Magnetic islands and singular currents at rational surfaces in 3D MHD equilibria

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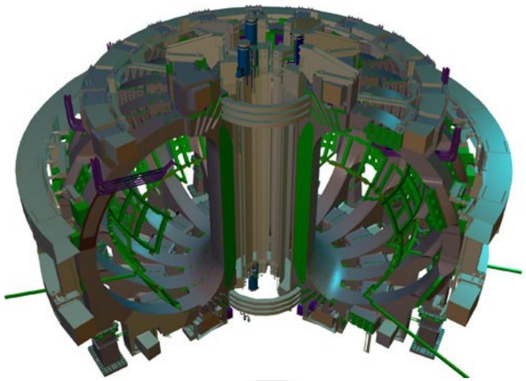
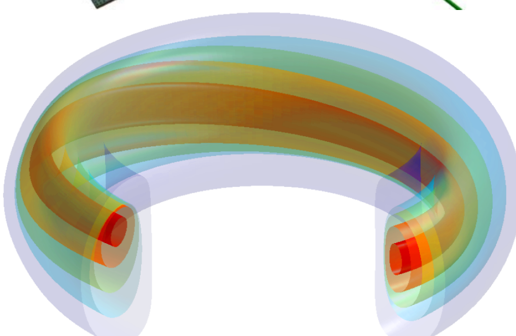
MPPC video-conference, December 8th 2014

3D MHD brings together tokamaks and stellarators



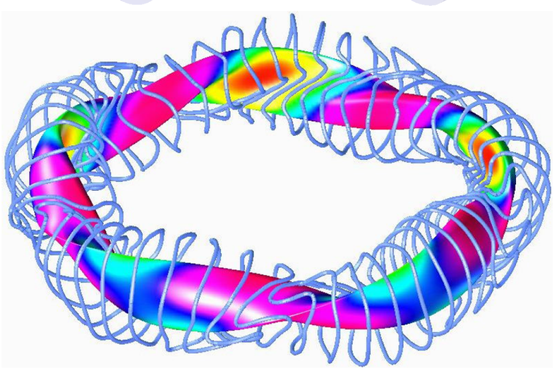
Tokamak non-axisymmetric designs

(magnetic ripple, resonant magnetic perturbations,…)



Tokamak MHD helical modes and bifurcations

(saturated internal kink, sawteeth)



Stellarator three-dimensional topology

Computational 3D MHD is an outstanding challenge

- Ideal MHD predicts the existence of singular currents in 3D
  - Critical for 3D equilibrium (magnetic islands, confinement)
  - Critical for 3D macroscopic stability (kink modes, sawteeth)

- Computation of 3D equilibria is an outstanding numerical challenge

① **Can we predict the magnitude of these singular currents?**

② **Can we compute 3D MHD equilibria with magnetic islands?**

Ideal MHD predicts singular currents

$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$
$$\mathbf{j} \times \mathbf{B} = \nabla p$$
$$\nabla \cdot \mathbf{j} = 0 \ , \ \ \nabla \cdot \mathbf{B} = 0$$

$$\Longrightarrow$$

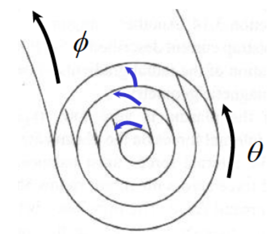
$$\mathbf{j}_\perp = (\mathbf{B} \times \nabla p)/B^2$$
$$\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$$

Magnetic coordinates

$$(\psi, \theta, \phi)$$

$$\xrightarrow{\text{ideal MHD}}$$

$$\sqrt{g}\ \mathbf{B} \cdot \nabla \equiv \iota \partial_\theta + \partial_\phi$$



Fourier decomposition

$$u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$$

$$\Longrightarrow$$

$$(\iota m - n) u_{mn} = i(\sqrt{g}\ \nabla \cdot \mathbf{j}_\perp)_{mn}$$

Equation type

$$x f(x) = h(x)$$
$$x \equiv \iota m - n, \ h(x) \sim p'$$

$$\Longrightarrow$$

$$\boxed{u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)}$$

Pfirsch-Schlutter current **Dirac δ-current**

How to calculate 3D MHD equilibrium?

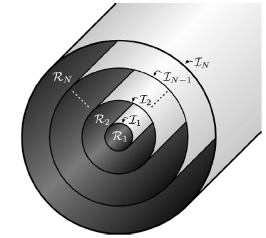
EITHER Insist on nested flux surfaces (ideal MHD)

- VMEC
  - based on energy functional with ideal constraints
  - cannot resolve rational surfaces

OR Relax assumption of flux surfaces

- Resistive MHD
  - initial value calculation, not based on energy minimum
  - islands develop and break rational surfaces
- Relaxed MHD
  - based on energy functional with fewer constraints
  - islands develop at rational surfaces

Multiregion Relaxed MHD



Taylor's theory	MRxMHD	Ideal MHD

Fewer constraints → More constraints

Helicity is conserved globally	**Helicity is conserved discretely**	**Helicity is conserved locally**
$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\, dV}_{H} - H_0\Big)$	$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\big(H_l - H_{l0}\big)\Big]$	$W = \int_V \Big(\frac{p}{\gamma - 1} + \frac{B^2}{2}\Big) dV$
Topology: $\mathbf{B}\cdot\nabla\psi\big\vert_{\partial V} = 0$	Topology: $\mathbf{B}\cdot\nabla\psi\big\vert_{\partial V_l} = 0$	Topology: $\mathbf{B}\cdot\nabla\psi = 0$
Given $p, \Delta\psi, H_0$	Given $p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$	Given $p(\psi), \psi_p(\psi)$
$\delta F = 0 \Longrightarrow \boxed{\nabla\times\mathbf{B} = \mu\mathbf{B}}$	$\delta F = 0 \Longrightarrow \boxed{\begin{array}{c}\nabla\times\mathbf{B} = \mu_l\mathbf{B} \\ [[p + B^2/2]] = 0\end{array}}$	$\delta W = 0 \Longrightarrow \boxed{\mathbf{j}\times\mathbf{B} = \nabla p}$
[Taylor, 1974]	[Dewar, Hole, Hudson, 2006]	[Kruskal, Kulsrud, 1958]

Existence of stepped-pressure equilibria?

