A new class of three-dimensional ideal-MHD equilibria with current sheets

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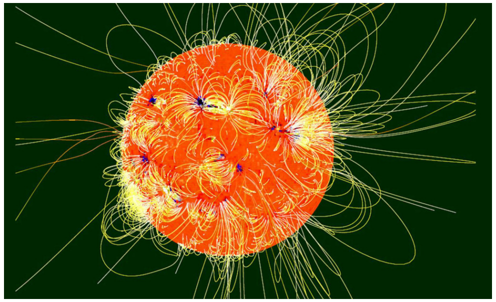
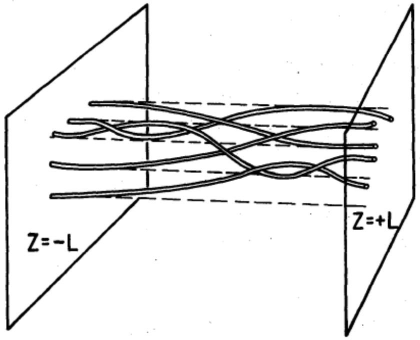
Current sheets in fusion and astrophysical plasmas

Current sheets predicted to form in 3D ideal-MHD equlibria…

…in the solar corona, where ideal plasma convection on the surface produces field entanglement.

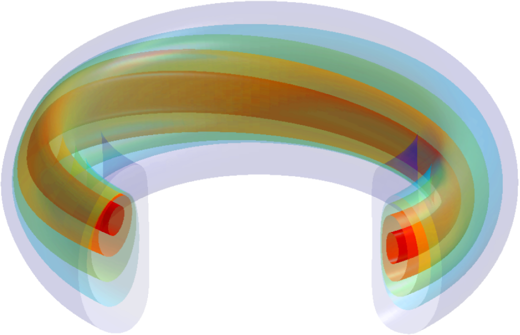
[Parker, 1972]

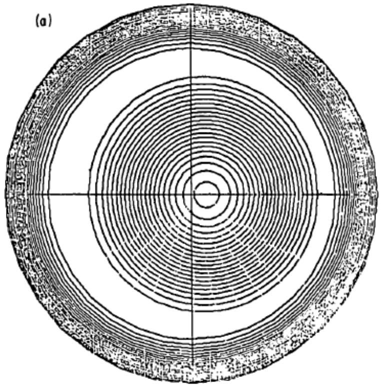




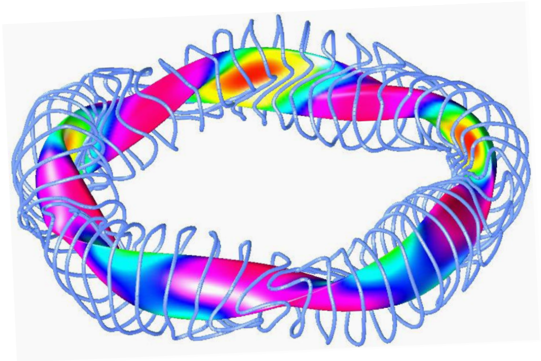
…in toroidally confined plasmas, where ideal kink instabilities bring the plasma to resonant 3D states.

[Rosenbluth, 1973]

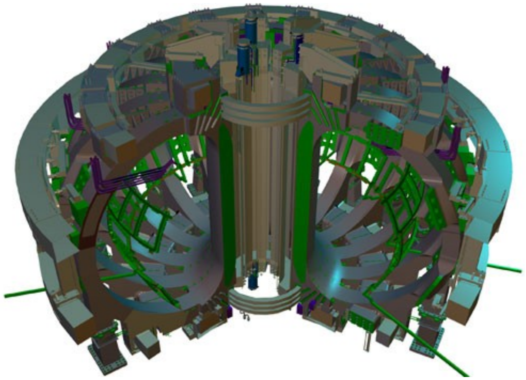




3D MHD brings together tokamaks and stellarators

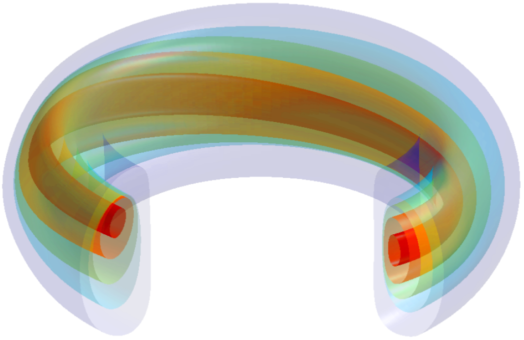


Stellarator three-dimensional topology



Tokamak non-axisymmetric designs

(magnetic ripple, resonant magnetic perturbations,...)



Tokamak MHD helical modes and bifurcations

(saturated internal kink, sawteeth)

On the menu today

① Origin of singular current densities in 3D MHD with nested surfaces.

② Questioning the existence of 3D ideal-MHD equilibria.

③ Exact computation of singular currents.

④ A new class of 3D ideal-MHD equilibria.

⑤ Application to resonant magnetic perturbations in fusion devices.

