ψ, $\psi_0(x|\mu) \equiv \dfrac{B_0}{\mu}(1 - \cos\mu x)$

is plane slab solution, $\overline{F}$ is the cross-sectional average of B_z ,

and $\widehat{\psi}$ obeys a *homogeneous* Beltrami equation: $(\nabla^2 + \mu^2)\widehat{\psi} = 0$

with boundary conditions such that ψ is constant on boundary and on cuts.

Extension of HK shielding solution

Helicity conservation requires *three extensions of HK solution*

Instead of the HK harmonic component ψ_1 we use ansatz

$$\widehat{\psi}(x,y) \equiv \frac{2\alpha\psi_a}{\sinh k_1 a}\left(|\sinh k_1 x| \cos ky + \gamma_{\mathrm{S}} \frac{k_1}{\mu} |\sin \mu x|\right) - \overline{\psi} \cos \mu x$$

where:

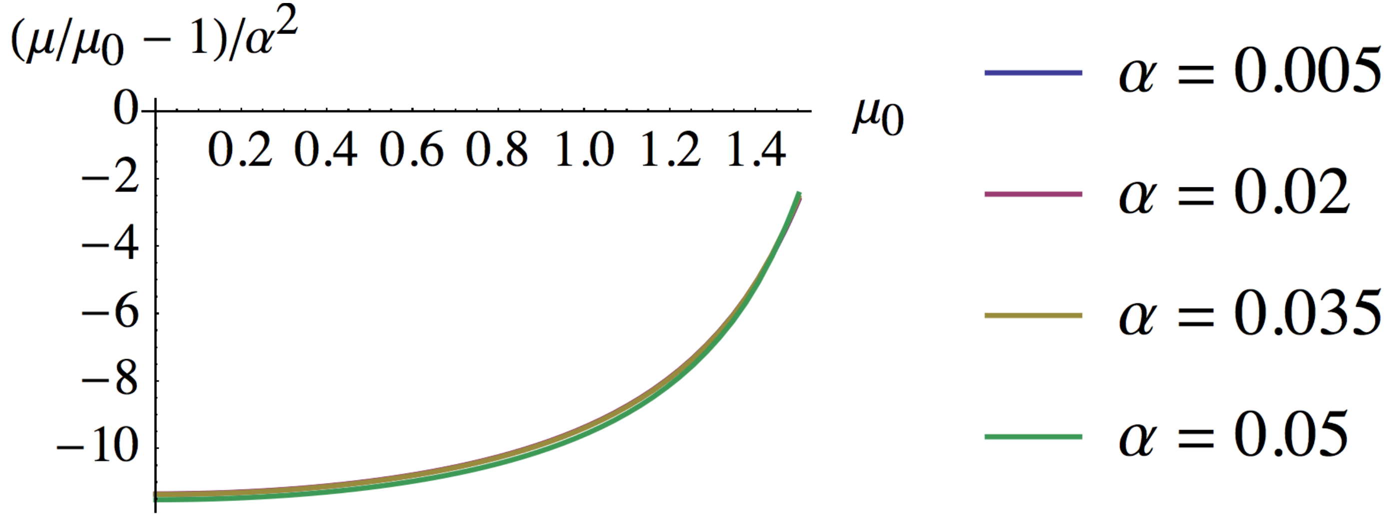
1. $\widehat{\psi}$ is a solution of the *Beltrami equation* $(\nabla^2 + \mu^2)\widehat{\psi} = 0$
   It is only *harmonic* in the *small-μ limit.* Likewise
   $k_1(\mu) \equiv (k^2 - \mu^2)^{1/2} \to k \text{ only as } \mu \to 0$
2. The term in γ_S was introduced in Dewar *et al.* 2013 to allow control of the *total current* in the sheet
3. The term in $\overline{\psi}$ is required for poloidal flux conservation

μ is *not fixed*

- In plane slab, *before* ripple is turned on, the *unperturbed* equilibrium flux function is