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure

OSCAR P. BRUNO
California Institute of Technology

PETER LAURENCE
Universita di Roma "La Sapienza"

We establish an existence result for the three-dimensional MHD equations

$$\begin{aligned} (\nabla \times \mathbf{B}) \times \mathbf{B} &= \nabla p \\ \nabla \cdot \mathbf{B} &= 0 \\ \mathbf{B} \cdot n|_{\partial T} &= 0 \end{aligned}$$

with $p \neq$ const in tori T *without symmetry*. More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

Communications on Pure and Applied Mathematics, Vol. XLIX, 717–764 (1996)

Stepped-Pressure Equilibrium Code (SPEC)

- Implementation of MRxMHD

- Finds minimum of

$$F = \sum_{l=1}^{N} \left[W_l + \frac{\mu_l}{2} \Big(H_l - H_{l0} \Big) \right]$$

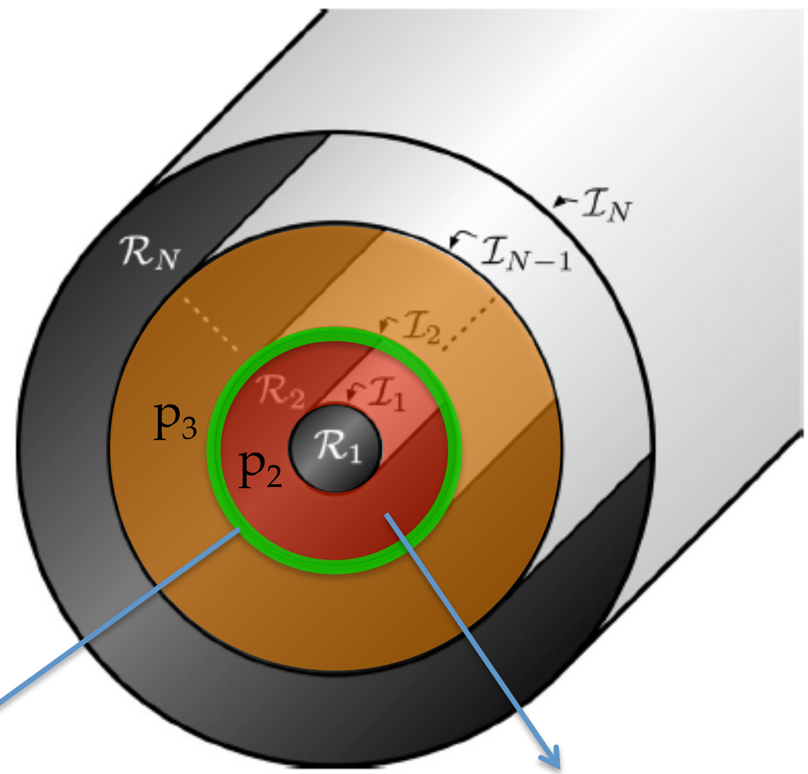
$$\text{Topology:} \;\; \mathbf{B} \cdot \nabla \psi \big|_{\partial V_l} = 0$$

