A comparison of linear and nonlinear solutions of ideal MHD equilibria in perturbed cylindrical geometry

S.R.Hudson, J. Loizu & S. Lazerson,
Princeton Plasma Physics Laboratory, Max Planck Institute for Plasma Physics;

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- METHOD : Ideal MHD equilibria in arbitrary geometry can be computed using the Stepped Pressure Equilibrium Code (SPEC) in the "ideal-limit"; i.e., as $N \to \infty$.
- PROBLEM : Ideal equilibria with nested flux surfaces and ***continuous*** rotational-transform (with continuous pressure) have unphysical currents near rational surfaces.
- Ideal equilibria with nested flux surfaces and ***continuous*** rotational-transform (with or without pressure) ***are not*** analytic functions of the boundary.
  - $\to$ Linear perturbation theory breaks down !
  - $\to$ The ideal MHD equilibrium model with continuous rotational-transform does not provide self consistent equilibrium solutions in 3D !
- SOLUTION: Ideal equilibria with nested flux surfaces and ***discontinuous*** rotational-transform ***are*** analytic functions of the boundary.
  - $\to$ Ideal MHD equilibria with smooth pressure profiles in 3D only generally exist (i.e. arbitrary boundary) if the transform is discontinuous.
  - $\to$ Resonant components of perturbation fields are ***not*** completely shielded at the rational surface.

1 MHD Energy Functionals

Minimize energy subject to flux, helicity & topological constraints

1.1 Taylor-Relaxed

- conserved flux and helicity in $N = 1$ volumes:

$$F \equiv \int_{\mathcal{V}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv + \frac{\mu}{2} \left(\int_{\mathcal{V}} \mathbf{A} \cdot \mathbf{B} \, dv - K \right) \tag{1}$$

- ideal constraint, $\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, used only at boundary $\equiv$ global relaxation.
- $\delta F = 0$ when $\boxed{\nabla \times \mathbf{B} = \mu \mathbf{B}}$.
- flat pressure profile

1.2 MRxMHD

- Multi-Region, relaXed MHD
- split the plasma into N volumes separated by "ideal" interfaces, $\mathbf{B} \cdot \mathbf{n} = 0$.
- conserved fluxes and helicities in N volumes:

$$F \equiv \sum_{i=1}^{N} \left[\int_{\mathcal{V}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv + \frac{\mu_i}{2} \left(\int_{\mathcal{V}_i} \mathbf{A} \cdot \mathbf{B} \, dv - K_i \right) \right] \tag{2}$$

- ideal constraint, $\delta \mathbf{B} = \nabla \times \boldsymbol{\xi} \times \mathbf{B}$, used at each ideal interface $\equiv$ local relaxation.
- $\delta F = 0$ when $\boxed{\nabla \times \mathbf{B} = \mu \mathbf{B}}$ and $\boxed{[[p + B^2/2]] = 0}$.
- arbitrary "stepped" pressure profile

1.3 Ideal MHD

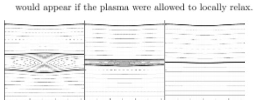
- conserved fluxes and helicities in $N = \infty$ volumes

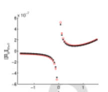
$$F \equiv \sum_{i=1}^{\infty} \int_{\mathcal{V}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv \tag{3}$$

- ideal "topological" constraint, $\delta \mathbf{B} = \nabla \times \boldsymbol{\xi} \times \mathbf{B}$, used globally $\equiv$ no relaxation.
- $\delta F = 0$ when $\boxed{\nabla p = \mathbf{j} \times \mathbf{B}}$.
- arbitrary smooth pressure profile? Maybe!

1.4 MRxMHD $\to$ Ideal MHD as $N \to \infty$

- Proven theoretically and demonstrated numerically (for axisymmetric equilibria) [Dennis et al. PoP, 20:032509, 2013]
- For decades, singular current densities have been predicted, but not observed, in ideal 3D MHD equilibria, $\nabla p = \mathbf{j} \times \mathbf{B}$
- From $\nabla \cdot \mathbf{j} = \nabla \cdot (\sigma \mathbf{B} + \mathbf{j}_\perp) = 0$, where $\mathbf{j}_\perp \equiv \mathbf{B} \times \nabla p / B^2$, derive $\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp$.
- Assuming straight-fieldline coordinates, $\mathbf{B} = \nabla \psi \times \nabla \theta + \iota(\psi) \nabla \zeta \times \nabla \psi$, solution for $\sigma = \frac{\mathbf{j} \cdot \mathbf{B}}{B^2}$, is $\sigma_{m,n} = \underbrace{\frac{(\sqrt{g} \nabla \cdot \mathbf{j})_{m,n}}{\iota m - n}}_{\text{Pfirsch-Schlüter}} + \underbrace{\Delta_{m,n} \, \delta(\psi - \psi_s)}_{\delta\text{-function}}$.
- Numerical proof using SPEC [Loizu et al. PoP, 22:022501, 2015]
- Shown below is a sequence of equilibria, with $N = 3$, with perturbed boundary, in which the flux and transform constraints are used to "shield" the magnetic island that would appear if the plasma were allowed to locally relax.





