Ideal MHD in the nested flux surface limit

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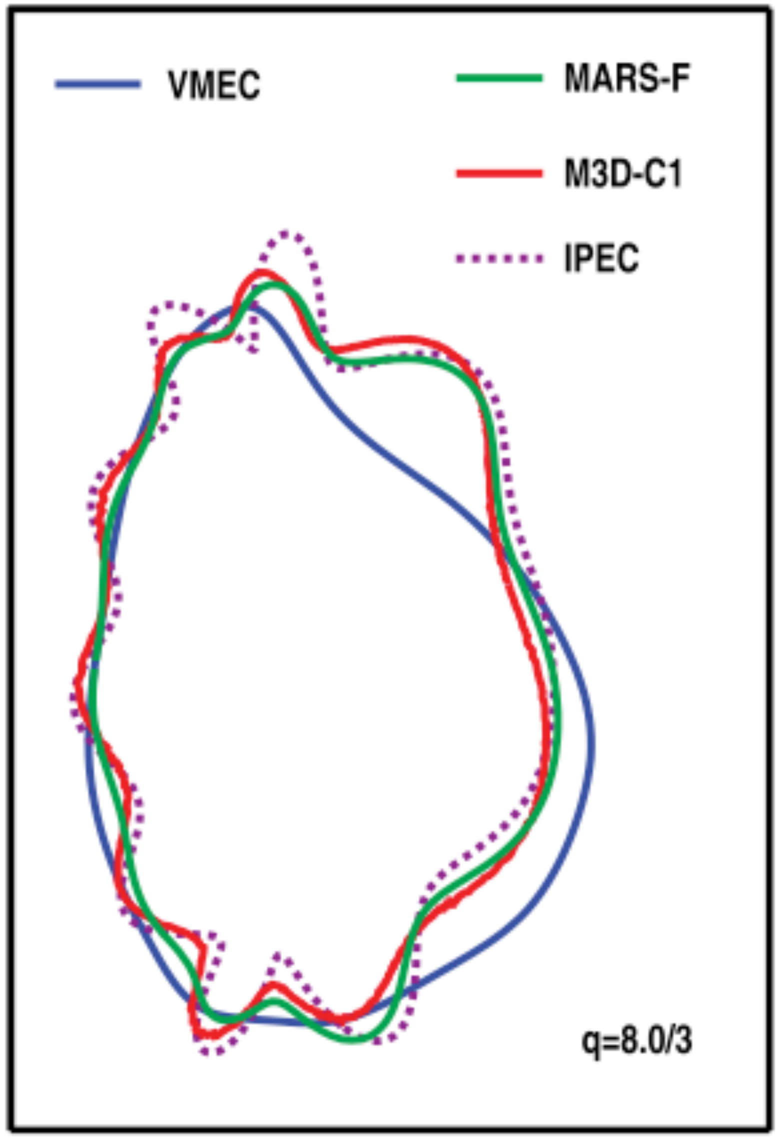
Motivation

Differences were found between VMEC and linear code responses at rational surfaces (Turnbull et al. 2013)



Turnbull A D et al. 2013 Comparisons of linear and nonlinear plasma response models for non-axisymmetric perturbations Phys. Plasmas 20 056114

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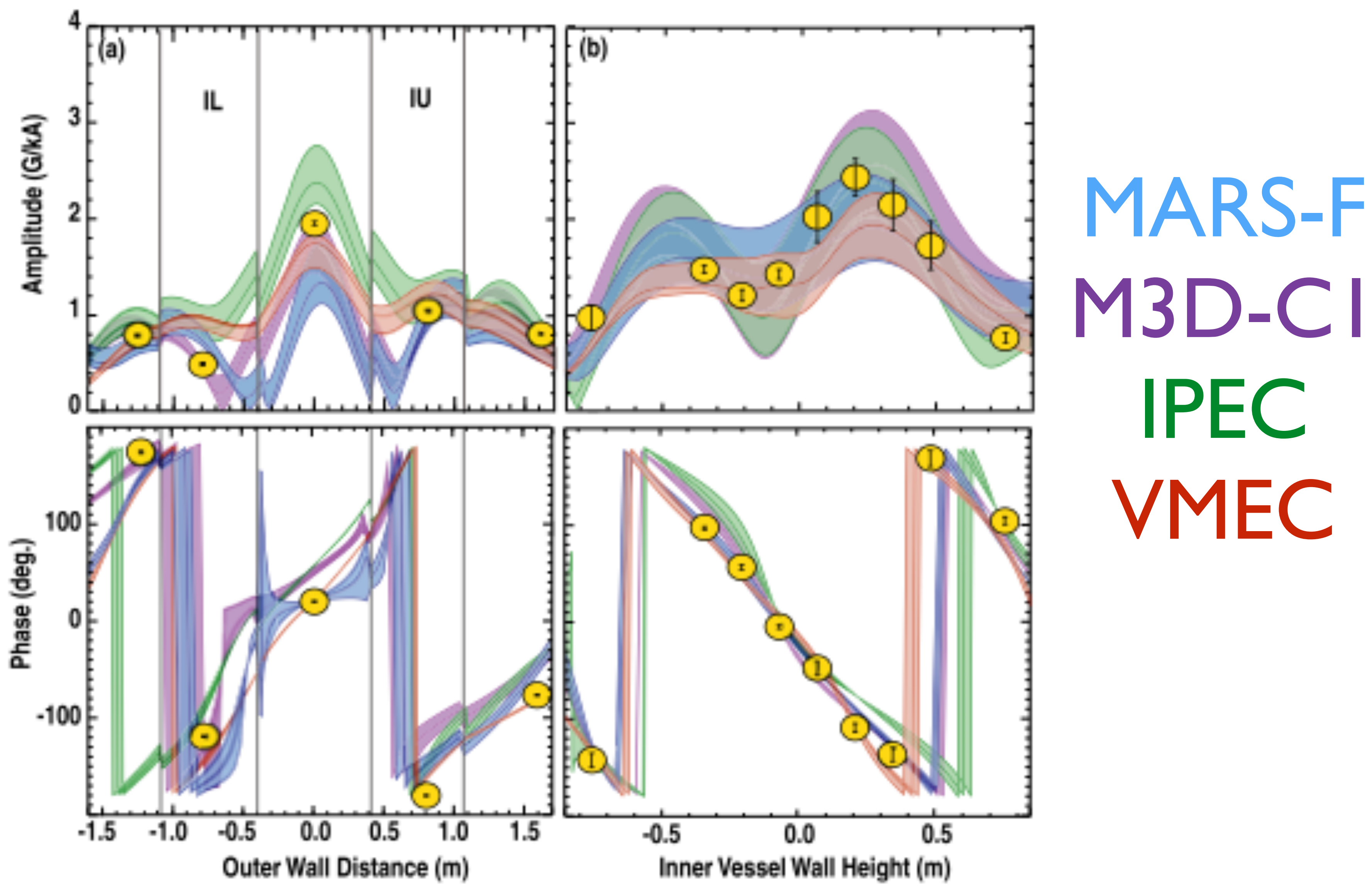
DIII-D 3D magnetic diagnostics did not discriminate plasma models (King, 2015)



King J D et al. 2015 Experimental tests of linear and nonlinear three-dimensional equilibrium models in DIII-D Phys. Plasmas 22 072501

