The non-linear plasma response in the nested flux surface limit

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Introduction

In order to ensure the utility of equilibrium models a set of tests must be preformed. These tests include[1]

- Verification: Demonstration that a code is correctly solving its set of equations
- Validation: Demonstration that a code is producing results consistent with experiment
- Benchmark: Comparison of how different codes through a common test problem

In this work a problem is described which attempts to verify and benchmark the ideal MHD response to applied 3D fields. The problem is examined using non-linear (VMEC[2], NSTAB[3]) and linear (MARS-F[4]) codes. Examples validating the nonlinear and linear ideal 3D MHD models are also shown.

Validation tests using the Dirac-δ Response

The equations of ideal MHD predict that singular currents will form at resonant rational surfaces, if nested flux surfaces are to be preserved

$$\vec{j} \equiv \mu\vec{B} + \vec{j}_\perp \quad \nabla\cdot\vec{j}=0 \quad \nabla\cdot\vec{B}=0 \quad \vec{j}\times\vec{B}=\nabla p \;\longrightarrow\; \vec{B}\cdot\nabla\mu = -\nabla\cdot\vec{j}_\perp \quad \vec{j}_\perp = \frac{\vec{B}\times\nabla p}{B^2}$$

In magnetic coordinates the gradient operator becomes

$$(\psi,\theta,\phi) \;\longrightarrow\; \sqrt{g}\vec{B}\cdot\nabla \equiv \iota\partial_\theta + \partial_\phi$$

Fourier decomposition leads to

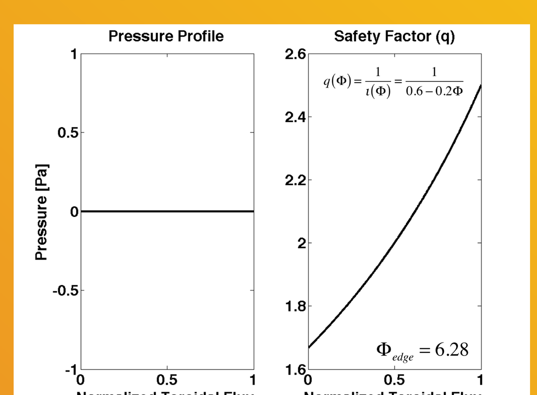
$$\mu = \sum_{m,n}\mu_{mn}e^{i(m\theta-n\phi)} \;\longrightarrow\; (\iota m - n)\mu_{mn} = i\left(\sqrt{g}\nabla\cdot\vec{j}_\perp\right)_{mn}$$

This gives us an equation of the form

$$\mu_{mn}(x) = \frac{h(x)}{x} + \hat{j}_{mn}\delta(x) \qquad h(x)\sim\frac{dp}{d\psi} \quad x\equiv\iota m-n$$

Pfirsch-Schlütter resonance   Dirac-δ

The Pfirsch-Schlütter resonance diverges across a rational surface which implies infinite total current. Thus pressure gradients must vanish at all rational surfaces (pressure becomes pathological). [H. Grad 1967]



To avoid this difficulty we choose our problem to have zero pressure and a fixed rotational transform profile. To further simplify the problem, a circular cross section equilibrium is considered.

$$R = R_0 + r(\theta,\zeta)\cos(\theta)$$

$$Z = Z_0 + r(\theta,\zeta)\sin(\theta)$$

A helical perturbation is then applied to the axisymmetric equilibrium

$$r(\theta,\zeta) = r_0 + \delta r\cos(m_1\theta - n_1\zeta)$$

Which can be redefined in terms of R and Z harmonics

$$R = R_0 + r_0\cos(\theta) + \delta r\cos(m_1\theta-n_1\zeta)\cos(\theta) = R_0 + r_0\cos(\theta) + \frac{\delta r}{2}\cos((m_1+1)\theta-n_1\zeta) + \frac{\delta r}{2}\cos((m_1-1)\theta-n_1\zeta)$$

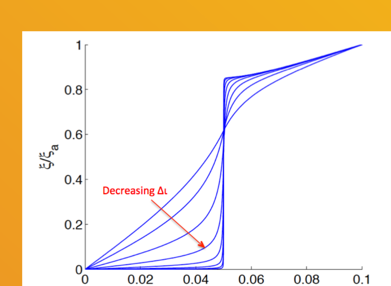
$$Z = Z_0 + r_0\sin(\theta) + \delta r\cos(m_1\theta-n_1\zeta)\sin(\theta) = R_0 + r_0\sin(\theta) + \frac{\delta r}{2}\sin((m_1+1)\theta-n_1\zeta) - \frac{\delta r}{2}\sin((m_1-1)\theta-n_1\zeta)$$

If we examine Newcomb's equation we find that only in the limit of continuous iota should we recover full shielding of the resonant perturbation (Loizu talk and Hudson poster) with overlapping surfaces, thus a minimum Δ-iota is required and the perturbation penetrates.