Singular current densities come in two flavours

magnetic differential equation

$$\nabla \cdot \mathbf{j} = 0\ ,\ \ \nabla \cdot \mathbf{B} = 0$$
$$\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$$
$$\mathbf{j} \times \mathbf{B} = \nabla p$$

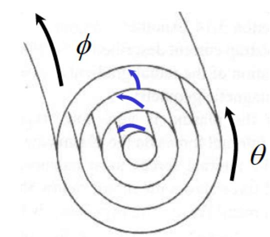
$$\Longrightarrow \quad \mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$$
$$\mathbf{j}_\perp = (\mathbf{B} \times \nabla p)/B^2$$

nested surfaces

Magnetic coordinates

$$(\psi, \theta, \phi)$$

$$\Longrightarrow \quad \sqrt{g}\ \mathbf{B} \cdot \nabla \equiv \iota\partial_\theta + \partial_\phi$$



Fourier decomposition

$$u = \sum_{m,n} u_{mn} e^{i(m\theta - n\phi)}$$

$$\Longrightarrow \quad (\iota m - n)u_{mn} = i(\sqrt{g}\ \nabla \cdot \mathbf{j}_\perp)_{mn}$$

Equation type

$$x f(x) = h(x)$$
$$x \equiv \iota m - n,\ \ h(x) \sim p'$$

$$\Longrightarrow \quad \boxed{u_{mn}(x) = h(x)/x + \hat{j}_{mn}\delta(x)}$$

Pfirsch-Schlüter current **Dirac δ-current**

Existence of 3D ideal-MHD equilibria?

- $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$ is not the current, but the current density [A/m^2] .

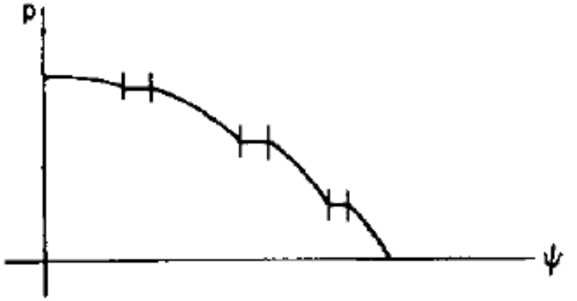
- Physically-valid equilibrium if the current $J = \int_\Sigma \mathbf{j} \cdot \mathbf{d}\sigma$ across any surface is finite (weak formulation of the problem).

- Problem: Pfirsch-Schlüter current diverges across certain surfaces.

- Historical conclusion: pressure gradients cannot be supported at resonant rationals and thus pressure is either fractal or stepped.

The function p is continuous but its derivative is pathological. We have obtained an equilibrium solution without infinite currents, but at the price of a very pathological pressure distribution.

[H. Grad, 1967]



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[...] More precisely, our theorems insure the existence of sharp boundary solutions for tori whose departure from axisymmetry is sufficiently small; they allow for solutions to be constructed with an arbitrary number of pressure jumps.

[Bruno and Laurence, 1996]

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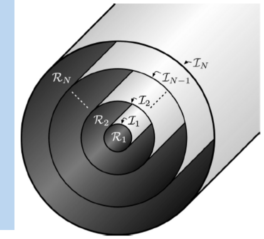
How to compute 3D ideal equilibria with current sheets?

Are there 3D MHD equilibria with nested surfaces & smooth pressure?

Multiregion Relaxed MHD



N = 1

N –> **∞**

[Dennis, 2013]

Taylor's theory	MRxMHD	Ideal MHD

Fewer constraints → More constraints

Helicity is conserved globally	Helicity is conserved discretely	Helicity is conserved locally
$F = W + \frac{\mu}{2}\Big(\underbrace{\int_V \mathbf{A}\cdot\mathbf{B}\; dV}_{H} - H_0\Big)$	$F = \sum_{l=1}^{N}\Big[W_l + \frac{\mu_l}{2}\Big(H_l - H_{l0}\Big)\Big]$	$W = \int_V \Big(\frac{p}{\gamma - 1} + \frac{B^2}{2}\Big) dV$
$\delta F = 0 \Longrightarrow \boxed{\nabla\times\mathbf{B} = \mu\mathbf{B}}$	$\delta F = 0 \Longrightarrow \boxed{\begin{array}{c}\nabla\times\mathbf{B} = \mu_l\mathbf{B}\\ [[p + B^2/2]] = 0\end{array}}$	$\delta W = 0 \Longrightarrow \boxed{\mathbf{j}\times\mathbf{B} = \nabla p}$
[Taylor, 1974]	[Dewar, Hole, Hudson, 2006]	[Kruskal, Kulsrud, 1958]

Stepped-Pressure Equilibrium Code (SPEC)

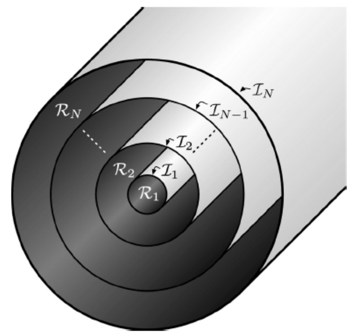
An implementation of MRxMHD

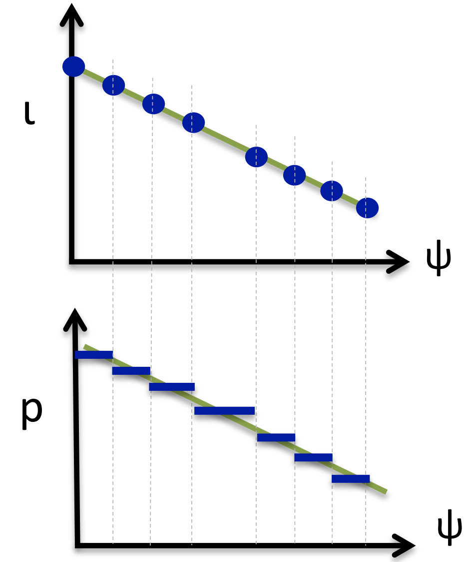
$$\mathcal{R}_l : \quad \nabla \times \mathbf{B} = \mu_l \mathbf{B}$$