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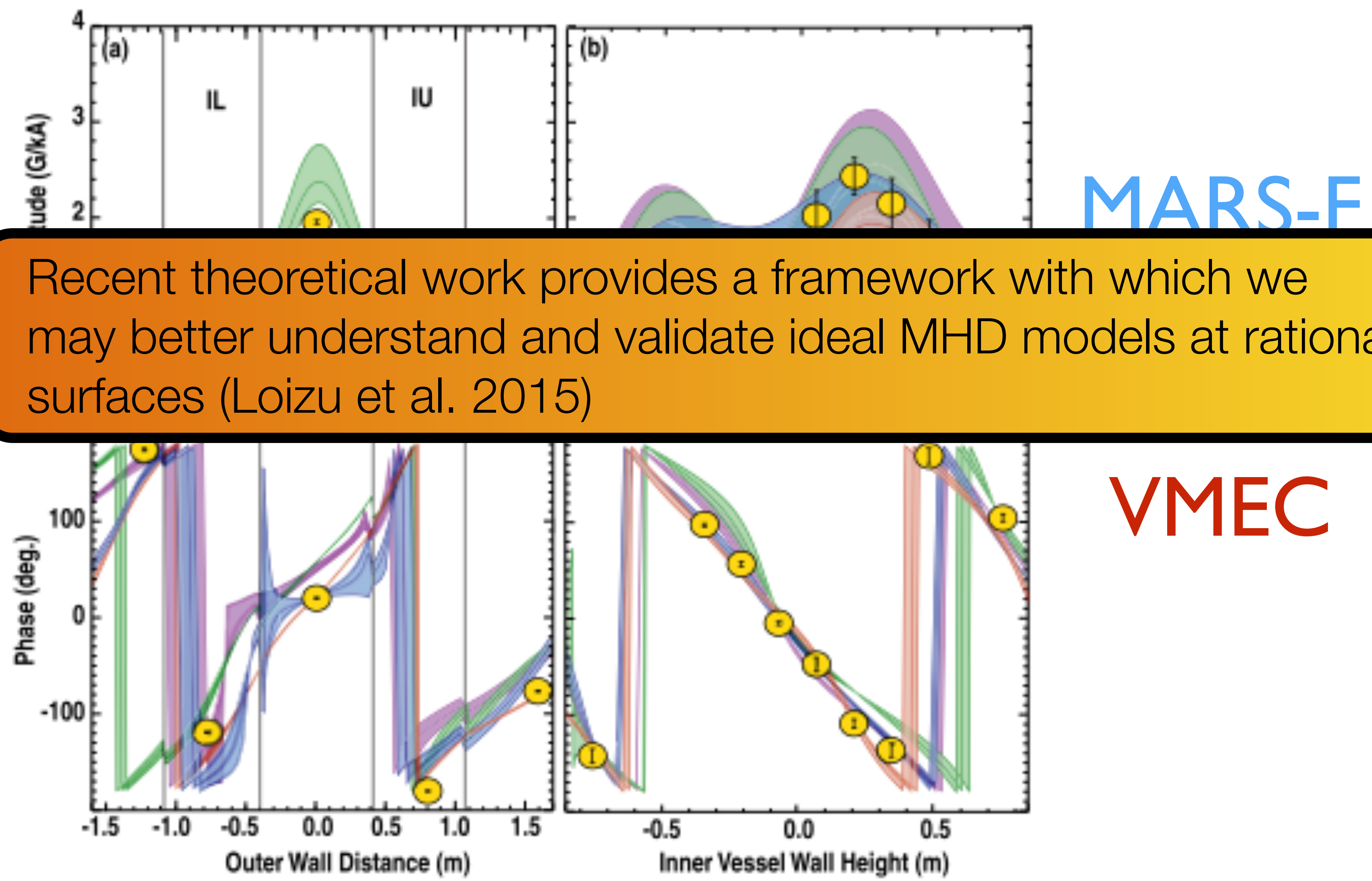

Recent theoretical work provides a framework with which we may better understand and validate ideal MHD models at rational surfaces (Loizu et al. 2015)

King J D et al. 2015 Experimental tests of linear and nonlinear three-dimensional equilibrium models in DIII-D Phys. Plasmas 22 072501

Outline of talk

- A new solution to Newcomb's equation
- The VMEC solution to a screw-pinch
- Finite beta effects
- Current sheets in real equilibria
- Concluding remarks

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$$\jmath\vec{B} \cdot \nabla = \iota\!\!\!-\,\partial_\theta - \partial_\zeta$$

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Expanding the parallel current in Fourier modes we then find

$$u_{mn} = \frac{h_{mn}(x)}{x} + \Delta_{mn}\delta(x)$$

$$x = \iota\!\!\!-\, m - n$$

$$h_{mn}(x) \equiv i\left(\jmath \nabla \cdot \vec{j}_\perp\right)_{mn}$$

$$\vec{j} \times \vec{B} = \nabla p \Rightarrow \vec{j}_\perp = \frac{\vec{B} \times \nabla p}{B^2}$$

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0 for p=0

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This reduces to Newcomb's equation

$$\frac{d}{dr}\left(f \frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = B_Z^2 \left(\iota - \iota_s\right)^2 \frac{r^3}{R^2 + r^2\iota^2}$$

$$g = B_Z^2 \left[\left(\iota - \iota_s\right)\left(k^2 r^2 + m^2 - 1\right)\bar{k} + \left(\iota - \iota_s\right) 2\bar{k}^2 r\right]$$

$$k = -\frac{n}{R} \qquad \iota_s = \frac{n}{m} \qquad \bar{k} = \frac{r}{R^2 + r^2 \iota_s^2}$$

Now consider a screw pinch

Newcomb's equation is singular where

$$\iota(r_s) = n/m$$

Resulting in a discontinuous plasma displacement, resulting in overlap of surfaces

$$|d\xi/dr| > 1$$

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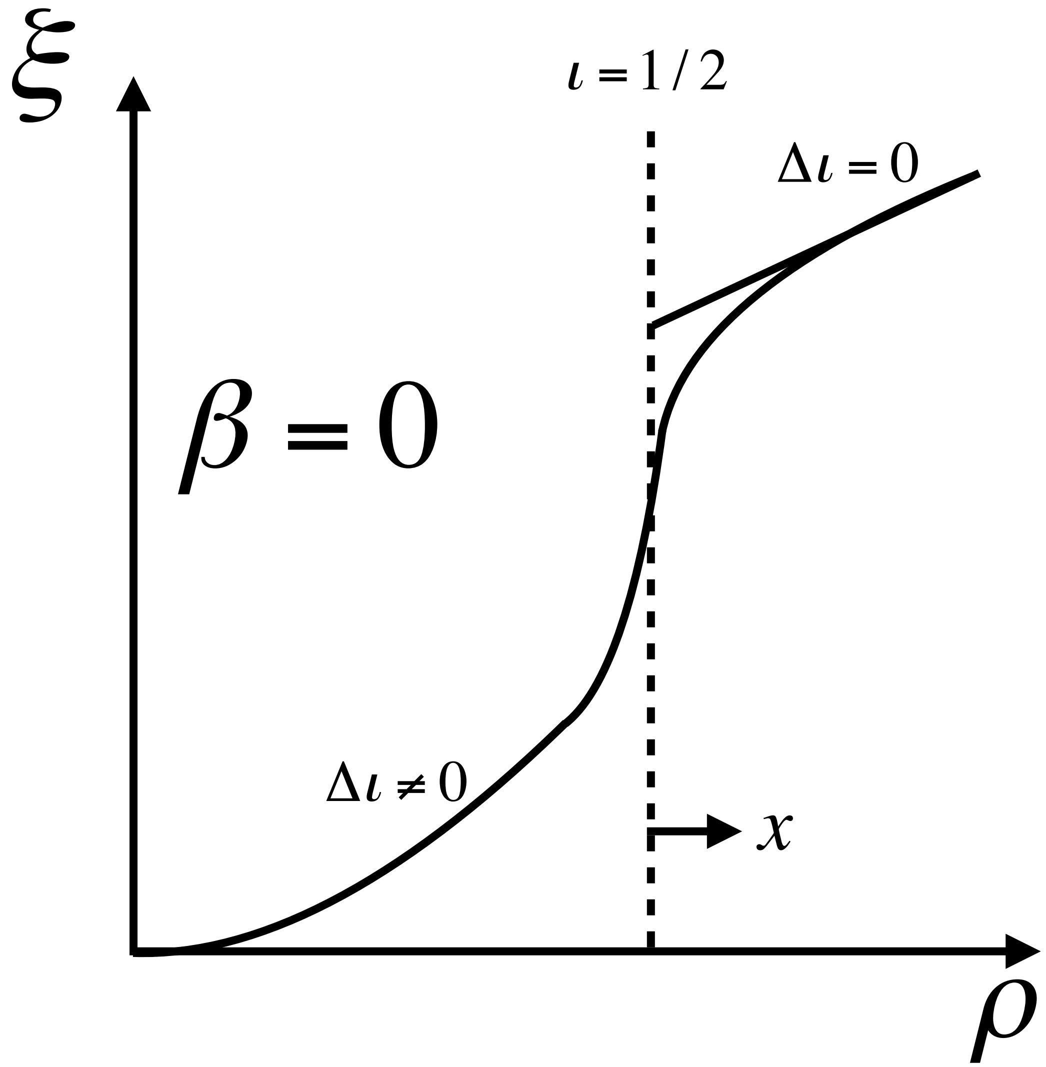
A resolution to this inconsistency has been formulated

- Ideal MHD equilibria with resonant surfaces are not analytic functions of the 3D boundary.
- An equilibrium model with fractal radial grid is unattractive.
- It has been recognised that infinite shear prevents surfaces from overlapping at rational surfaces*.
- The resolution is to include a discontinuity in the rotational transform at the resonant surface (no longer rational).