$$\text{Given } p_l, \Delta\psi_l, \Delta\psi_{p,l}, H_{l0}$$

$$\Longleftrightarrow$$

$$\mathcal{R}_l : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$$

$$\mathcal{I}_l : \quad [[p + B^2/2]] = 0$$



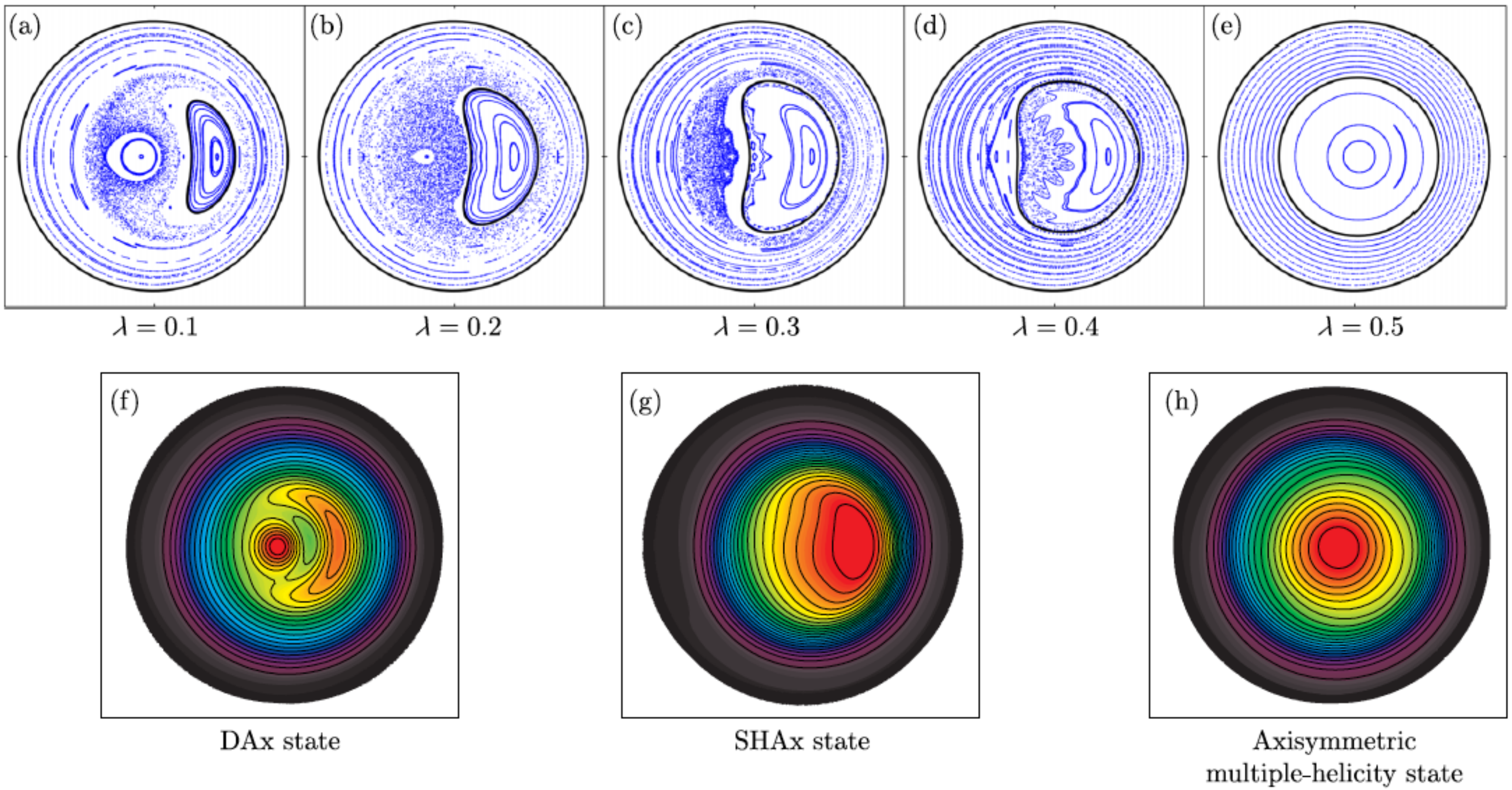
$\Delta\psi, \Delta\psi_p, H_0$

or $\Delta\psi, \Delta\psi_p, \mu$

or $\Delta\psi, \iota^{-}, \iota^{+}$

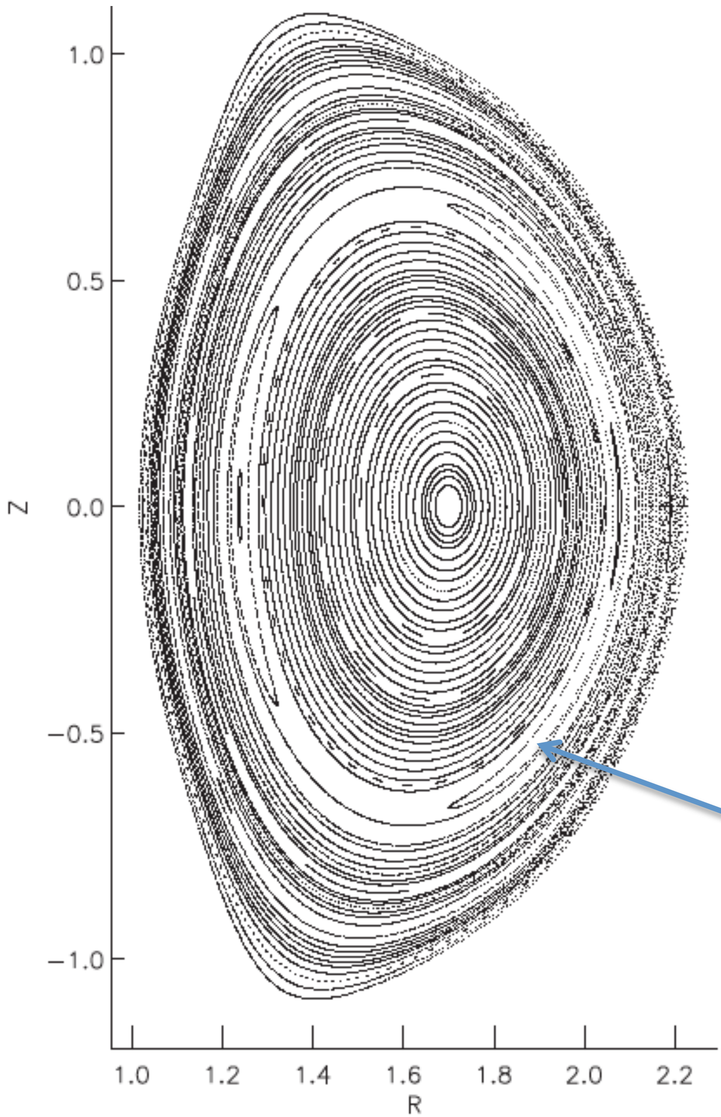
[Hudson et al, PoP, 2012]

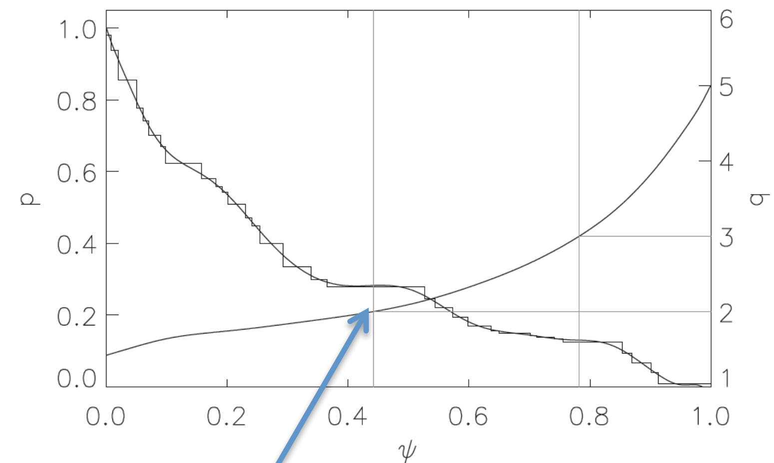
SPEC successfully reproduced helical states in RFPs



[Dennis et al, PRL, 2013]

SPEC computed DIII-D equilibrium with perturbed boundary

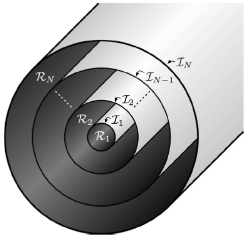




Magnetic island at q=2 rational surface

[Hudson et al, PoP, 2012]