$$\mathcal{I}_l : \quad [[p + B^2/2]] = 0$$

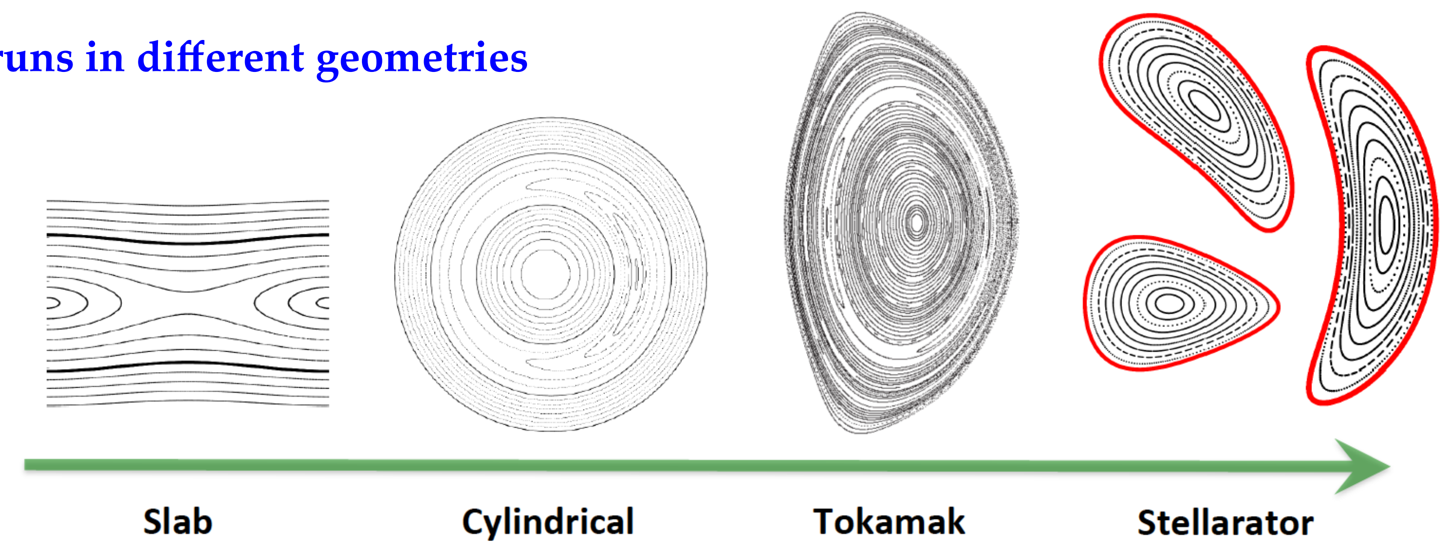
Given $p_l, \Delta\psi_l, \iota_l^+, \iota_l^-$

$l = 1, 2, \ldots N$





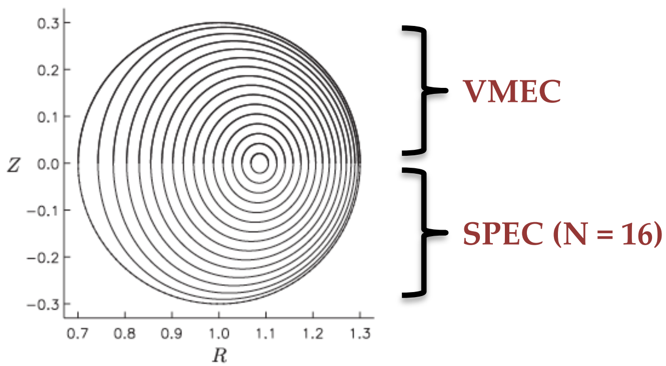
SPEC runs in different geometries



Stepped-**P**ressure **E**quilibrium **C**ode (SPEC)

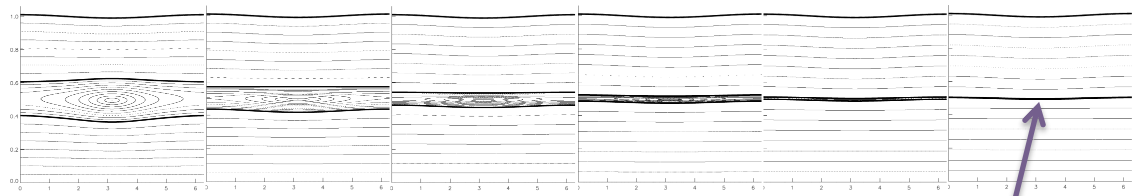
AXISYMMETRIC IDEAL MHD

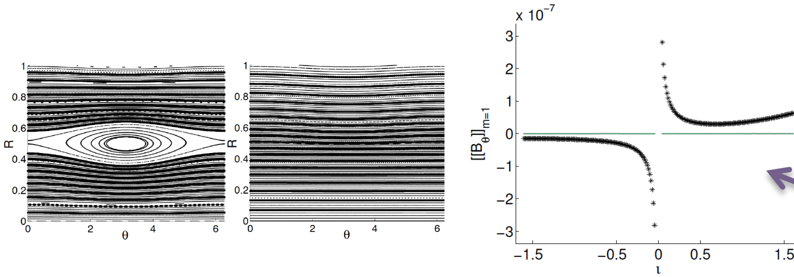
[Hudson et al, PoP, 2012]



NON-AXISYMMETRIC IDEAL MHD

[Loizu et al, PoP, 2015]