* Loizu J et al. 2015 Magnetic islands and singular currents at rational surfaces in three-dimensional magnetohydrodynamic equilibria Phys. Plasmas 22 022501

The perturbed screw-pinch



Reconsider the screw pinch, including a discontinuity in the rotational transform

$$\Delta\iota \equiv \iota(r_s^+) - \iota(r_s^-) > 0$$

Near the resonant surface we define

$$x = \left| \frac{\iota - \iota_s}{\partial\iota / \partial r} \right|$$

Expanding about the surface we find

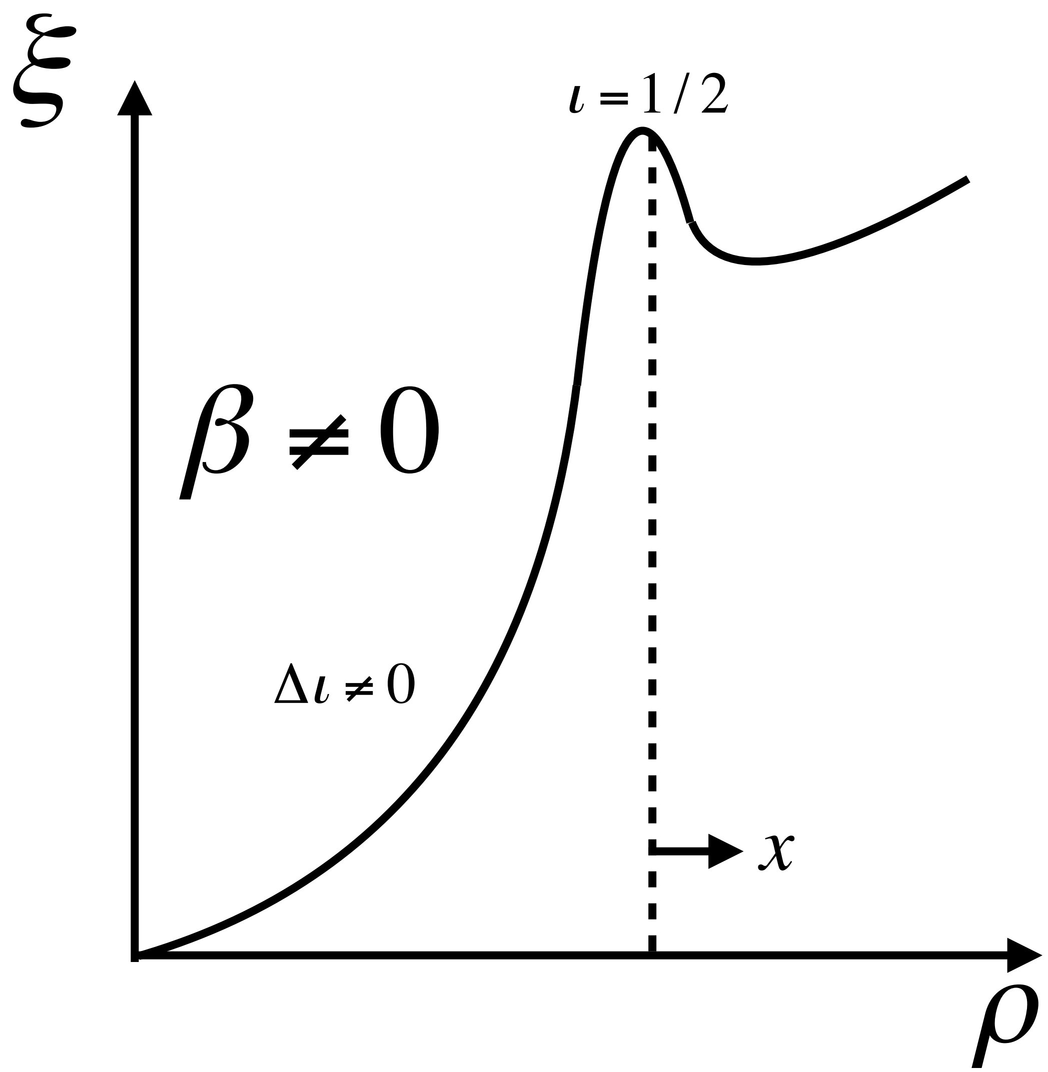
$$\xi \sim x^\alpha \qquad \alpha = -1, \cancel{0}$$

The maximum gradient of the displacement may then be written

$$|\xi'| = 2 \frac{\partial\iota_s}{\partial r} \frac{\xi}{\Delta\iota} \xrightarrow{|\xi'| < 1} \Delta\iota > \Delta\iota_{min} = 2\xi_s \iota_s'$$

Loizu J et al. 2015 Existence of three-dimensional ideal-magnetohydrodynamic equilibria with current sheets Phys. Plasmas 22 090704

The finite beta behaviour has been explored



Inclusion of pressure in the screw pinch problem modifies Newcomb's equation so that

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f = f_{\beta=0}$$

$$g = g_{\beta=0} + \bar{k}\iota_s^2 r^3 p'$$

Expanding about the surface we now find

$$\xi \propto x^\alpha \qquad \alpha = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - D_S} \qquad D_S = -\left(\frac{2p'\iota_s^2}{rB_Z^2\iota'^2}\right)_S < \frac{1}{4}$$

At finite beta this Suydam criterion returns

$$\xi \sim \lambda_1 x^{\alpha_1} + \lambda_2 x^{\alpha_2} \qquad -1 < \alpha_1 < -\frac{1}{2} < \alpha_2 < 0$$

Loizu J et al. 2015 Pressure driven amplification and penetration of resonant magnetic perturbations Phys. Plasmas (submitted)

Can these results be used to validate VMEC?

- VMEC assumes nested flux surface by construction.

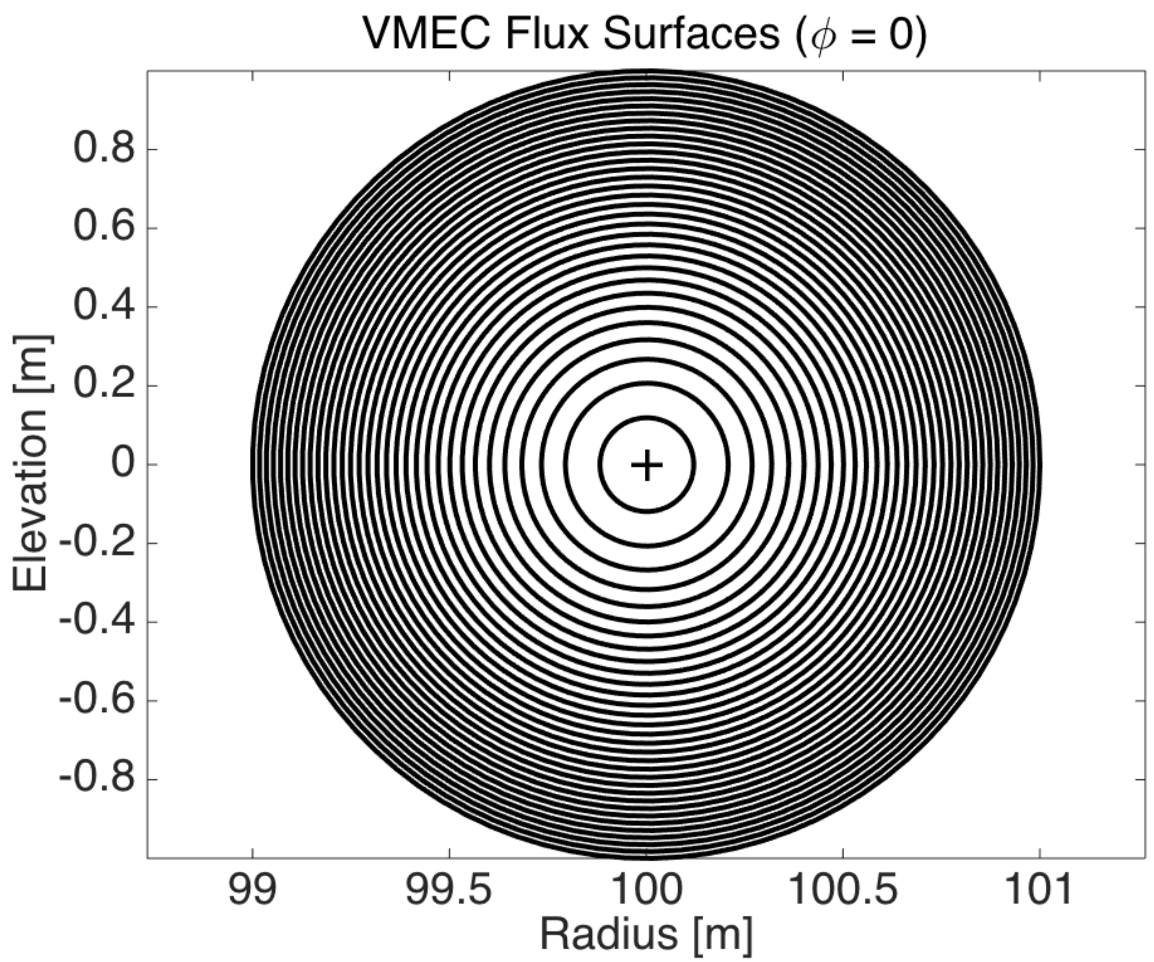
$$\vec{B} = \nabla\zeta \times \nabla\chi + \nabla\Phi \times \nabla\theta^*$$