Ideal MHD is a limit of MRxMHD



The infinite interface limit of multiple-region relaxed magnetohydrodynamics

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(Received 19 December 2012; accepted 4 March 2013; published online 15 March 2013)

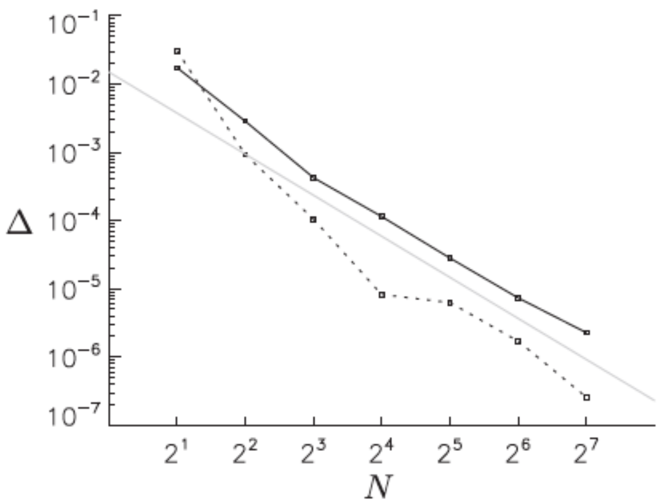
We show the stepped-pressure equilibria that are obtained from a generalization of Taylor relaxation known as multi-region, relaxed magnetohydrodynamics (MRXMHD) are also generalizations of ideal magnetohydrodynamics (ideal MHD). We show this by proving that as the number of plasma regions becomes infinite, MRXMHD reduces to ideal MHD. Numerical convergence studies illustrating this limit are presented. © *2013 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4795739]

Corollary for large N:

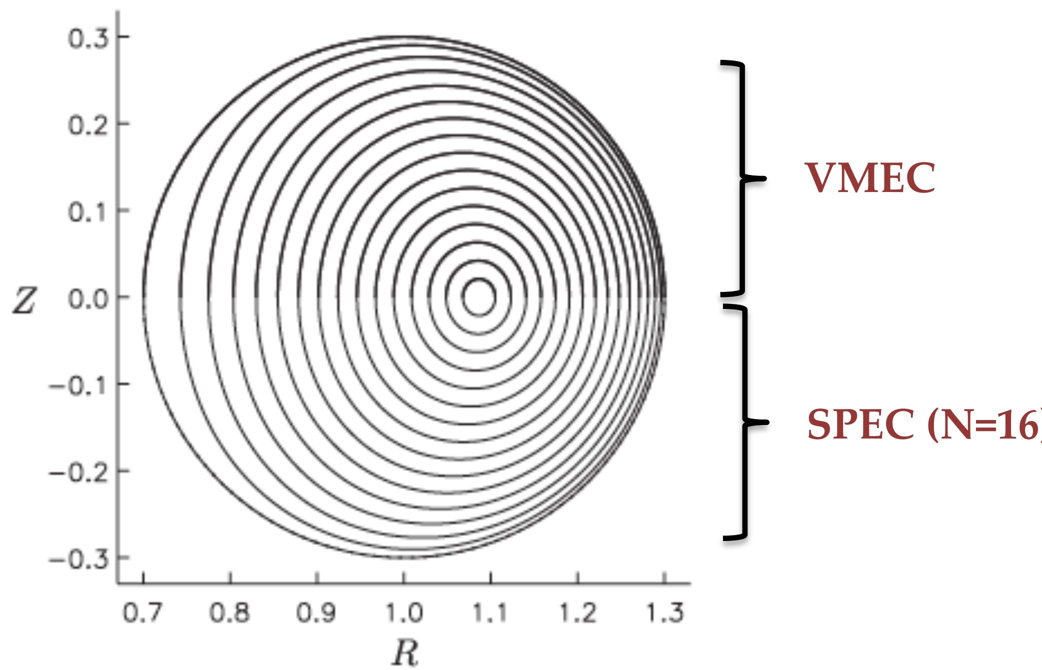
① **Axisymmetry: Grad-Shafranov equilibrium should be retrieved**

② **Non-axisymmetry: singular currents should arise** $j \sim [[\mathbf{B}]] \times \mathbf{n}\ \delta(x)$

SPEC benchmarked against VMEC in axisymmetry



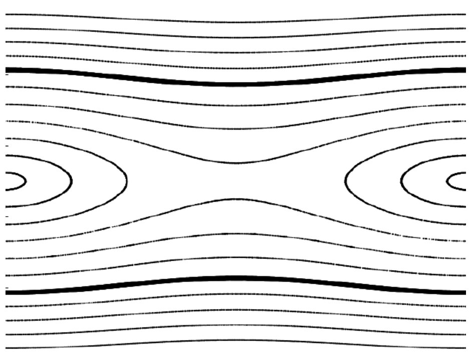
$$\Delta \equiv \int_0^{2\pi} d\theta \left| \mathbf{x}_{\mathrm{VMEC}}(\theta) - \mathbf{x}_{\mathrm{SPEC}}(\theta) \right|$$

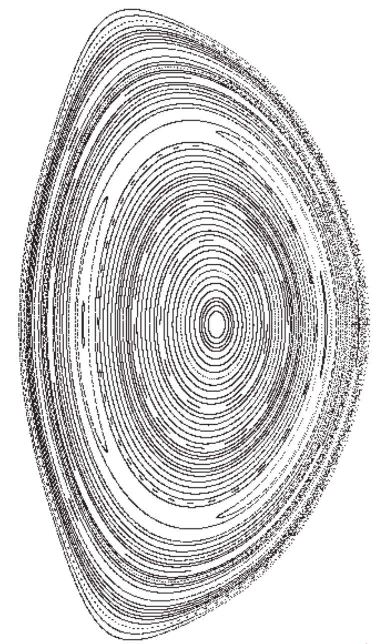


[Hudson et al, PoP, 2012]