1.5 Ideal MHD equilibria have unphysical currents

- If the pressure, p, is continuous, ideal MHD equilibria have unphysical currents.
- If the pressure gradient, p', is manipulated to avoid the singular currents, then
  - (i) p' must be fractal: exotic numerical methods are required, and the topology of the magnetic field *cannot* be constrained to match the given pressure profile
  - (ii) p must be discontinuous: see Stepped Pressure Equilibrium Code.
- Singular current densities **are** allowed within ideal MHD, but the total current passing through any finite area must not diverge.

1 Stepped Pressure equilibrium Code (SPEC)

Constructs extrema of MRxMHD energy functional

1.1 Arbitrary geometry

- Cartesian, cylindrical or toroidal geometry.
- Fixed-boundary; free-boundary almost completed.
- Stellarator symmetric, or non-stellarator symmetric.

1.2 Input profiles

- $\psi_i \equiv$ toroidal flux inside i-th interface;
- $p_i \equiv$ pressure in each relaxed volume;
- $\iota_i^-, \iota_i^+ \equiv$ rotational-transform on each side of each interface.
  - $\to$ for continuous transform choose $\iota_i^- = \iota_i^+$.
- can also use parallel current profile, μ_i, or helicity profile, K_i, to define equilibrium.

1.3 Algorithm

- Use Newton method to set force-balance vector to zero:
  1. degrees-of-freedom $\equiv$ Fourier harmonics of interfaces $\equiv \mathbf{x} \equiv \{R_{i,m,n}, Z_{i,m,n}\}$.
  2. in each volume, write $\mathbf{B} = \nabla \times (A_\theta \nabla \theta + A_\zeta \nabla \zeta)$,

$$\mathbf{B} \equiv \nabla \times \sum_{m,n,j} [A_{\theta,m,n,j} \nabla \theta + A_{\zeta,m,n,j} \nabla \zeta] \, T_j(s) \cos(m\theta - n\zeta) \tag{1}$$

  3. solve $\nabla \times \mathbf{B} = \mu \mathbf{B}$ in each volume as a matrix equation, $L[\mathbf{x}] \cdot \mathbf{a} = \mathbf{c}$, where $\mathbf{a} \equiv \{A_{\theta,m,n,j}, A_{\zeta,m,n,j}\}$
  4. construct force "imbalance" vector, defined at interfaces:

$$\mathbf{F}(\mathbf{x}) \equiv \{[[p + B^2/2]]_{i,m,n}\} \tag{2}$$

  5. iterate on linear correction to find $\mathbf{F}(\mathbf{x}) = 0$.

$$\mathbf{F}(\mathbf{x} + \delta \mathbf{x}) \approx \mathbf{F}(\mathbf{x}) + \nabla \mathbf{F} \cdot \delta \mathbf{x} = 0, \qquad \delta \mathbf{x} = -(\nabla \mathbf{F})^{-1} \cdot \mathbf{F}(\mathbf{x}). \tag{3}$$

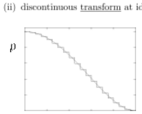
  6. derivative of magnetic field w.r.t. interface geometry given by matrix perturbation

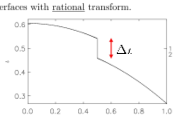
$$(L + \delta L) \cdot (\mathbf{a} + \delta \mathbf{a}) = \mathbf{c} + \delta \mathbf{c}, \qquad L \cdot \delta \mathbf{a} = \delta \mathbf{c} + \delta L \cdot \mathbf{a}, \tag{4}$$

- Can also use conjugate gradient method to minimize energy subject to constrained helicity (only if helicity profile is given and constrained).

1.4 Discontinuous pressure, and/or discontinuous transform

1. For 3D MHD equilibria to have well-defined solutions, must have either:
   - (i) discontinuous pressure at ideal interfaces with irrational transform, or
   - (ii) discontinuous transform at ideal interfaces with rational transform.





2. Hitherto, only SPEC allows discontinuous pressure and/or discontinuous transform.

1.5 Fundamental Theorem of 3D equilibria

- Well, not really a theorem; but, **flux surfaces must not overlap!**
- If the ideal interfaces overlap, SPEC will crash!