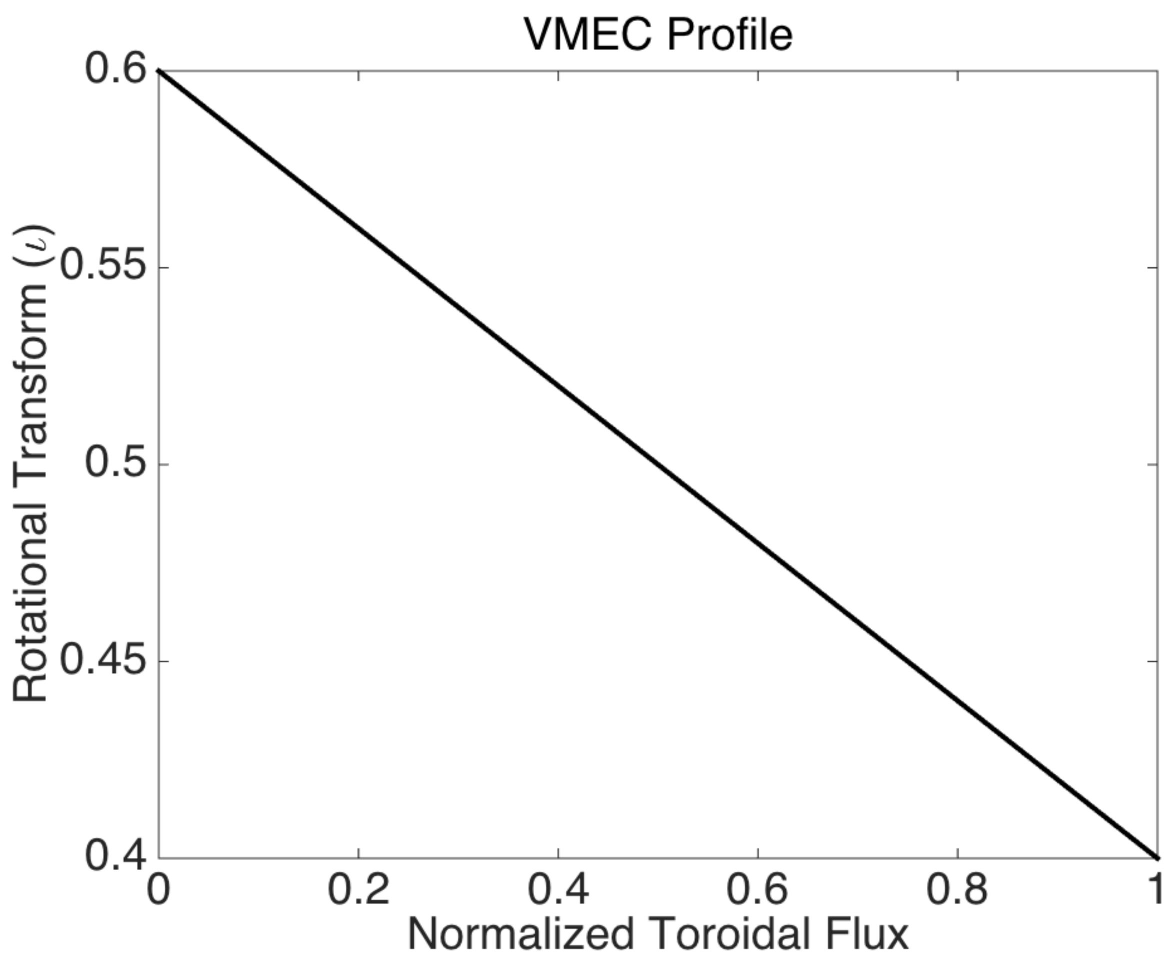
- Can approximate a cylindrical model.

$$\frac{R}{a} \to \infty \qquad N_{fp} \to \infty$$

- Continuous iota profiles.