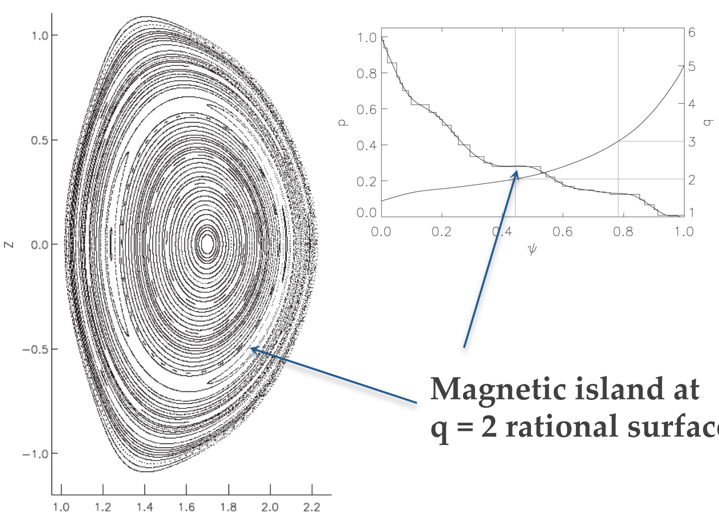


$$j \sim [[\mathbf{B}]] \times \mathbf{n}\ \delta(x)$$

$$j_{\parallel} \sim \frac{p'}{\iota m - n}$$

RMP IN DIII-D

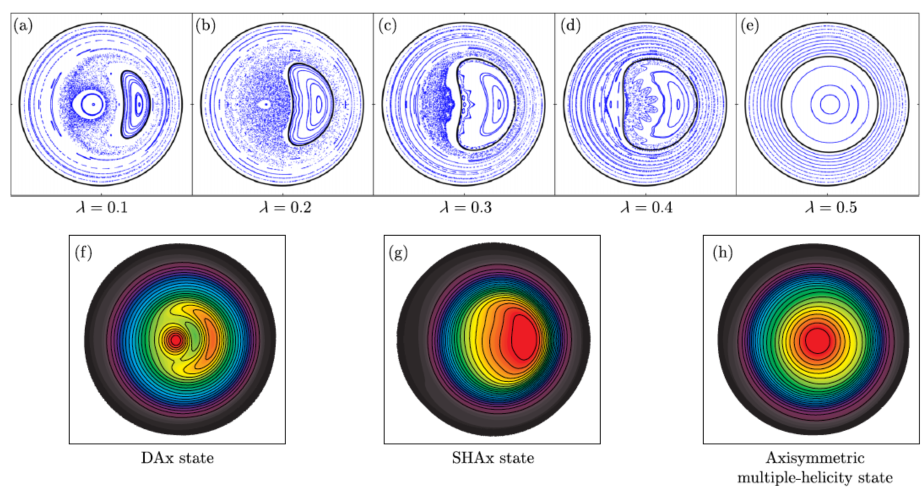
[Hudson et al, PoP, 2012]



Magnetic island at
q = 2 rational surface

HELICAL STATES IN RFP

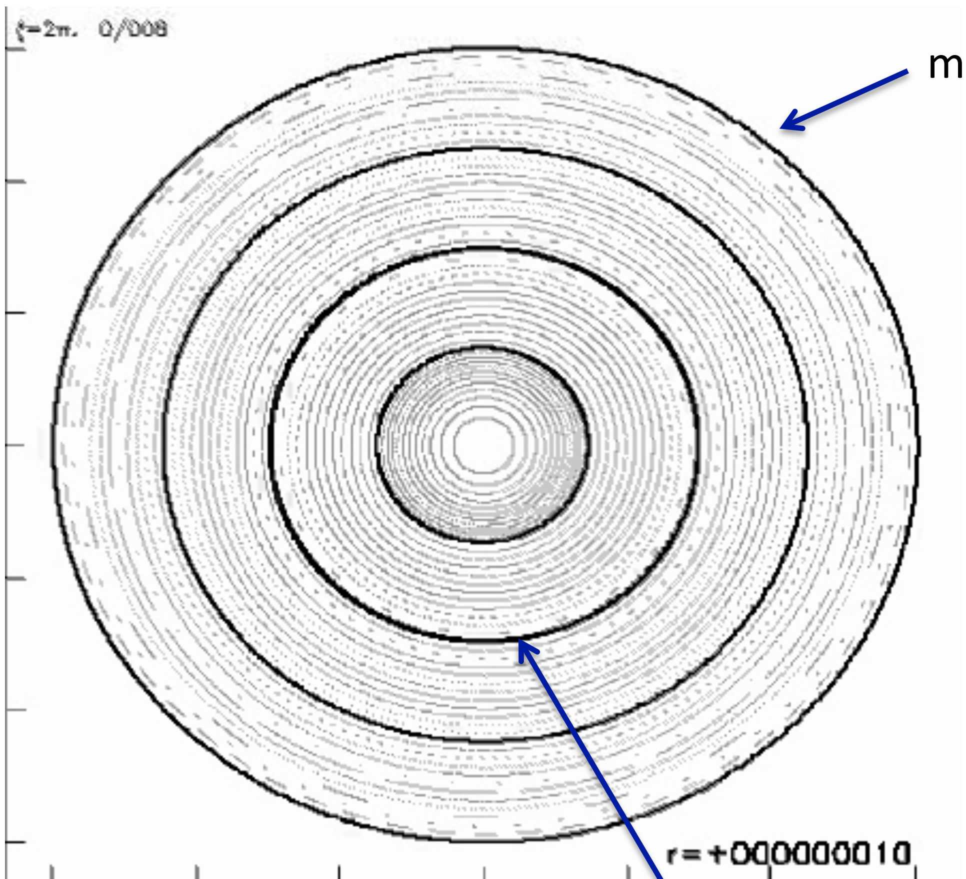
[Dennis et al, PRL, 2013]