1 Cylindrical, linearly perturbed

1.1 Standard linear theory

1. Cylindrical geometry (r, θ, ϕ).
2. Given "axisymmetric" equilibrium, $\nabla p = \mathbf{j} \times \mathbf{B}$;
   - i. shall hereafter ignore pressure; include constraint $\nabla \cdot \boldsymbol{\xi} = 0$
   - ii. require rotational-transform $\iota(r)$ to be given function.
3. Arbitrary displacement in geometry

$$\boldsymbol{\xi} \equiv \xi^r(r) \cos(m\theta - n\zeta) \, \mathbf{e}_r + \xi^\theta(r) \sin(m\theta - n\zeta) \, \mathbf{e}_\theta + \xi^\phi(r) \sin(m\theta - n\zeta) \, \mathbf{e}_\phi.$$

4. The linear "ideal" magnetic field perturbation is $\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$.
5. The perturbed current is $\delta \mathbf{j} = \nabla \times \delta \mathbf{B}$.
6. The perturbed force is $\delta \mathbf{f} = \underbrace{\delta \mathbf{j} \times \mathbf{B} + \mathbf{j} \times \delta \mathbf{B}}_{linear} + \underbrace{\delta \mathbf{j} \times \delta \mathbf{B}}_{ignored}$.
7. 3 unknowns: ξ^r, ξ^θ & ξ^ϕ; and 3 constraints: $\delta f^r = 0$, $\delta f^\theta = 0$ & $\delta f^\phi = 0$.
8. The linear equations can be represented as

$$\mathbf{A} \cdot \frac{d^2}{dr^2} \begin{pmatrix} \xi^r \\ \xi^\theta \\ \xi^\phi \end{pmatrix} + \mathbf{B} \cdot \frac{d}{dr} \begin{pmatrix} \xi^r \\ \xi^\theta \\ \xi^\phi \end{pmatrix} + \mathbf{C} \cdot \begin{pmatrix} \xi^r \\ \xi^\theta \\ \xi^\phi \end{pmatrix} = 0$$

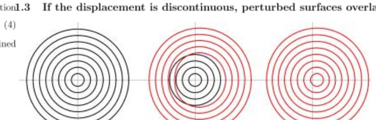
9. Can express $\xi^\theta = \alpha_\theta \xi^{r\prime} + \beta_\theta \xi^r$ and $\xi^\zeta = \alpha_\zeta \xi^{r\prime} + \beta_\zeta \xi^r$, and derive a single second-order, linear "Newcomb" equation for ξ^r :

$$e.g. \; \frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) - \frac{m^2}{r^2} \psi - \frac{J^{\zeta\prime}}{B^z(\iota - n/m)} \psi = 0, \quad \text{where} \quad \psi \equiv \xi^r B^\zeta (\iota - n/m). \tag{1}$$

1.2 Newcomb equation is singular!

1. If resonant rational surface, $\iota = n/m$, is present, then
   - i. Eqn(1) is singular!
   - ii. Nontrivial solutions for the perturbed geometry are discontinuous.

1.3 If the displacement is discontinuous, perturbed surfaces overlap



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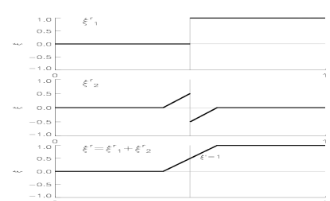
1. ***If*** the 1st order correction is discontinuous,
   and the non-linear solution has non-overlapping flux surfaces,
   then, the 2nd order term $\approx$ 1st order term near the discontinuity.
2. Breakdown of analyticity: if $f' dx \approx \frac{1}{2} dx f'' dx$, then $f'' \sim 1/dx$.
3. Linear perturbation theory is not valid rational surfaces are present,
   - i. nonlinear analysis is required [Rosenbluth, 1973].



1.5 Consider an equilibrium with discontinuous rotational-transform

1. No rational surfaces $\equiv$ no unphysical singularities!
2. Perturbed flux surfaces do not overlap if $\Delta\iota > \mathcal{O}(\delta |\mathbf{b}|)$.

1 SPEC, linearly perturbed

To compare directly with linearly perturbed equilibrium codes, SPEC has been linearized.

1.1 Linear displacement

1. Begin with e.g. an axisymmetric equilibrium (e.g. cylindrical geometry).
2. The geometry of the interior interfaces depends on the geometry of the boundary.
3. Construct total-pressure "imbalance" vector, defined at interfaces:

$$\mathbf{F}_\mathbf{b}(\mathbf{x}) \equiv \{[[p + B^2/2]]_{i,m,n}\} \tag{1}$$

$\mathbf{x} = \{$geometry of internal interfaces$\}$,
$\mathbf{b} = \{$geometry of external boundary$\}$.