The VMEC screw-pinch



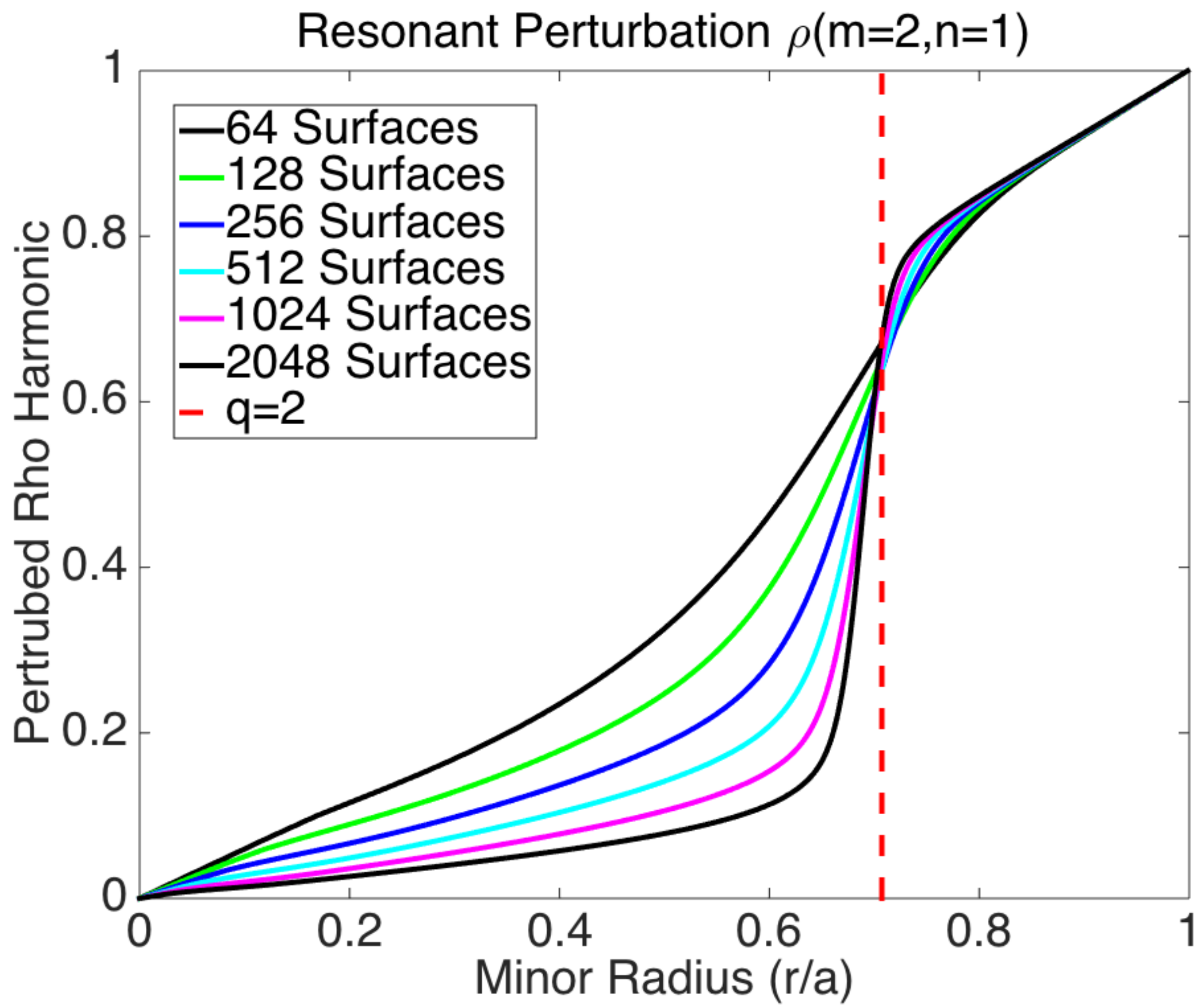


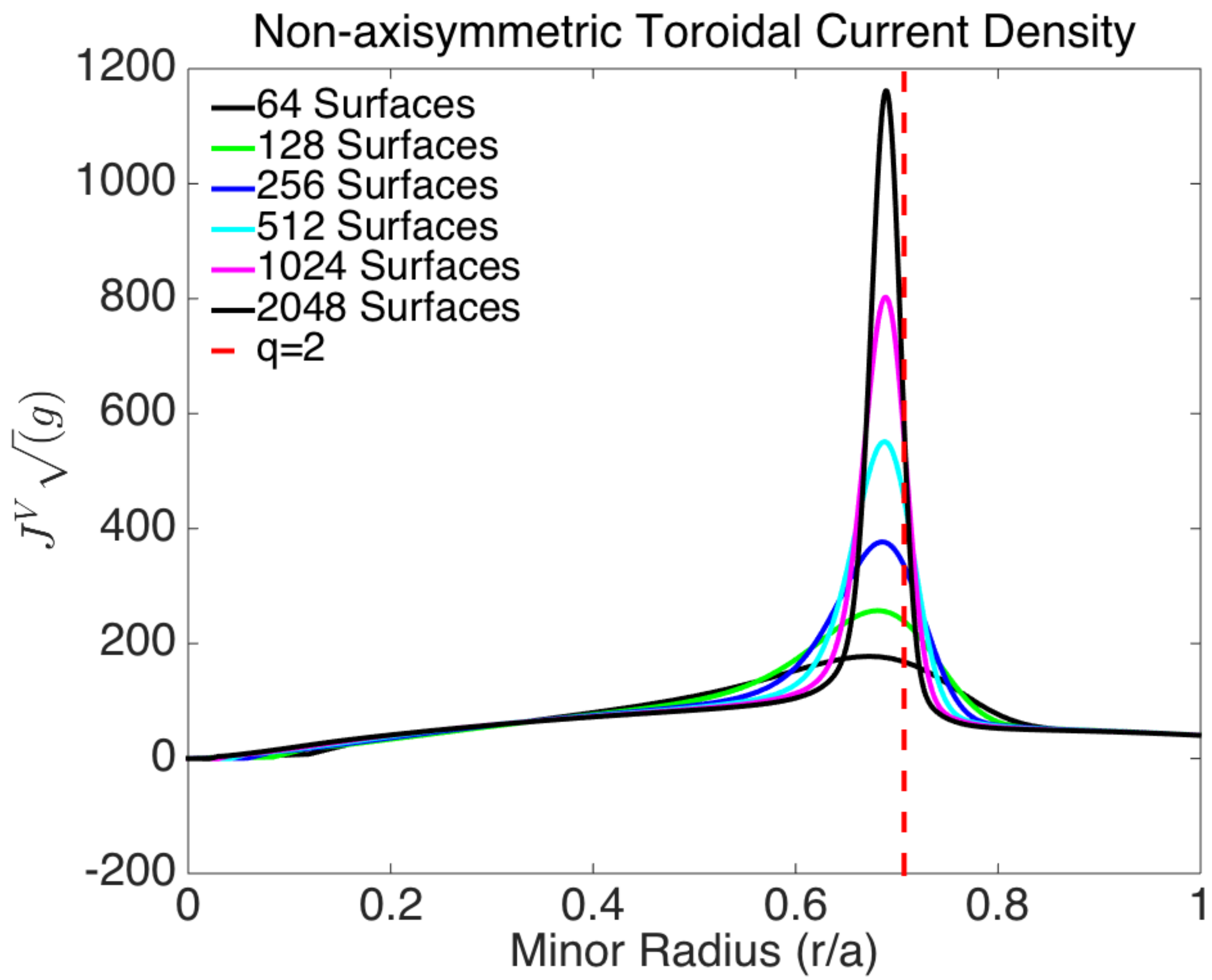
$$R = R_{00} + a\cos\theta + \frac{\delta\rho_{n_1,m_1}}{2}\cos\left[(m_1+1)\theta - n_1\zeta\right] + \frac{\delta\rho_{n_1,m_1}}{2}\cos\left[(m_1-1)\theta - n_1\zeta\right]$$

$$Z = a\sin\theta + \frac{\delta\rho_{n_1,m_1}}{2}\sin\left[(m_1+1)\theta - n_1\zeta\right] - \frac{\delta\rho_{n_1,m_1}}{2}\sin\left[(m_1-1)\theta - n_1\zeta\right]$$