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

$$f(r) = \frac{r\left(krB_z + mB_\theta\right)^2}{k^2r^2+m^2}$$

$$g(r) = \frac{1}{r}\left(krB_z + mB_\theta\right)^2\frac{k^2r^2+m^2-1}{k^2r^2+m^2} + \frac{2k^2r}{(k^2r^2+m^2)^2}\left(k^2r^2B_z^2 - m^2B_\theta^2\right)$$



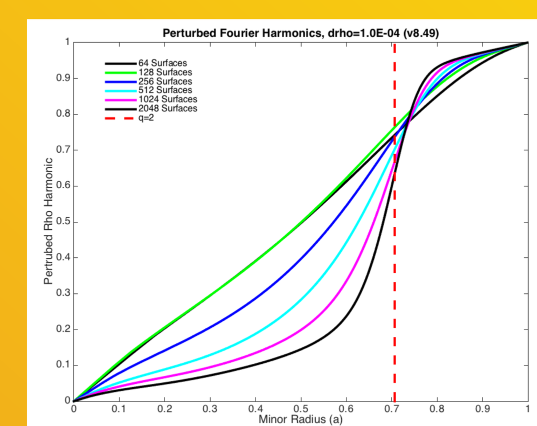
VMEC

The VMEC code solves for nonlinear ideal MHD plasma equilibrium by minimizing the MHD energy. It uses a magnetic field parameterization which enforces that all flux surfaces are nested.

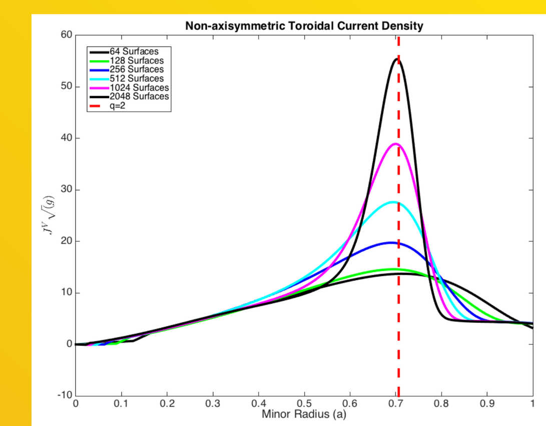
$$W = \int\frac{B^2}{\mu_0} + \frac{p}{\gamma-1}dV \qquad \vec{j}\times\vec{B} - \nabla p = 0$$

$$\vec{B} = \nabla\zeta\times\nabla\chi + \nabla\Phi\times\nabla\theta^* \qquad \theta^* = \theta + \lambda(\Phi,\theta,\zeta)$$

The test problem was attempted using a perturbation of amplitude 1E-4, minor radius of 1.0, and major radius of 100.0. Scans in radial resolution indicate a shielding like response at the q=2 surface.

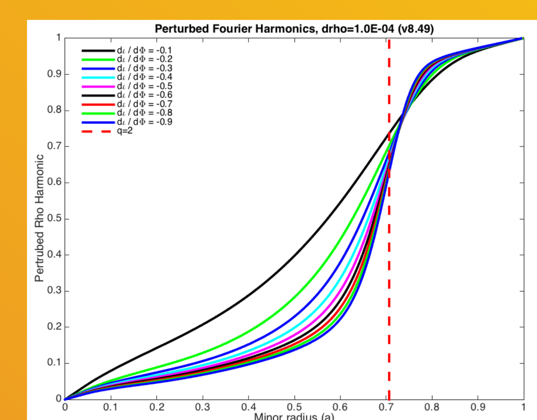


Radial displacement of VMEC flux surfaces for the perturbed cylindrical equilibrium. The slope of the displacement at the rational surface appears to scale with radial grid resolution.

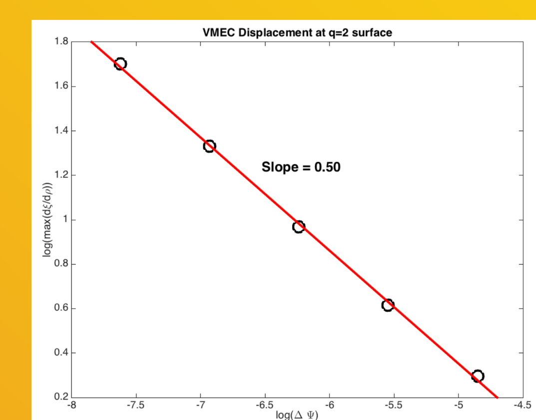


The toroidal current density indicates a response at the q=2 surface.

At all times the displacement derivative remains significantly below unity indicating that surfaces have not overlapped. The dependence on local shear is weak.

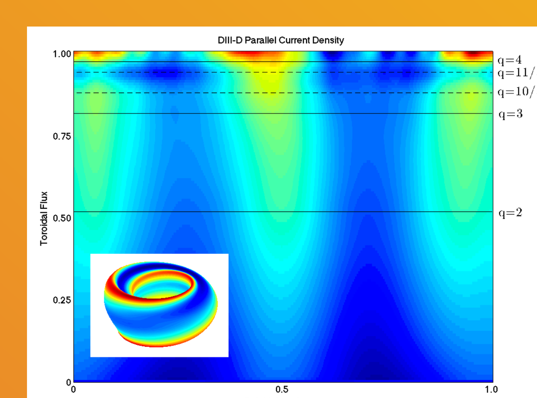


Radial displacement as a function of shear across the q=2 surface (512 radial grid points).