Philosophy: build up understanding by steps of increasing complexity

- There are two superimposed singularities in the parallel current
  - Start with constant-pressure plasma to isolate the δ-current
- Geometry introduces most of the complexity
  - Start with slab geometry





Hahm-Kulsrud-Taylor available solution (1985)

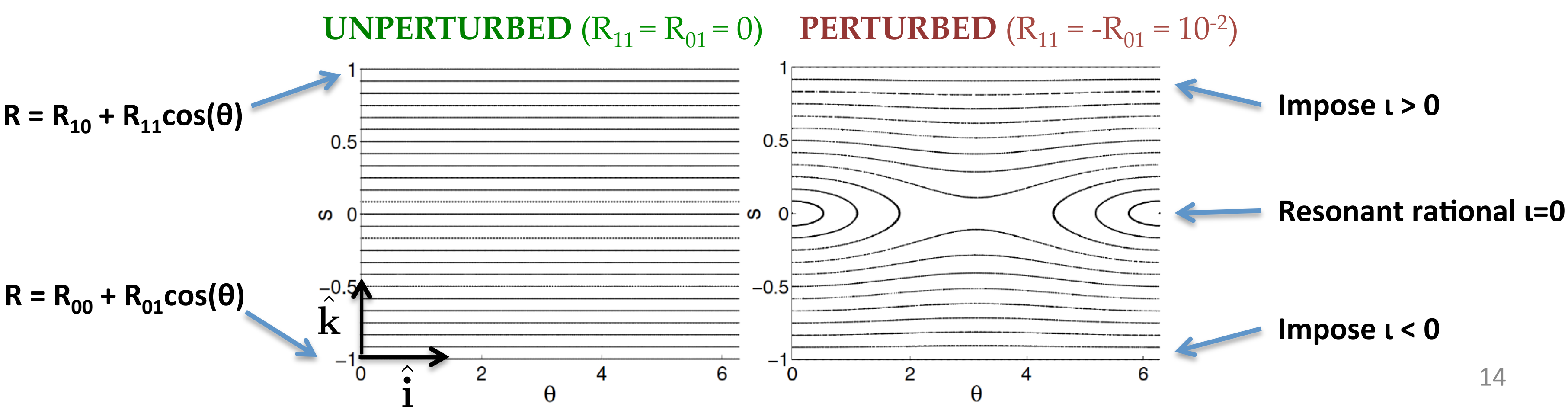
Rosenbluth-Dagazian-Rutherford available solution (1973)

MRxMHD in slab can be treated analytically

SINGLE RELAXED VOLUME

- Slab torus $\mathbf{r} = \theta\hat{\mathbf{i}} + \zeta\hat{\mathbf{j}} + R(s,\theta,\zeta)\hat{\mathbf{k}}$, $s \in [-1, 1]$.
- $R(s,\theta,\zeta)$ interpolates boundary interfaces.
- $\nabla \times \mathbf{B} = \mu\mathbf{B}$ with $\mathbf{B} \cdot \nabla s = 0$ at boundaries.
- Unperturbed solution: $\mathbf{B}_u$ depends on
  $(\mu, \Delta\Psi_p, \Delta\Psi_t)$ **or** $(H, \Delta\Psi_p, \Delta\Psi_t)$ **or** $(\iota^+, \iota^-, \Delta\Psi_t)$
- Perturbed boundary (n=0,m=1): $\mathbf{B}_u + \delta\mathbf{B}$ depends on
  $(\iota^+, \iota^-, \Delta\Psi_t)$ **and** $(\Delta, R_{0,1}, R_{1,1})$

$\Big\}$ linearized solution ($R_{11} \ll \Delta$)



MRxMHD in slab can be treated analytically

MULTIPLE RELAXED VOLUMES

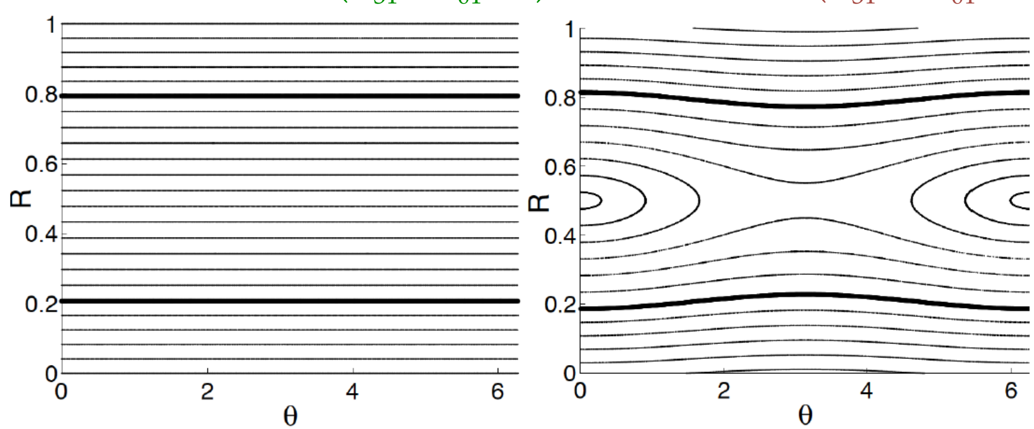
- $\nabla \times \mathbf{B} = \mu \mathbf{B}$ in each volume $\mathcal{V}_l$, $l = 1 \ldots N$.
  - $\mathbf{B}(t_l^{\pm}, \Delta\Psi_{t,l})$ needs interfaces shape $\{\Delta_l, R_{l,1}\}$
- $[[p + B^2/2]] = 0$ across each interface.
  - $N - 1$ nonlinear equations $F_l(\Delta_l, \Delta_{l+1}) = 0$
  - $(N-1)^2$ matrix system $\mathcal{M}\mathbf{R} = \mathbf{S}$

$$\lim_{\substack{\Delta\Psi_2 \to 0 \\ t_2^{\pm} \to 0}} \left[\det \mathcal{M}\right] = 0 \qquad \lim_{\substack{\Delta\Psi_2 \to 0 \\ t_2^{\pm} \to 0}} \mathbf{S} = 0$$

$$\lim_{\substack{\Delta\Psi_2 \to 0 \\ t_2^{\pm} \to 0}} \mathbf{R} \sim \lim_{x,y \to 0} x/y$$