Aspect Ratio	100
Field Periodicity	100

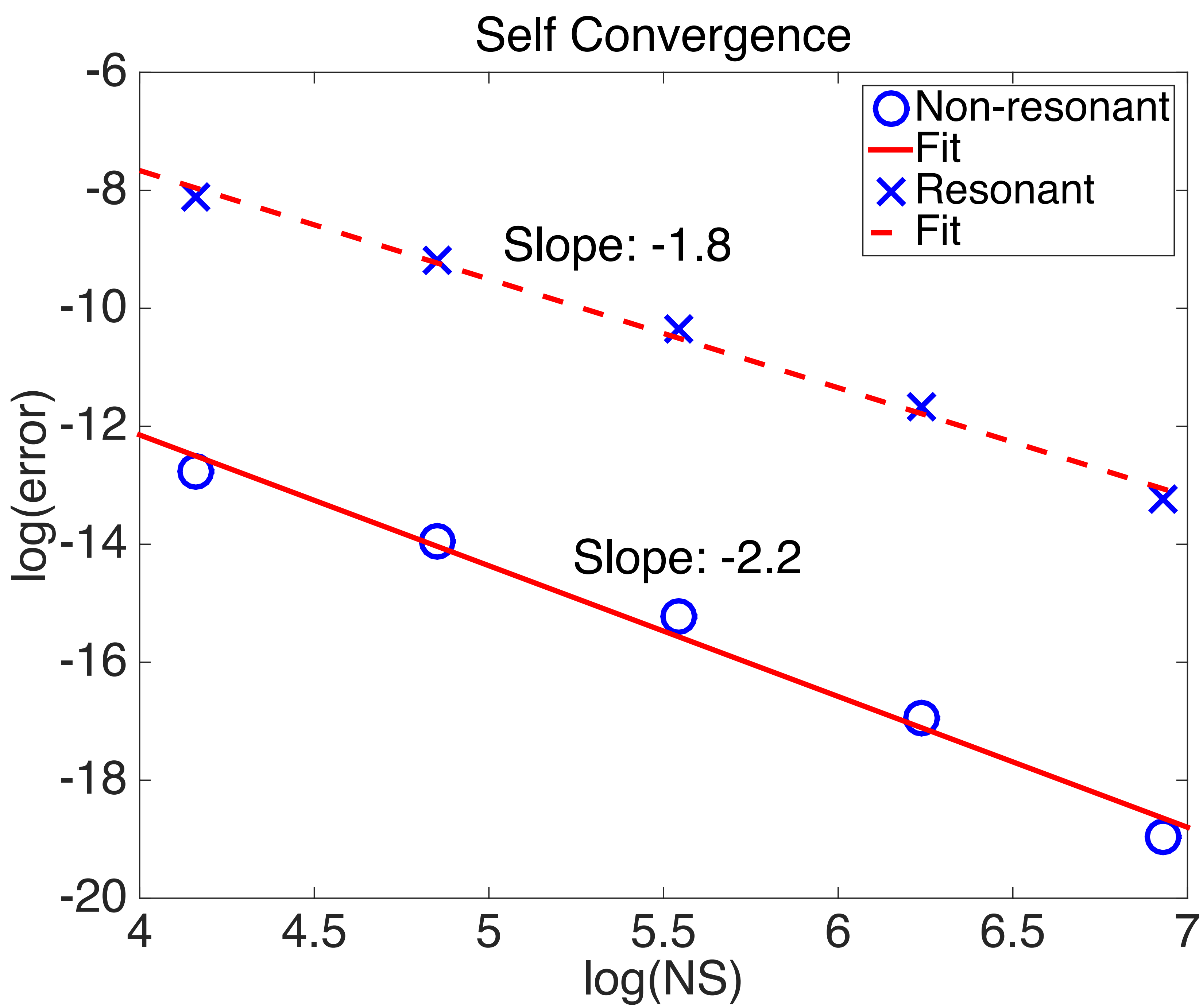
Resonant response



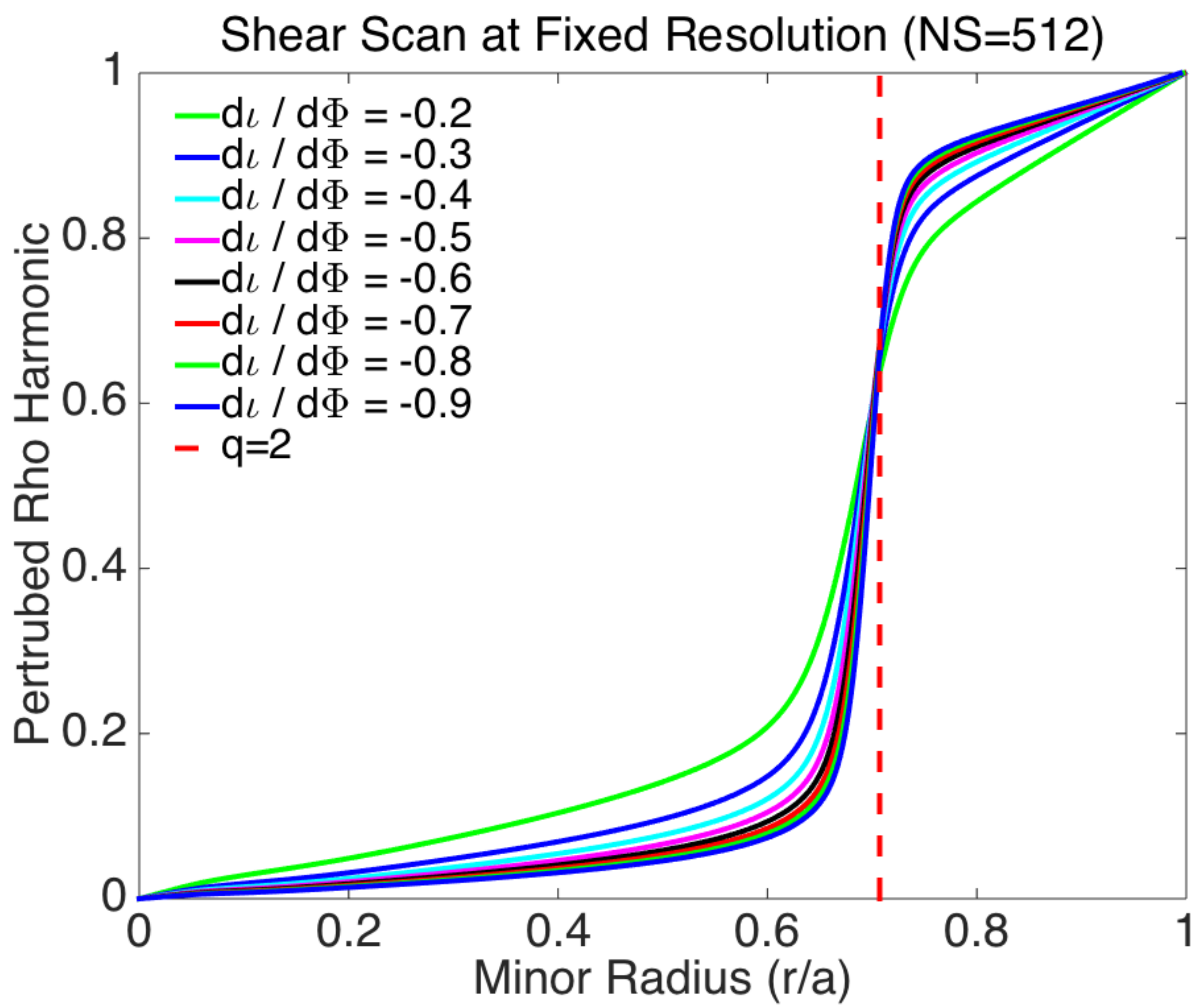


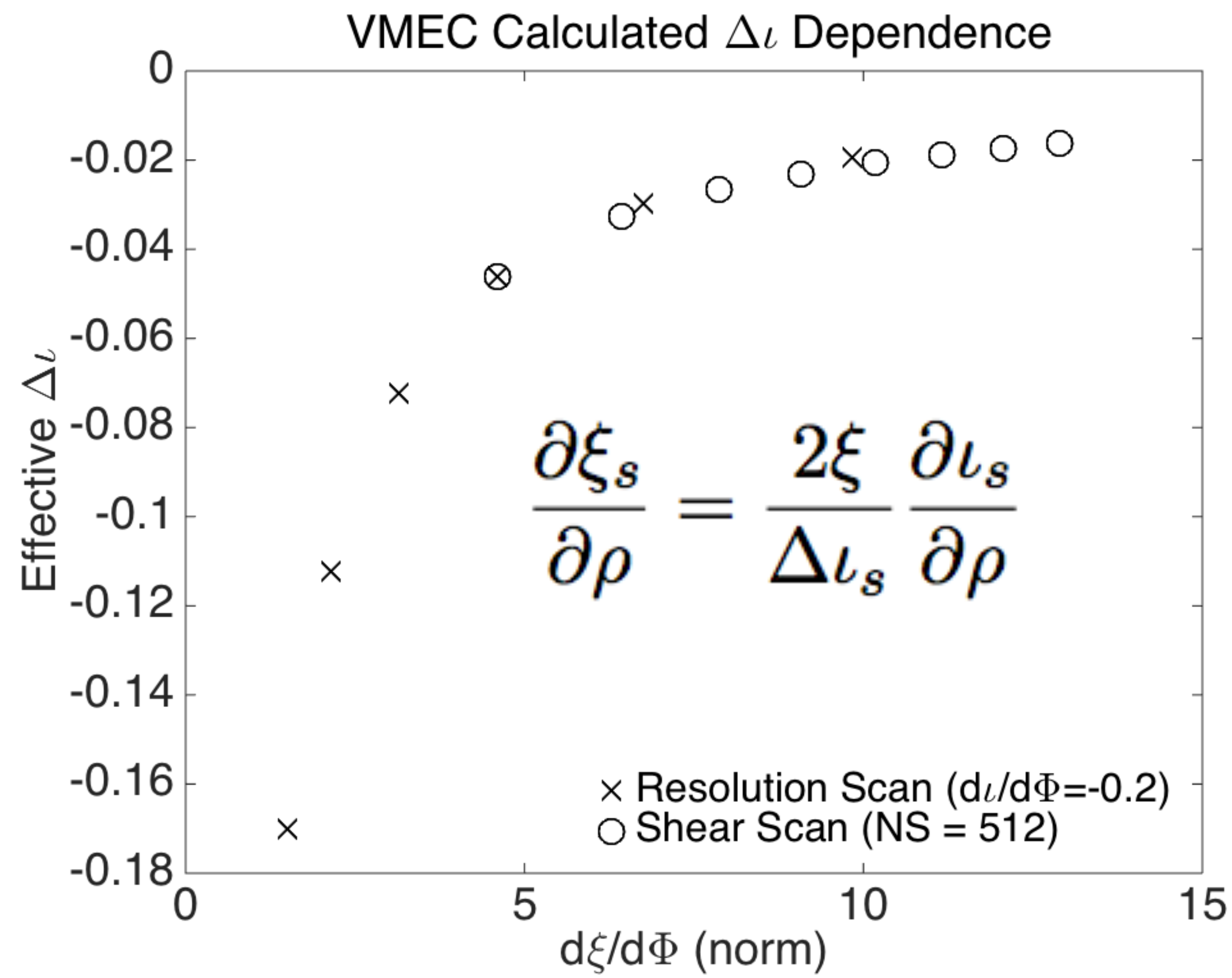
The displacement shows a response across the rational surface which scales with radial resolution.