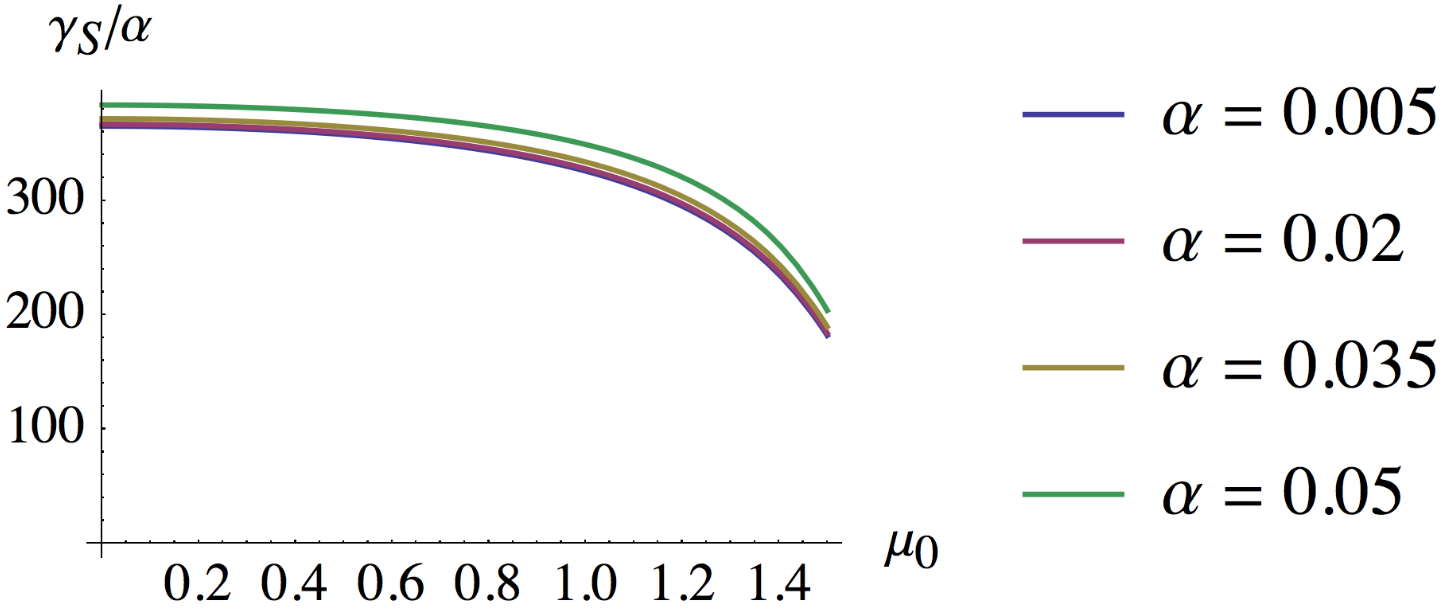
$$\psi_0(x|\mu_0) \equiv \frac{B_0}{\mu_0}(1 - \cos \mu_0 x)$$

- As amplitude parameter α is increased from 0, μ must *change* to preserve helicity and fluxes:

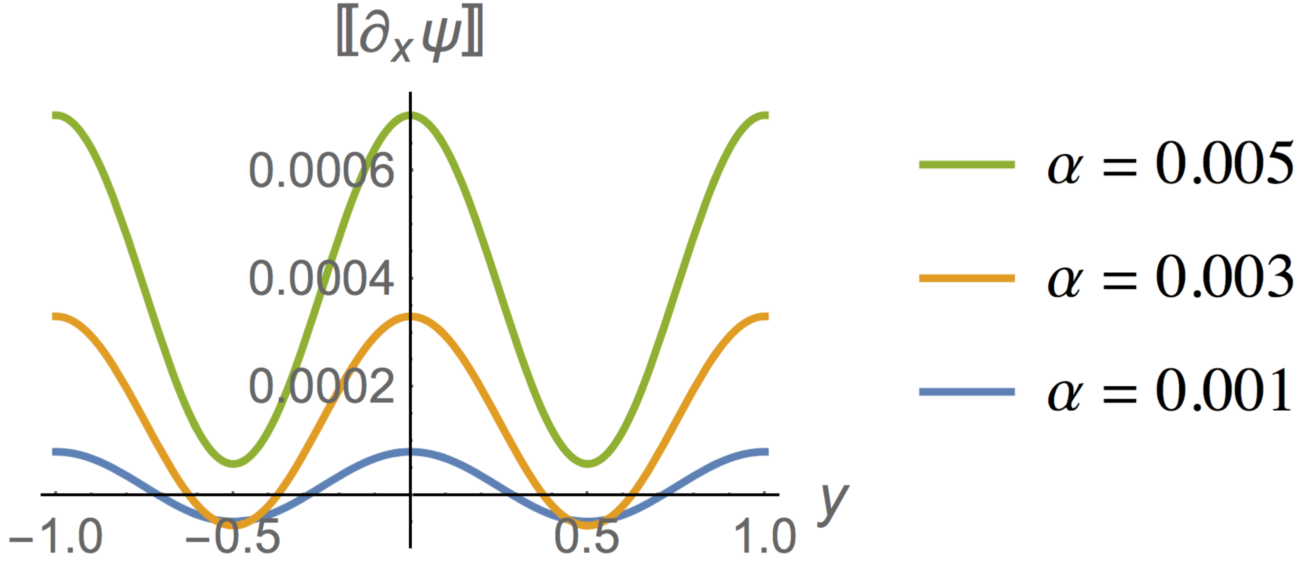


Current sheet has a strong d.c. component

- HK implicitly assumed the total current in the sheet was zero, but MRxMHD switch-on shows there is a *nonzero* total current $J = \dfrac{2\alpha\psi_a k_1 \lambda}{\sinh k_1 a}\gamma_{\rm S}$ proportional to $\gamma_{\rm S}$:

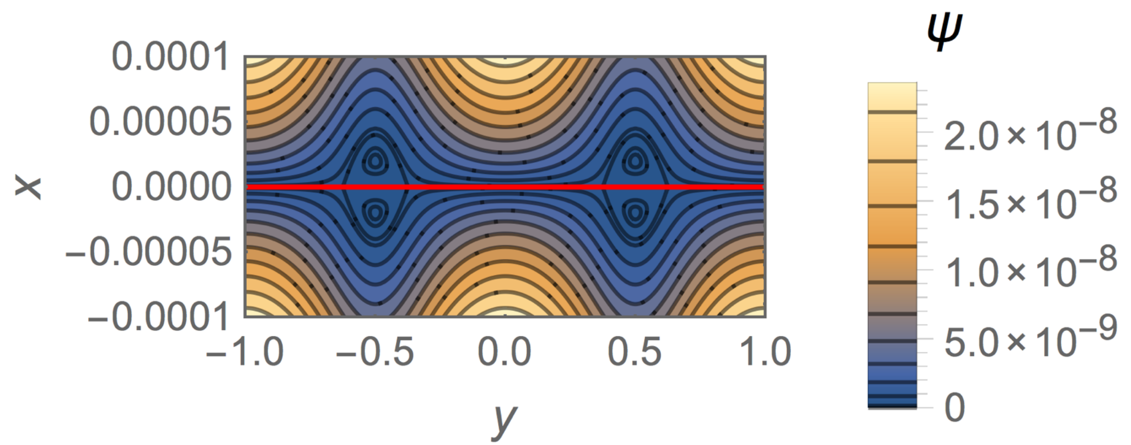


Current sheet reverses for small perturbations



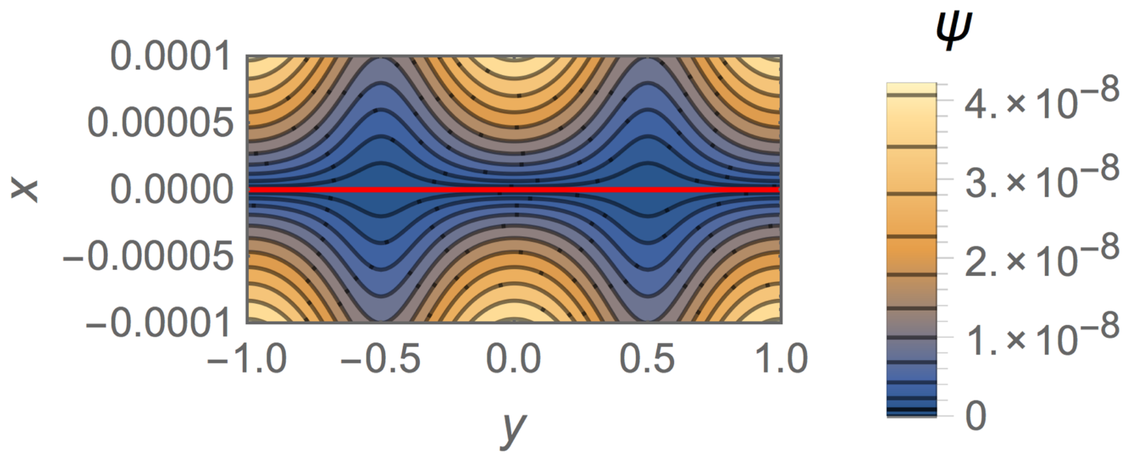
Fully shielded case: Plots of the jump in the gradient of ψ, vs. *y* for μ_0 = 1.4 and selected small values of α, showing the occurrence of current-density reversal for the two smallest values.