Need to specify how we take the limit

UNPERTURBED ($R_{31} = R_{01} = 0$) **PERTURBED** ($R_{31} = -R_{01} = 10^{-2}$)

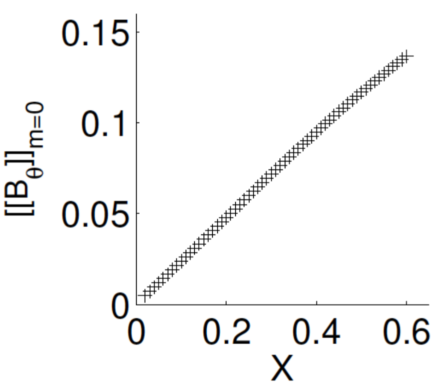


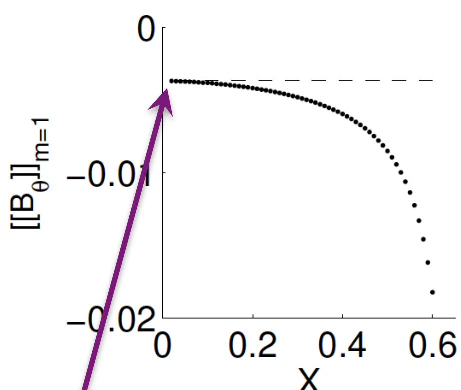
IDEA: shield the island by reducing flux and transform in the resonant volume

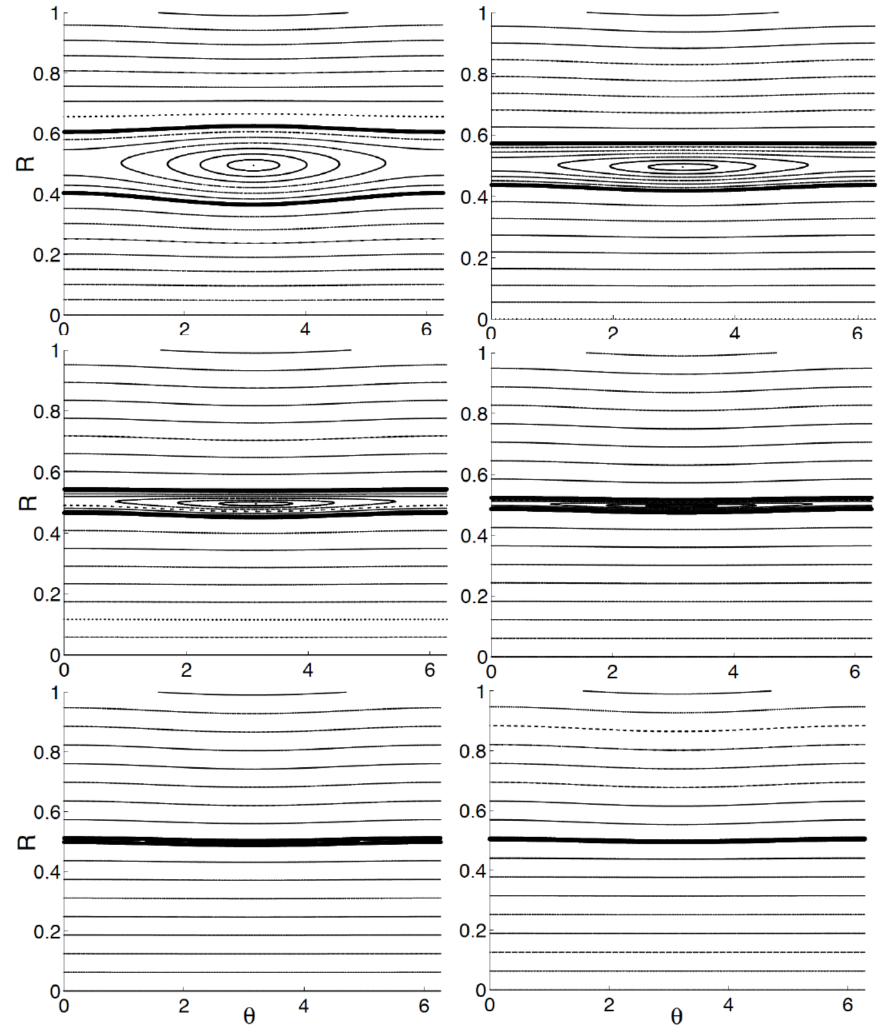
MRxMHD in slab can be treated analytically

ISLAND SHIELDING AND $\delta(x)$-CURRENT

- Squeeze island with N=3.
  - $t_2^{\pm} = \pm X^{\alpha}$
  - $\Delta\Psi_{t,2} = X^{\beta}$
  - $X \to 0$
- Prediction $\lim_{X\to 0}[[\mathbf{B}]] \neq 0$
  - $\mathbf{j} = [[\mathbf{B}]] \times \mathbf{n}\, \delta(t - t_{res})$