Self-convergence consistent with finite difference



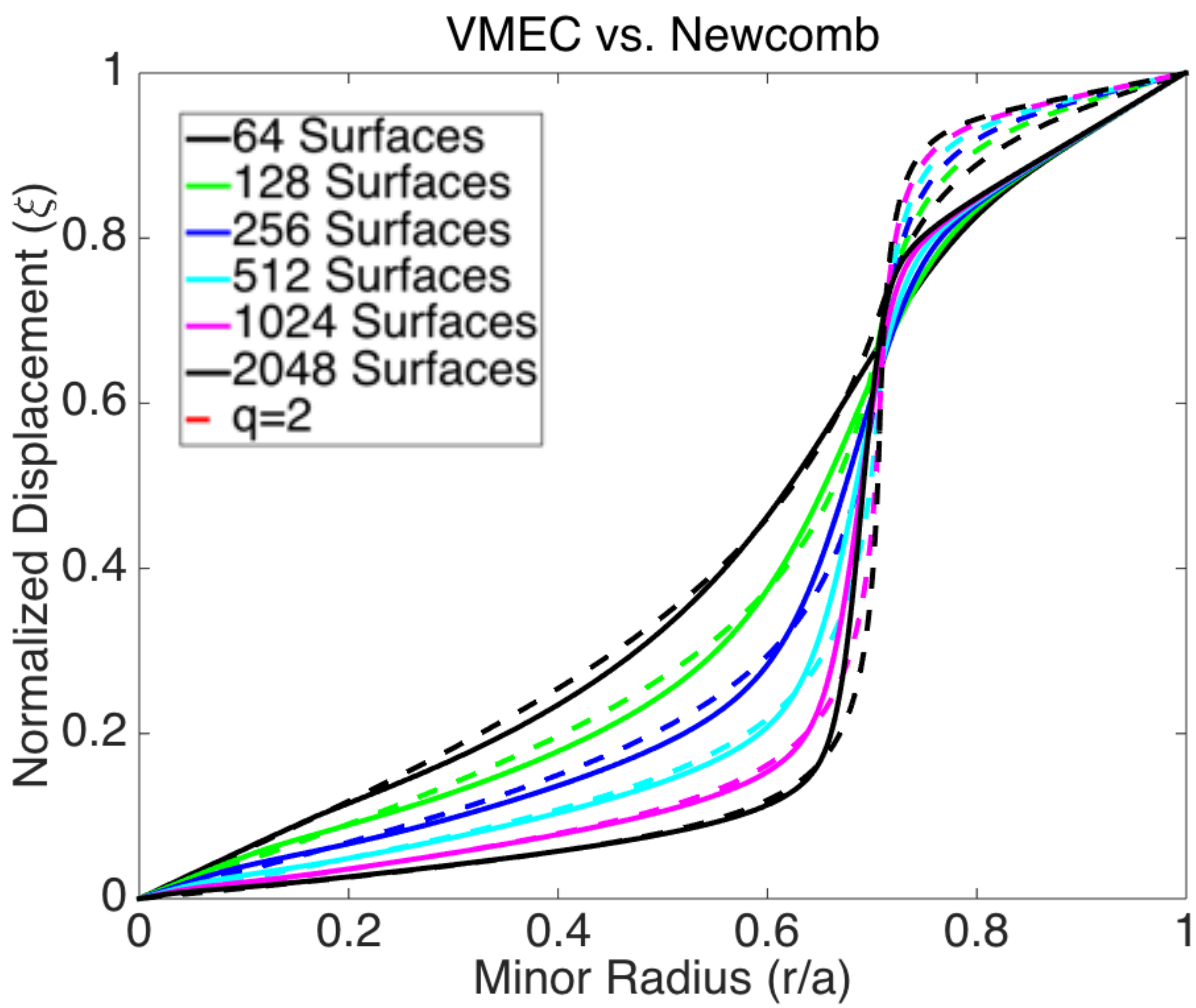
Response shows some shear dependence

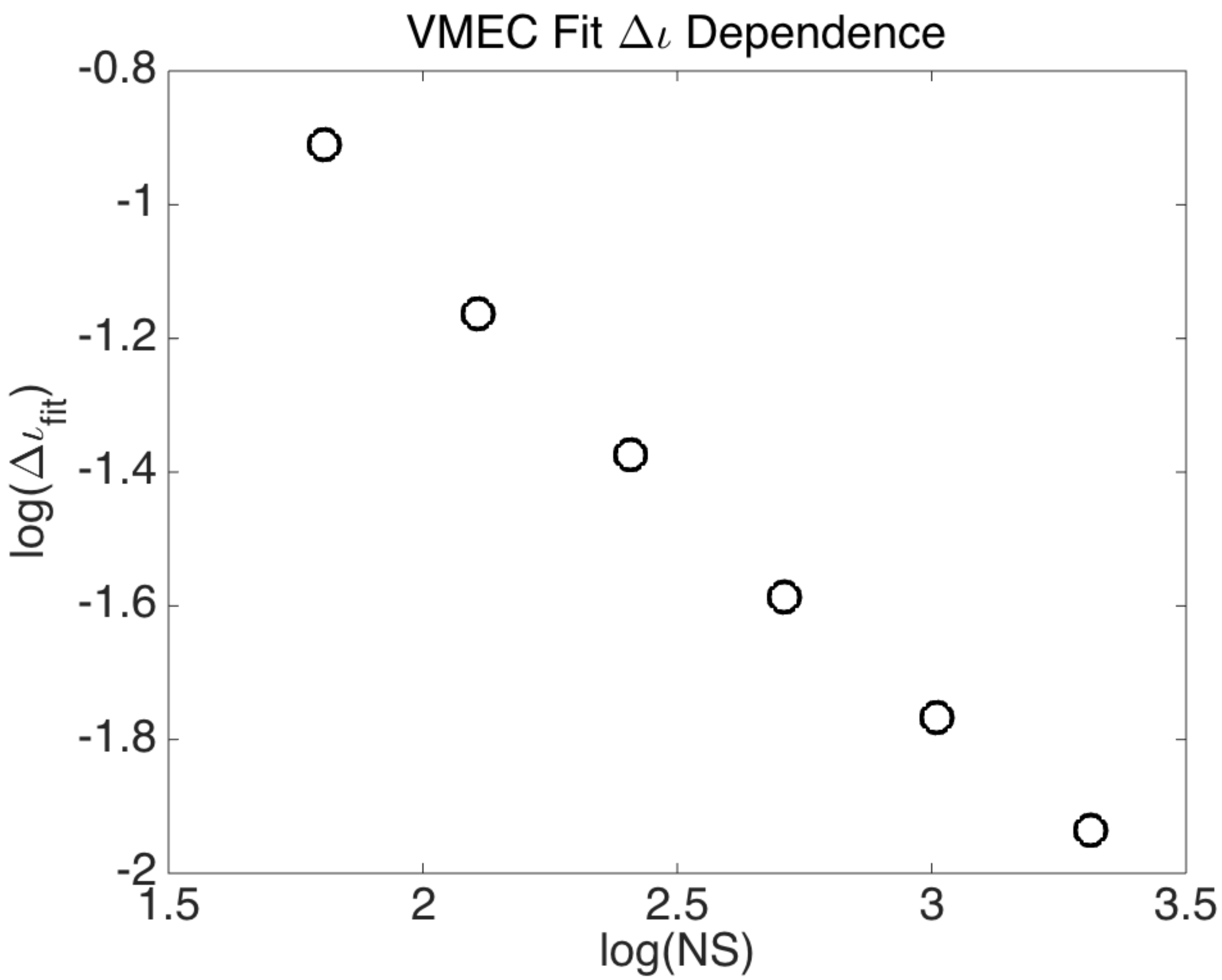




Shear and radial resolution play a role. However the effective discontinuity is greater than the minimum necessary for nested flux surfaces and that which can be attributed to the radial finite difference.

Qualitative agreement with theory

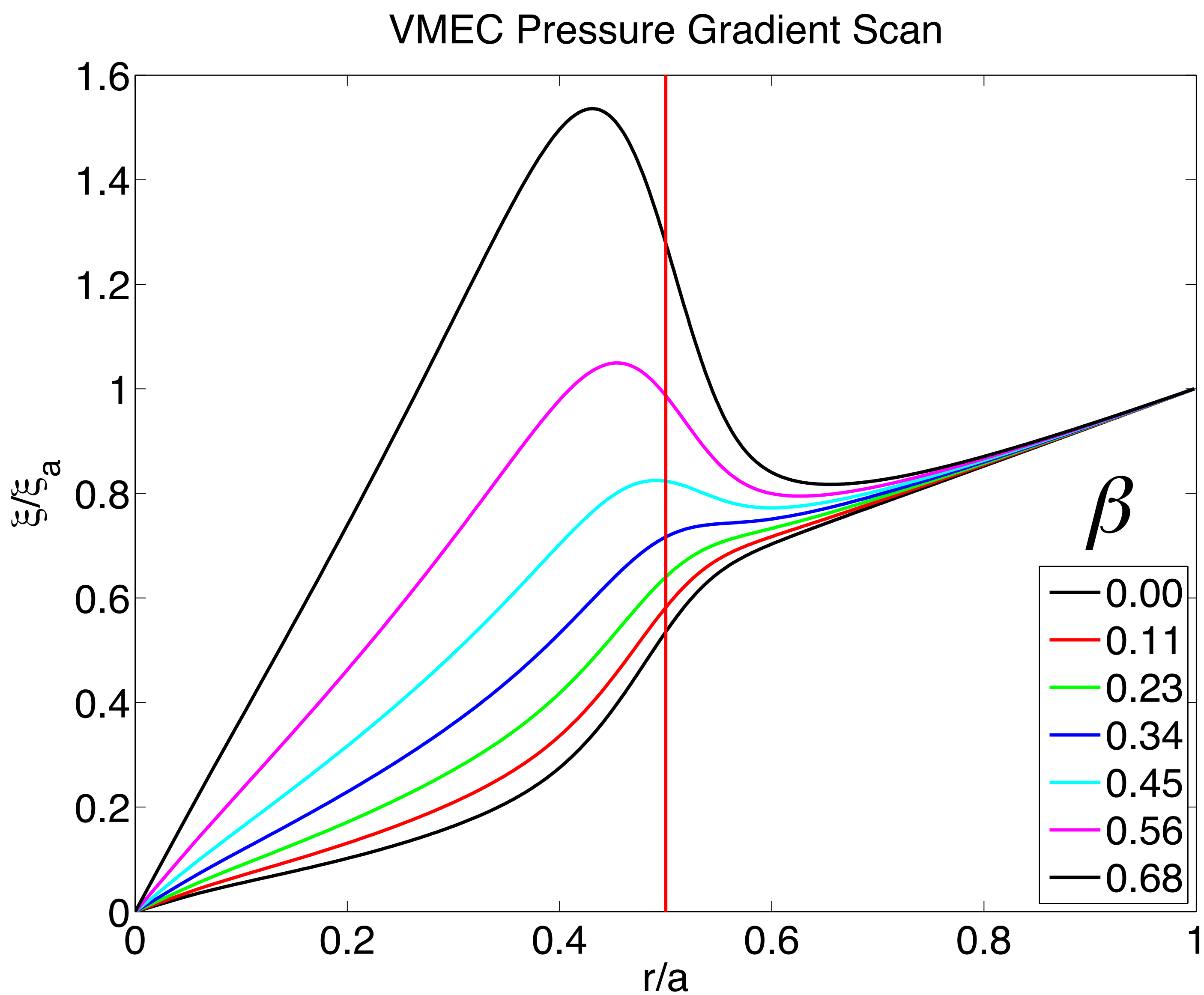




Comparison with the linear model suggests that despite the absence of a discontinuous iota profile, VMEC is obtaining qualitatively similar solutions.