Current reversal causes “half-islands”



α = 0.003

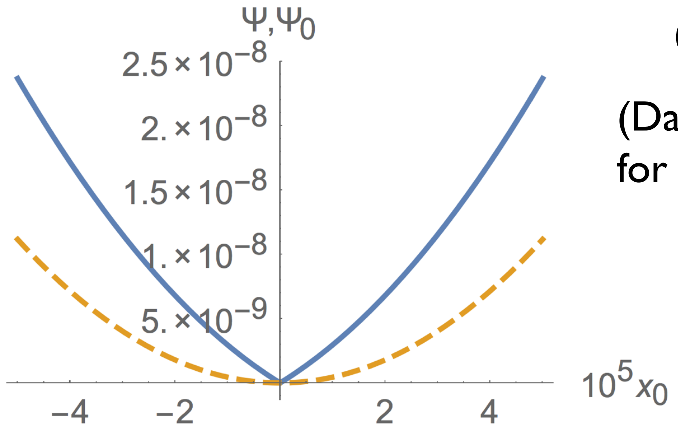
Fully shielded case: Level surfaces of ψ (magnetic surfaces) in the case $\mu_0 = 1.4$, α = 0.003, showing the occurrence of a small half-islands bisected by the reversed-current section of the current sheet.



α = 0.005

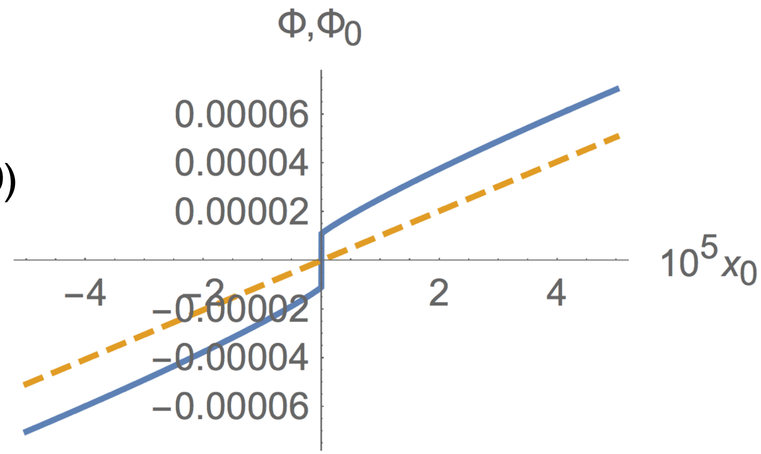
No current reversal — no half-islands

Fluxes and rotational transform I



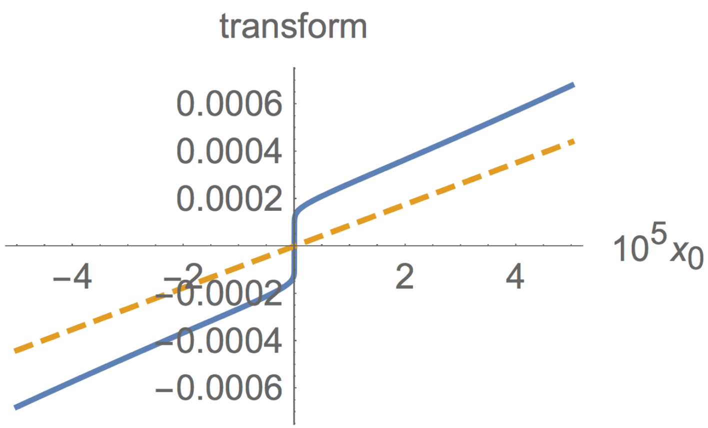
$\alpha = 0.001$

(Dashed curves are for plane slab, $\alpha = 0$)



Poloidal flux as a function of x_0 (= x along y-axis), showing discontinuity in slope at $x = 0$ caused by current sheet

Toroidal flux as a function of x along y-axis, showing discontinuity at $x = 0$ caused by half-island.