Complete shielding requires discontinuous transform

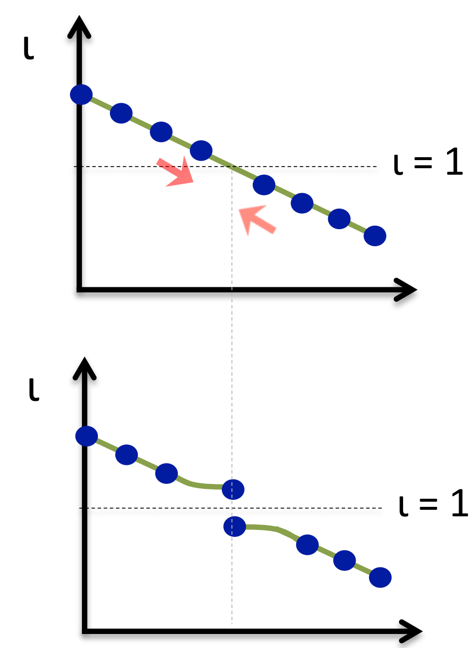
m=1, n=1 perturbation: $\xi_{11}/a = 10^{-2}$

Z



R

$[[B_\theta]]_{n=1,m=1} \neq 0$

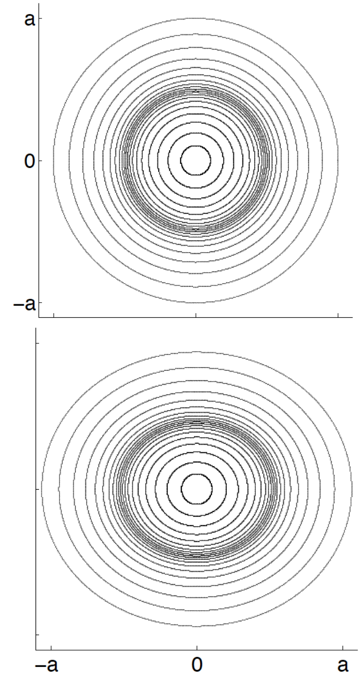


A new class of 3D MHD equilibria

- Consider equilibria with discontinuous transform across resonances.

- This class of equilbria allows for
  - Nested surfaces
  - Arbitrary 3D geometry
  - Arbitrary continuous and smooth pressure
  - Integrable current sheets

  [Loizu et al, Phys Plasmas 22 090704, 2015]



- This class of ideal-MHD states may be accessed when island-healing mechanisms are at play. [Bhattacharjee PoP 1995, Hegna PoP 2012]

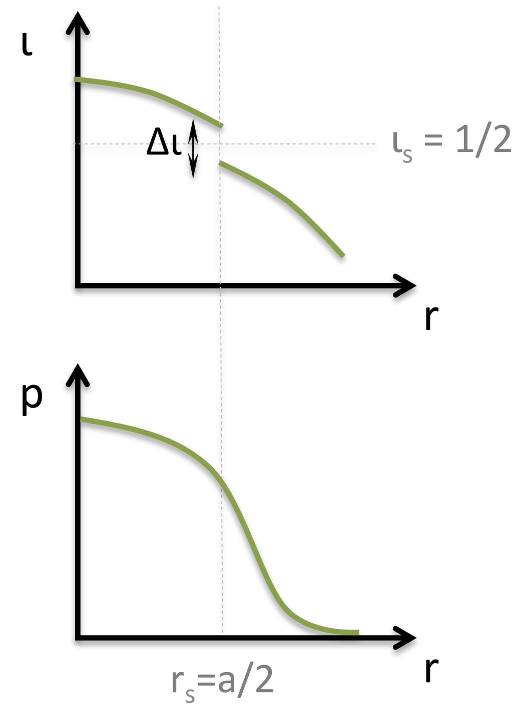
Application: resonant magnetic perturbations

- Consider a screw-pinch axisymmetric equilibrium:

$$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z^2(1+\iota^2\frac{r^2}{R^2})\right] + \frac{r\iota^2 B_z^2}{R^2} = 0$$

- Choose equilibrium profiles:

$$\iota(r) = \iota_0 - \iota_1 (r/a)^2 \pm \Delta\iota/2\ ,$$
$$p(r) = p_0[1 - 2(r/a)^2 + (r/a)^4]$$

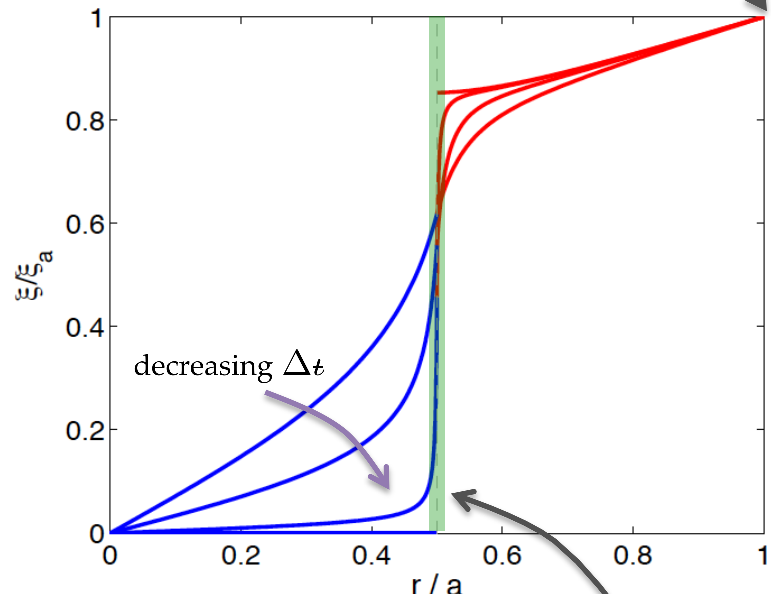


- **Outstanding question**: what is the ideal response to a resonant boundary perturbation ?