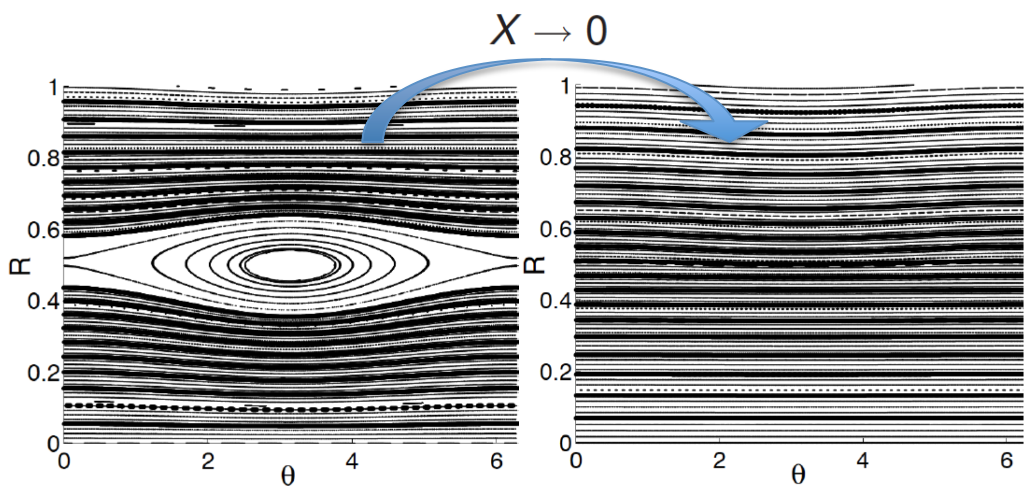




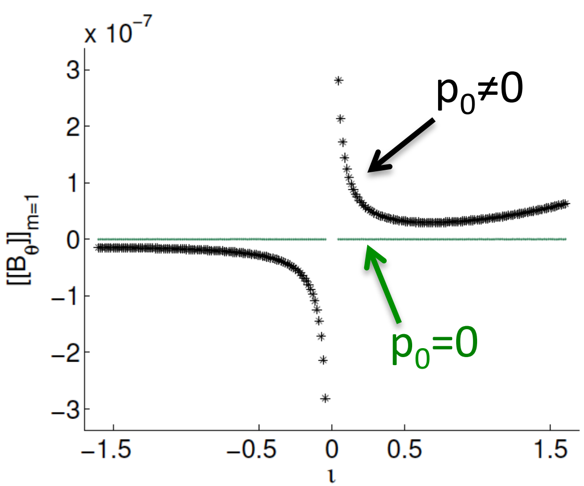
retrieves exactly HKT solution in the limit of small shear

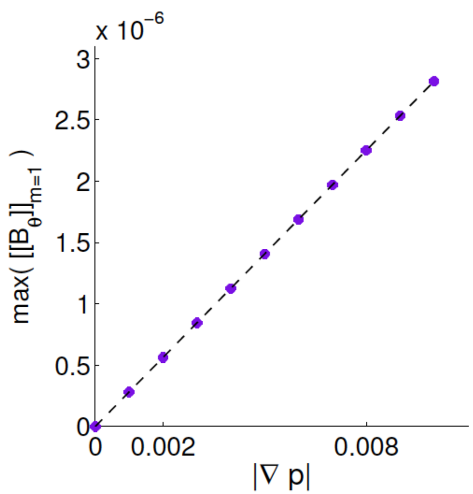
MRxMHD in slab can be treated analytically

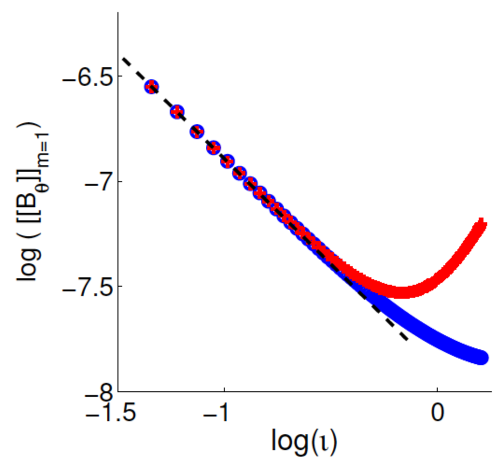
ISLAND SHIELDING AND PFIRSCH-SCHLUTTER CURRENT



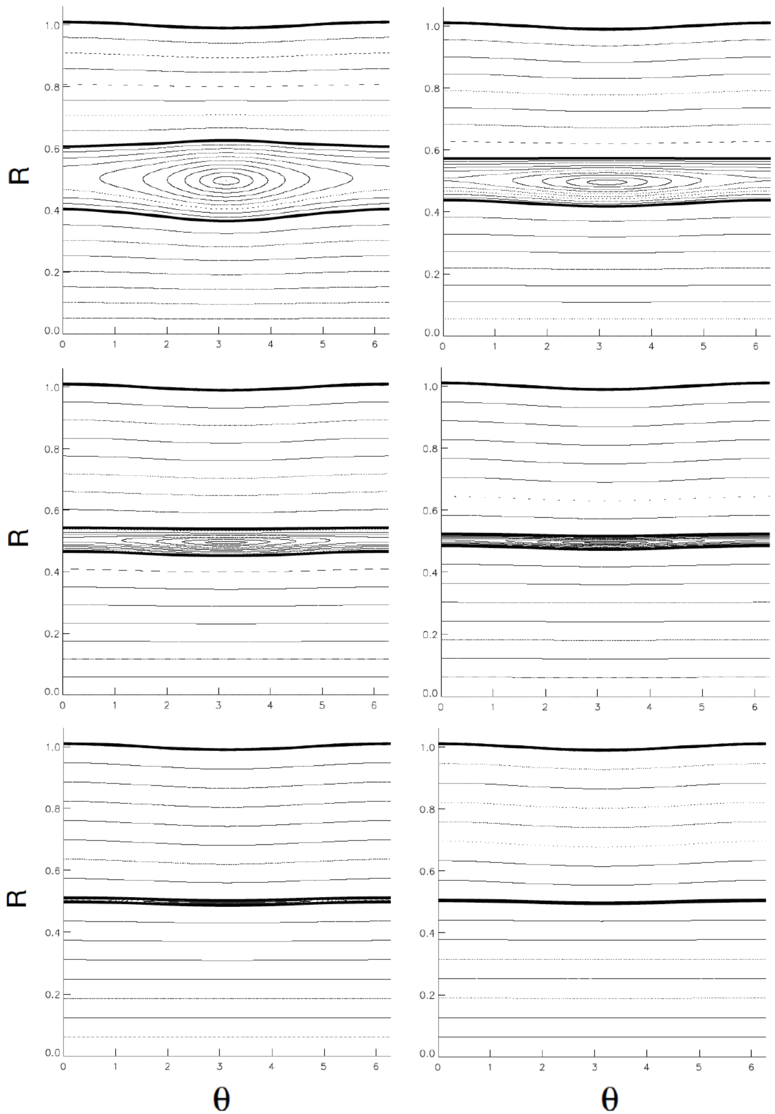
$$p(\Psi) = p_0(1 - \Psi)$$
$$\iota(\Psi) = \iota_0(2\Psi - 1)$$