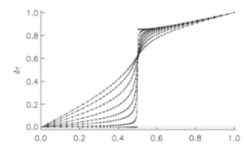
4. The linear correction to the internal geometry for a change in the boundary is

$$\mathbf{F}_{\mathbf{b}+\delta\mathbf{b}}(\mathbf{x} + \delta \mathbf{x}) \approx \mathbf{F}(\mathbf{x}) + (\nabla_\mathbf{x} \mathbf{F}) \cdot \delta \mathbf{x} + (\nabla_\mathbf{b} \mathbf{F}) \cdot \delta \mathbf{b} = 0, \tag{2}$$

solution $\delta \mathbf{x} = -(\nabla_\mathbf{x} \mathbf{F})^{-1} \cdot (\nabla_\mathbf{b} \mathbf{F}) \cdot \delta \mathbf{b}$.

1.2 Verification with semi-analytic solution

1. Linearized-SPEC may be compared to semi-analytic solutions to Newcomb equation
2. Excellent agreement!
3. As $\Delta\iota \to 0$, linear solution $\to$ discontinuous.



1 Sine qua non condition

1.1 Latin

1. *Sine qua non*, or *condicio sine qua non.*
2. Plural: *condiciones sine quibus non*
3. Definition: an indispensable and essential action, condition or ingredient.
4. Translation: "without which it could not be".
5. First known use, 1602. First known use in plasma physics, 2015.

1.2 Fundamental assumption: flux surfaces cannot overlap

1. The *total* displacement must satisfy the

$$\textit{sine qua non} \text{ condition:} \quad \xi^{r\prime} \text{ continuous}, \quad \xi^{r\prime} < 1. \tag{1}$$

1.3 The rotational transform *must* be discontinuous

1. Technically, there are no rational surfaces.
2. The discontinuity, $\Delta\iota$, must exceed a minimum value, $\Delta\iota > \mathcal{O}(\delta |\mathbf{b}|)$, to ensure the *sine qua non* condition is satisfied.
3. All the problems with ideal MHD equilibria in 3D vanish!
4. Ideal MHD equilibria in arbitrary 3D geometry with smooth pressure exist!
   - i. Proof: construct a convergent perturbation expansion.
5. VMEC and NSTAB should be modified to allow discontinuous transform.
6. Magnitude of $\Delta\iota$ must be provided by transport theory, conservation of helicity, ... Recall that:
   - i. equilibrium codes require the pressure, $p(\psi)$, and transform, $\iota(\psi)$, profiles to be *given* functions of toroidal flux, ψ.
   - ii. Physically meaningful $p(\psi)$ and $\iota(\psi)$ profiles are determined by transport theory.

1.4 With discontinuous transform, the perturbation penetrates past the rational surface!

1. Problems with linearly-perturbed equilibria:
   - i. inconsistent with non-overlapping perturbed flux surfaces,
   - ii. incorrect prediction that the resonant harmonic of the perturbation is completely shielded at the resonant surface.

1 SPEC, nonlinear

1.1 Nonlinear convergence

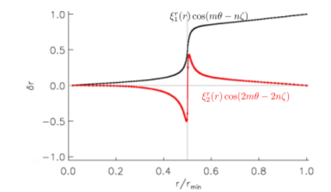
1. The nonlinear solution satisfies

$$\mathbf{F}_{\mathbf{b}+\delta\mathbf{b}}(\mathbf{x}) = 0 \tag{1}$$

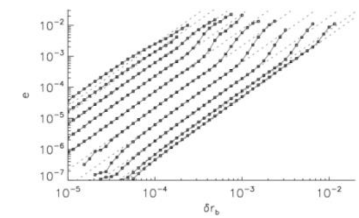
2. The error between the nonlinear solution and the linear approximation satisfies

$$\underbrace{(\mathbf{x}_0 + \delta \mathbf{x})}_{linear} - \mathbf{x} = \mathcal{O}(\delta \mathbf{b}^2)$$

3. Shown below is the first and second resonant harmonic of the displacement (not to scale).



3. Shown below is the numerical confirmation that the error between the SPEC nonlinear solution and the linear approximation is $\mathcal{O}(\delta \mathbf{b}^2)$.
   - i. Shown is the error for various values of $\Delta\iota$.



4. The magnitude of the error depends on $\Delta\iota$:
   - i. As $\Delta\iota \to 0$, the nonlinear solution cannot be well approximated by the linear solution, because the linear solution is discontinuous.