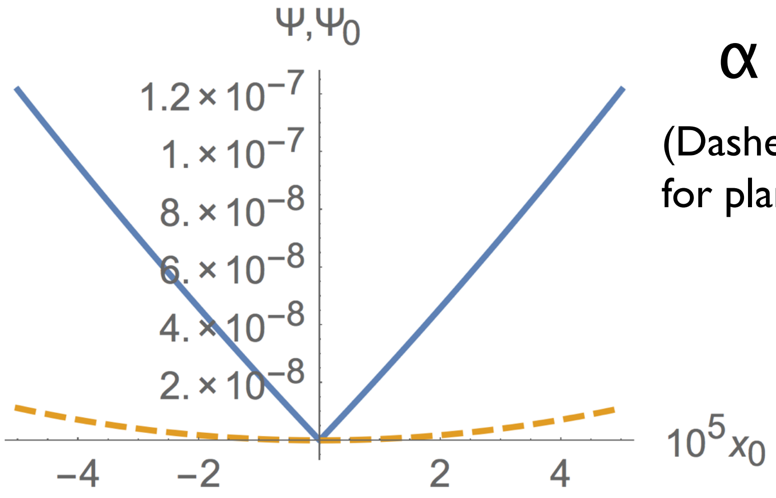


Rotational transform (1/q)

$$\Psi'(x_0)/\Phi'(x_0)$$

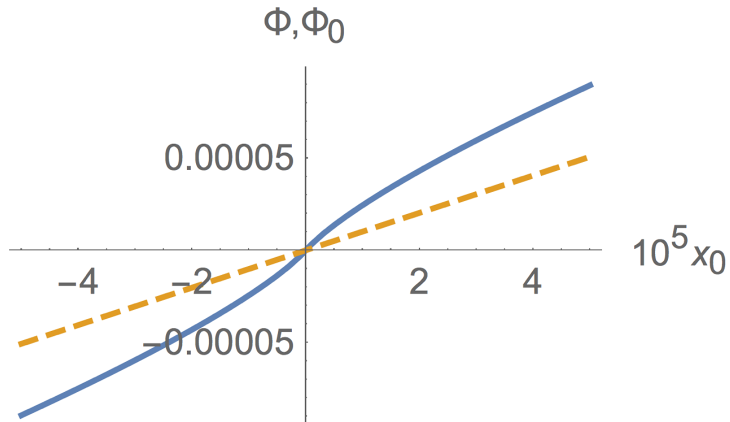
showing jump or large slope near $x_0 = 0$.

Fluxes and rotational transform II

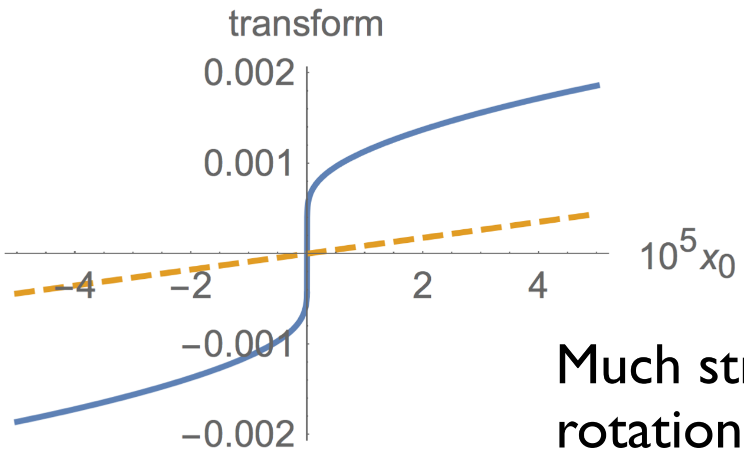


α = 0.005

(Dashed curves are for plane slab, α = 0)



Discontinuity in toroidal flux has gone as there are no half-islands above a threshold in α c. 0.0045



Much stronger jump in rotational transform

Conclusions

- Multi-region generalization of Taylor relaxation has been extended to a self-consistent dynamics through Hamilton's Principle of Stationary Action.
- A rippled slab model has been used to illustrate the formation of a resonant current sheet as boundary ripple is switched on
- For very small ripple amplitudes current reversal occurs in the current sheet and unperturbed sheared magnetic field exhibits topological change, with small half-islands, locking rotational transform to resonant value
- For larger ripple amplitude rotational transform jumps