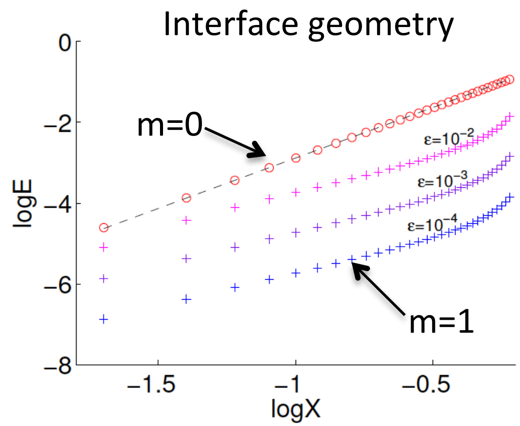


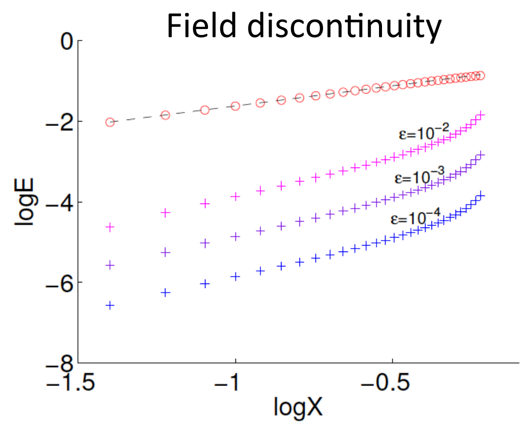




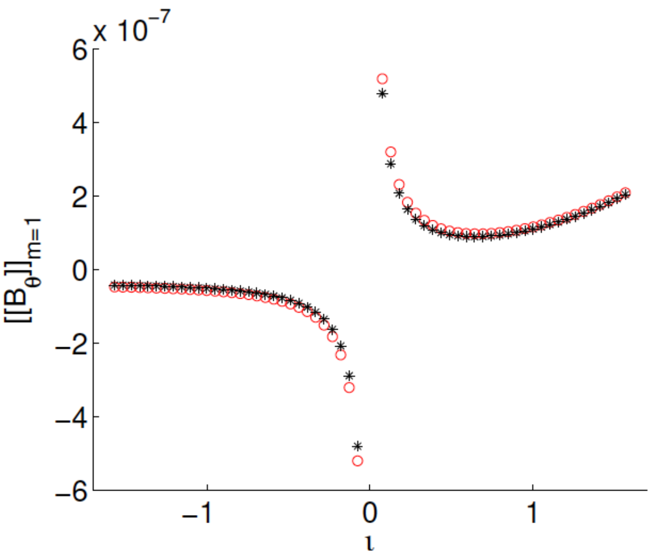
SPEC reproduces the analytical results





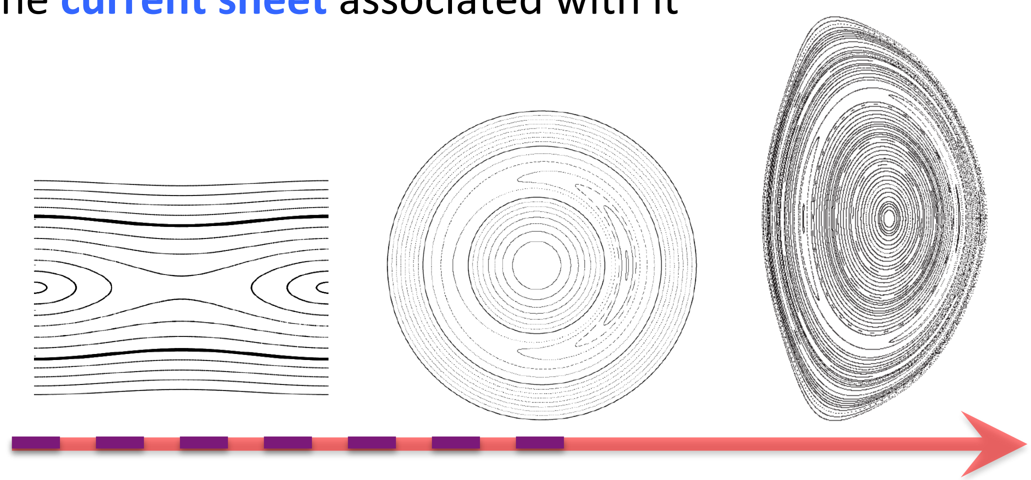


The agreement between SPEC and the linear analytical prediction improves with decreasing perturbation



Summary and outlook

- Ideal MHD predicts two types of singular currents at rational surfaces in 3D
    - We have provided **the first numerical proof** of their mutual existence
    - We have developed an **analytical linear slab model** that
      (1) describes the formation of islands around resonant rational surfaces
      (2) retrieves the ideal MHD limit in which magnetic islands are shielded
      (3) computes the subsequent formation of δ-currents and 1/x-currents
- SPEC is capable of computing 3D equilibria in slab, cylindrical, toroidal
    - Currently using SPEC to obtain the **nonlinearly saturated internal kink** and the **current sheet** associated with it



Summary and outlook