[Turnbull et al, PoP 2013; Reiman et al, NF 2015]

Ideal linear response to an RMP at β = 0

$\xi_a \cos(m\theta + kz)$



$\mathbf{j} = [[\mathbf{B}]] \times \hat{\mathbf{n}}\delta(r - r_s)$

- Perturbed equilibrium satisfies:

$$\delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 + \mathbf{j} \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$

- Reduces to Newcomb equation:

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

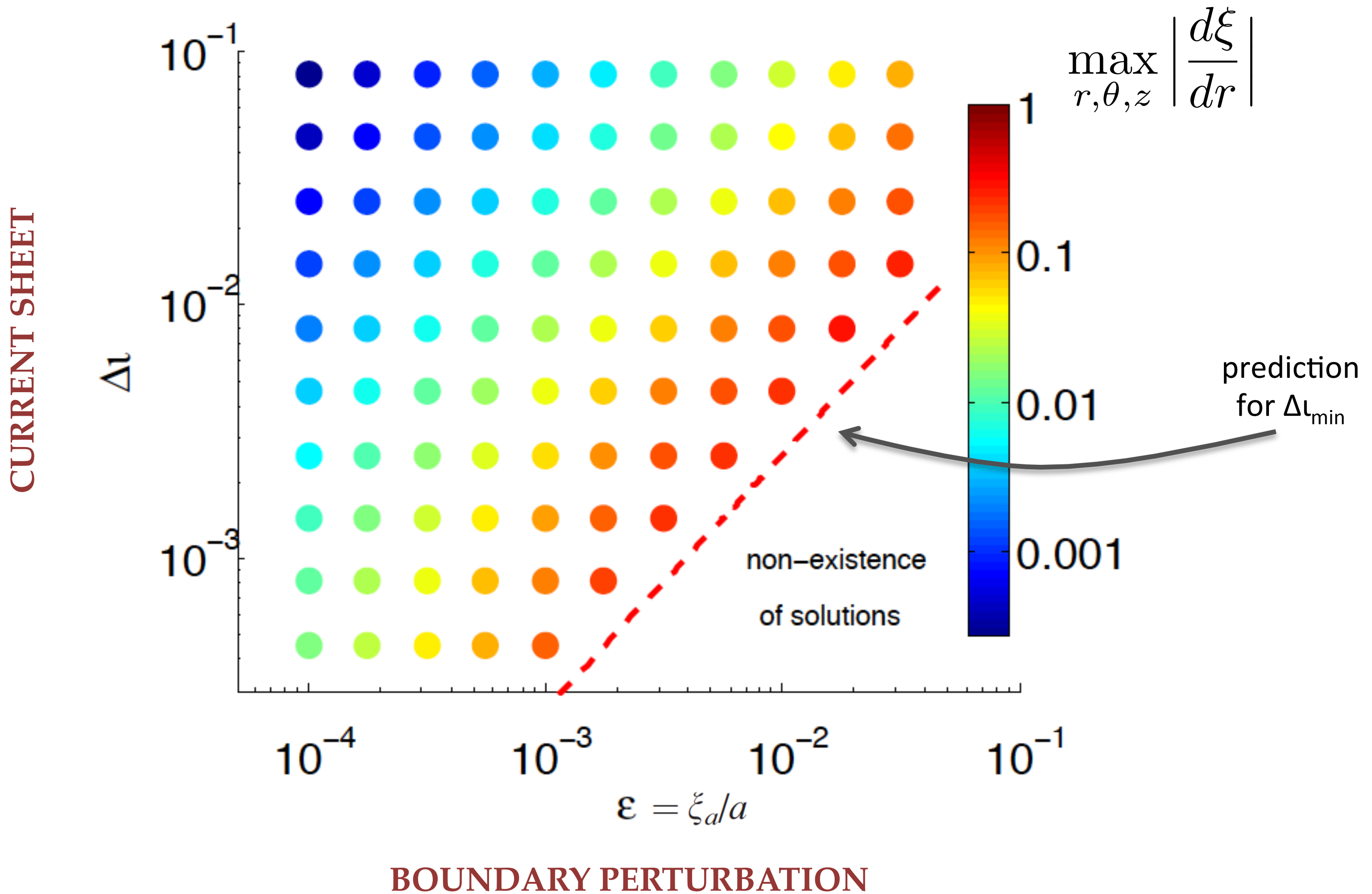
$$f = B_z^2(\iota - \iota_s)^2\bar{k}r^2 \ ,$$

$$g = \frac{f}{r^2}(k^2r^2 + m^2 - 1) + B_z^2(\iota_s^2 - \iota^2)2\bar{k}^2\iota_s^2 r$$

- *Sine qua non* condition for the existence of equilibria: $|\xi'| \leq 1$

- Implies **minimum current sheet**: $\Delta\iota \geq \Delta\iota_{min} = 2\iota_s'\xi_s$