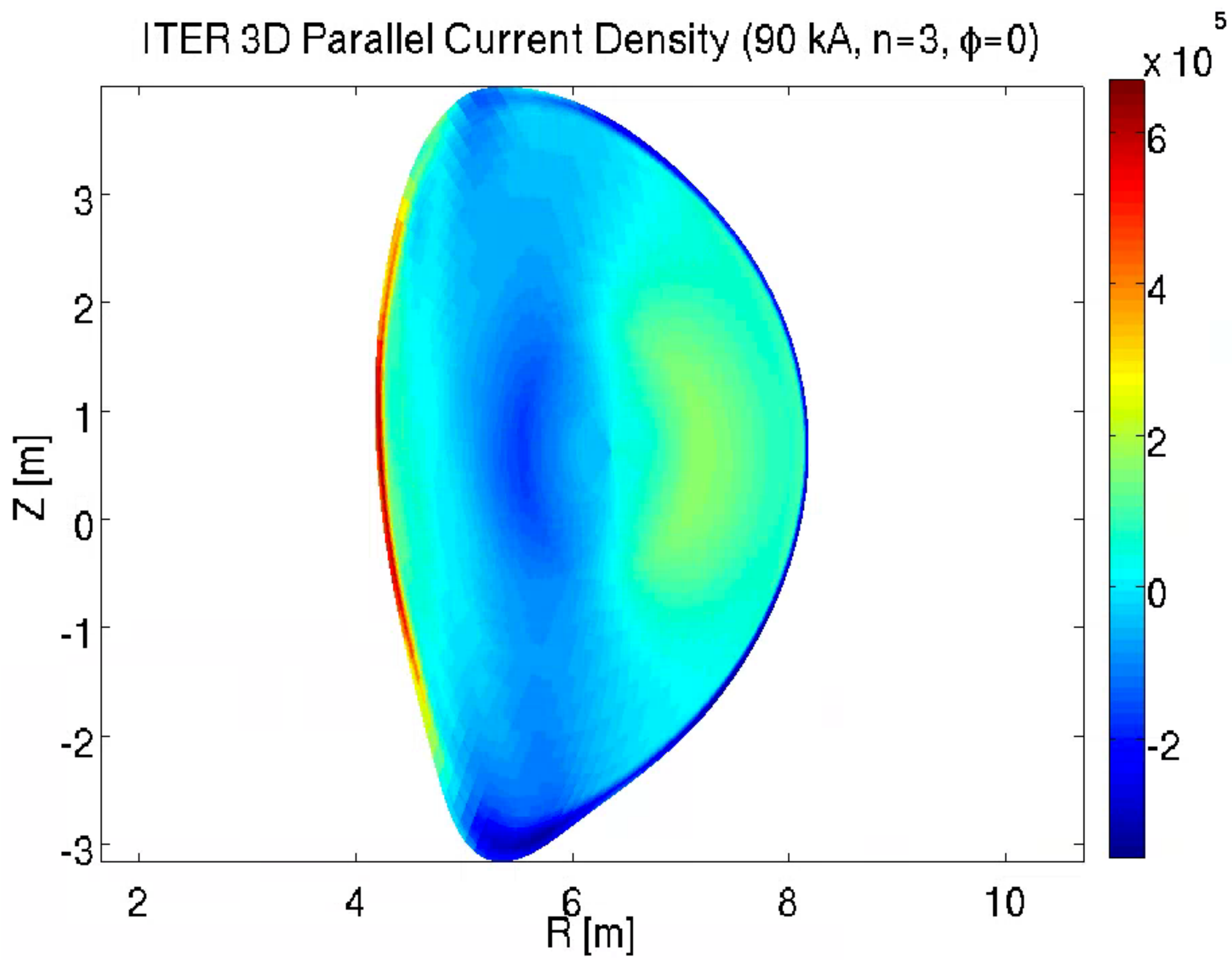
VMEC exhibits the finite beta response as well

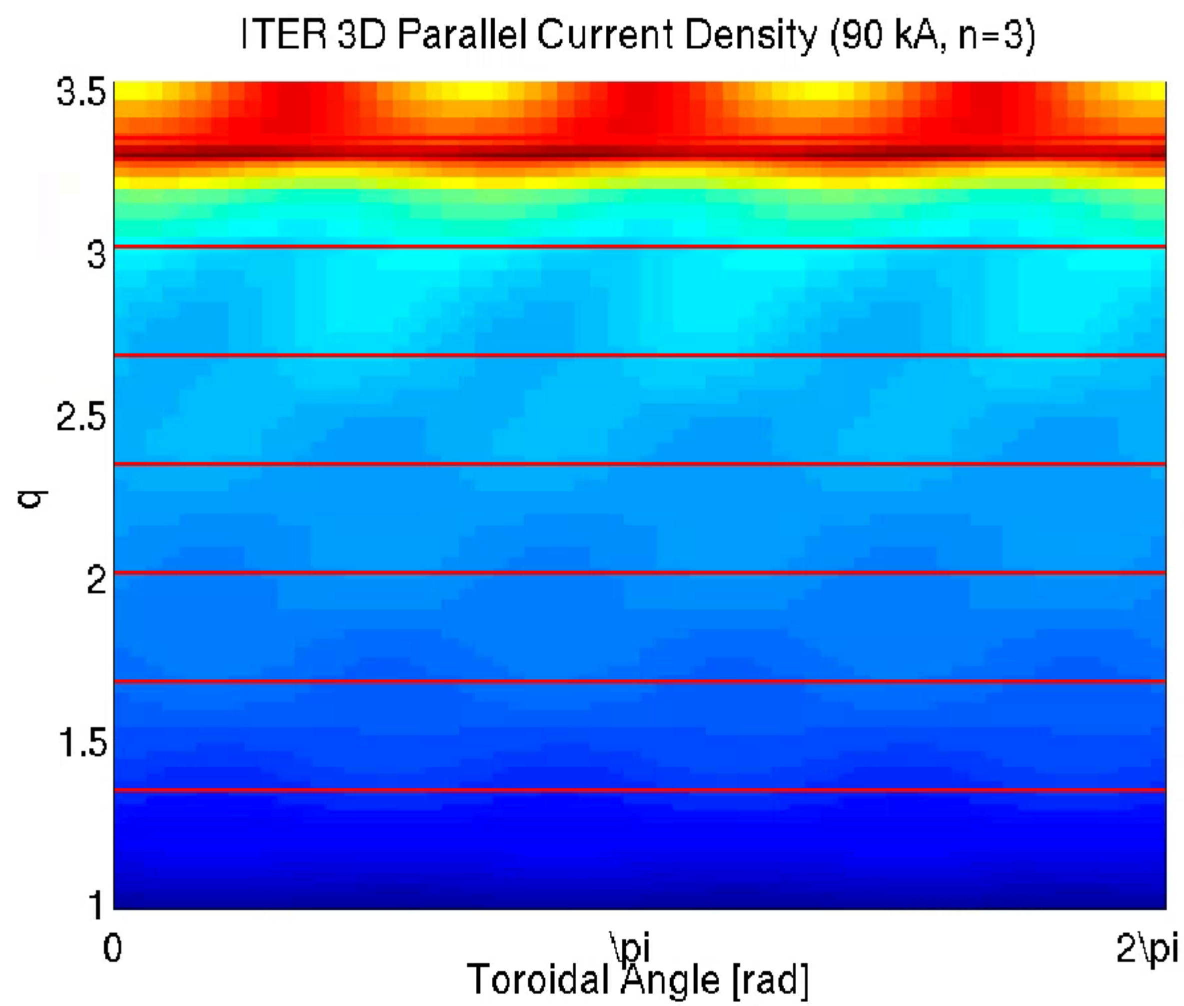
In this model pressure gradients can amplify resonant perturbations.



Effects present in real tokamak equilibria

The structure of the current density in the more realistic geometry of a ‘perturbed’ tokamak shows that this type of response is not limited to the screw-pinch case.





Lazerson S 2014 The ITER 3D magnetic diagnostic response to applied n= 3 and n= 4 resonant magnetic perturbations PPCF 56 095006

Conclusions

- The newly developed solutions to Newcomb's problem provide a means to validate the VMEC plasma response across rational surfaces.
- This new theoretical formulation suggests that VMEC is lacking transform discontinuities in it's formulation.
- Despite this shortcoming the VMEC code qualitatively reproduces the non-local response across a resonant surface.
- This response is also present in tokamak equilibria with applied RMP fields.