Existence space is reproduced in nonlinear calculations



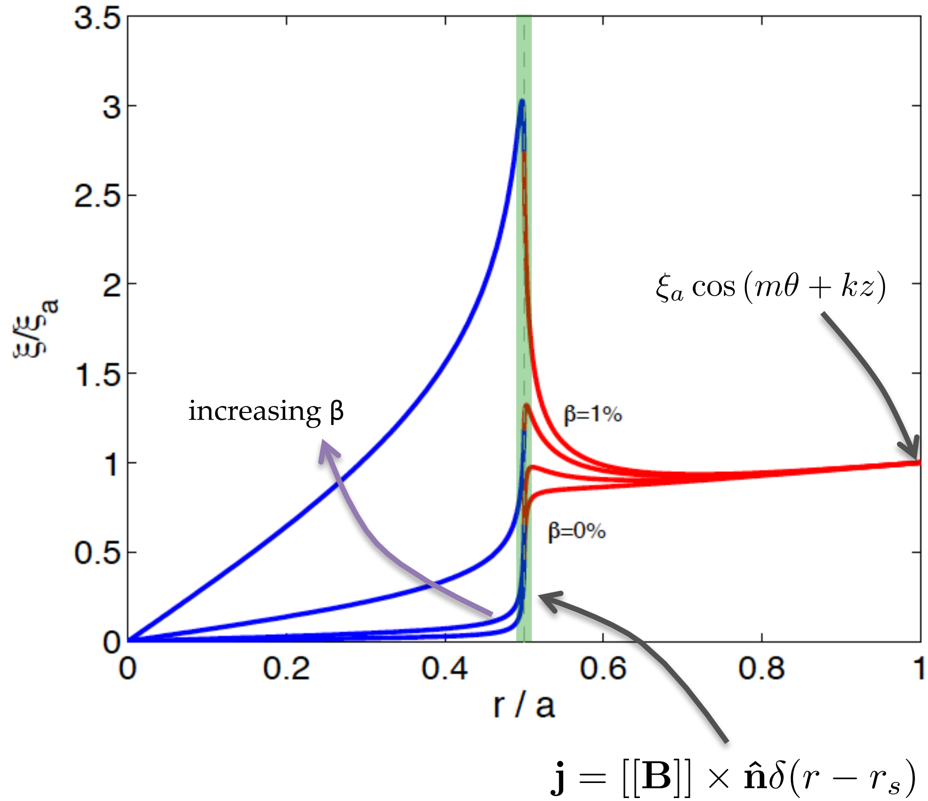
Ideal linear response to an RMP at β > 0

- Solve Newcomb equation:

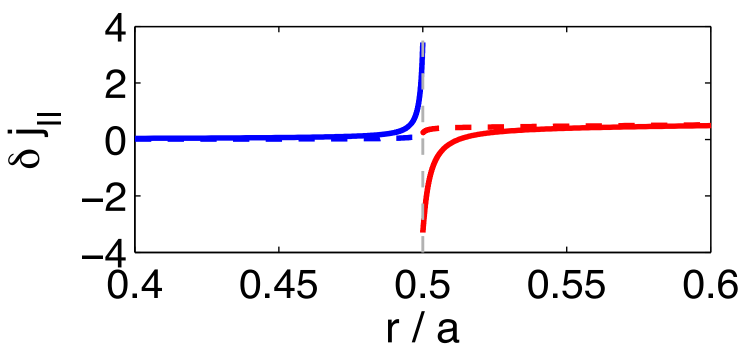
$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = f|_{\beta=0}\ ,$$

$$g = g|_{\beta=0} + \bar{k}\iota_s^2 r^3 p'$$

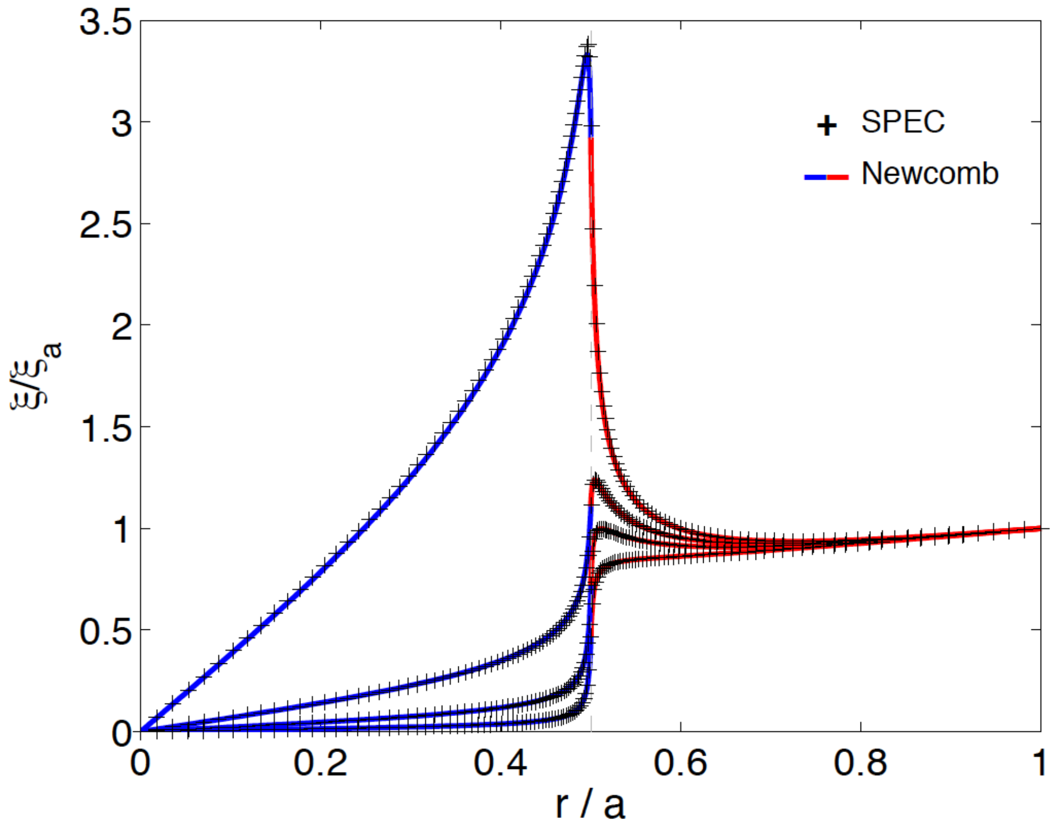


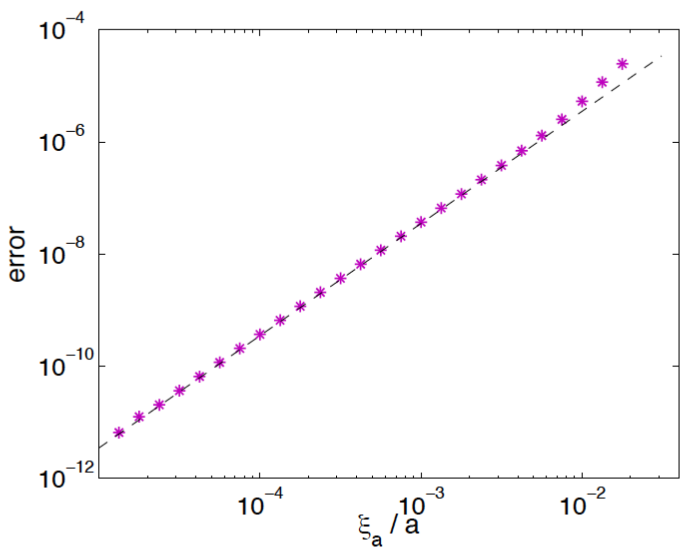
- Pfirsch-Schlutter current:



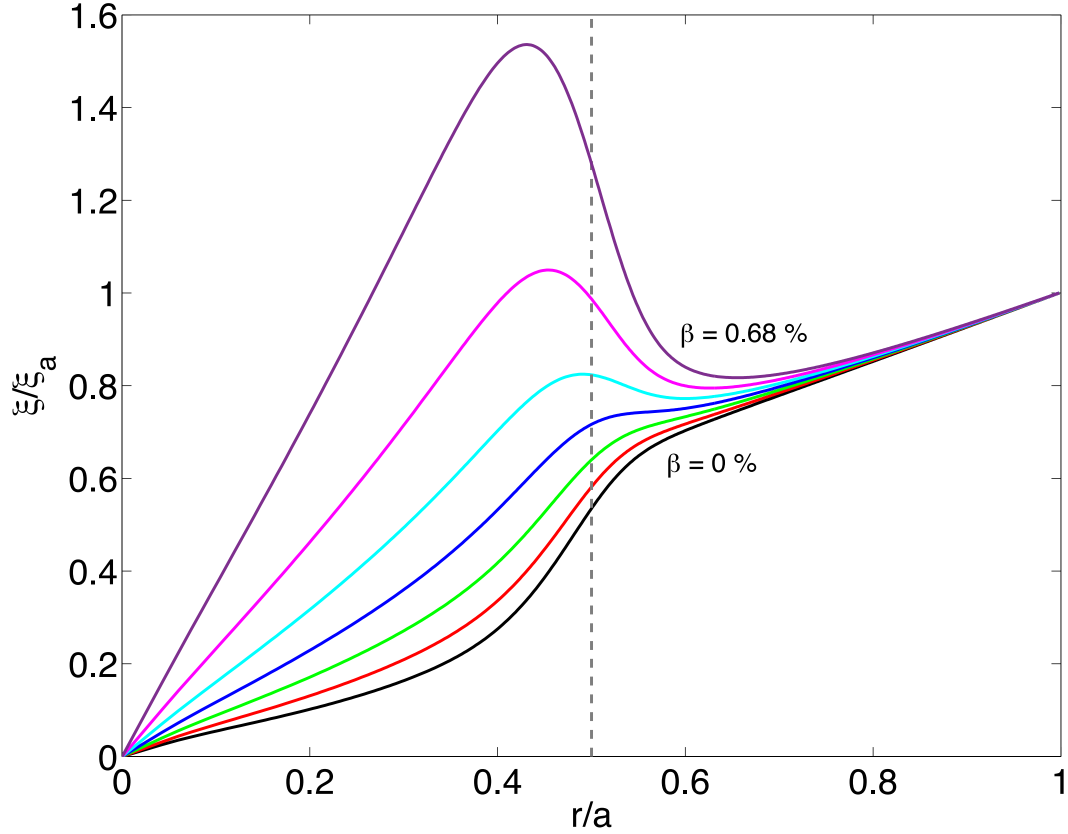
- Pressure-driven **amplification** and **penetration** of RMP in ideal MHD!

SPEC nonlinear calculations exactly verified





VMEC qualitatively reproduces same behaviour



An exact agreement with Newcomb's solutions may require explicit handling of discontinuities in the magnetic field.

Summary and perspectives

- First numerical proof of the existence of singular current densities.

  [Loizu et al, Phys Plasmas 22 022501, 2015]

- New class of 3D MHD equilibria allows for nested surfaces and smooth pressure.

  [Loizu et al, Phys Plasmas 22 090704, 2015]

- Novel prediction: amplification and penetration of RMP even within ideal MHD.

  [Loizu et al, Phys Plasmas, submitted]

- The questions

  (1) what sets the value of for $\Delta\iota$?

  (2) how are these states accessed ?

  remain to be investigated.