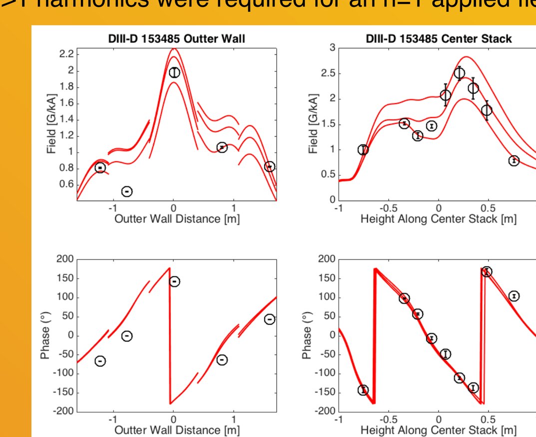


The maximum displacement derivative as a function of the grid resolution indicates at all times the surfaces do not overlap.

To validate the plasma response model the diagnostics response of a DIII-D discharge to applied RMP fields was evaluated. Good agreement was found although n>1 harmonics were required for an n=1 applied field.



Current density parallel to the magnetic field for DIII-D equilibrium 153485 showing peaked structures at various rational surfaces. Subplot shows exaggerated (x20) edge of VMEC equilibria with displacement colored (~1 [cm] peak).



Magnetic diagnostic response as calculated by DIAGNO for the VMEC equilibrium. Lines correspond to variations +/-5% in bootstrap current profile amplitude.

Evaluation of the current density has been recently improved in VMEC through an improved radial differencing technique. The current density is written

$$\mu_0 j^u \equiv \mu_0\sqrt{g}J^u = \frac{\partial B_s}{\partial v} - \frac{\partial B_v}{\partial s} \qquad \mu_0 j^v \equiv \mu_0\sqrt{g}J^v = \frac{\partial B_u}{\partial s} - \frac{\partial B_s}{\partial u}$$

However the covariant components of the field are calculated on a half radial grid

Odd m modes

$$\frac{\partial B_u^{\ i}}{\partial s} = \sqrt{s^i}\,\frac{b_u^{i+1/2} - b_u^{i-1/2}}{h_s} + \frac{b_u^{i+1/2} + b_u^{i-1/2}}{4\sqrt{s^i}}$$

$$B_s^i = \frac{b_s^{i+1/2} + b_s^{i-1/2}}{2\sqrt{s^i}}$$

Even m modes

$$\frac{\partial B_u^{\ i}}{\partial s} = \frac{B_u^{i+1/2} - B_u^{i-1/2}}{h_s}$$

$$B_s^i = \frac{B_s^{i+1/2} + B_s^{i-1/2}}{2}$$

where

$$b_u = \frac{B_u}{\sqrt{s}} \quad b_v = \frac{B_v}{\sqrt{s}}$$

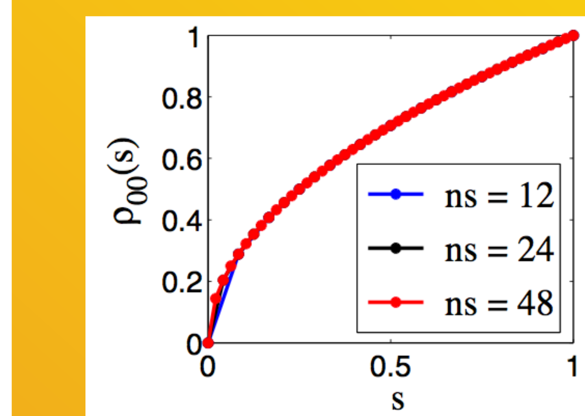
NSTAB

The NSTAB code solves for the nonlinear ideal MHD plasma equilibrium by minimizing the ideal MHD energy. It uses a magnetic field parametrization which enforces that all flux surfaces are nested.

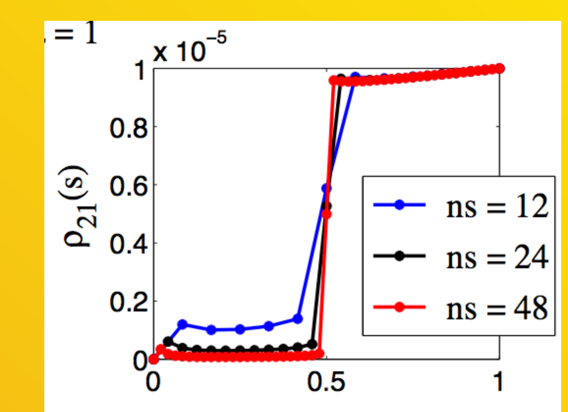
$$W = \int\frac{B^2}{\mu_0} + \frac{p}{\gamma-1}dV \qquad \vec{j}\times\vec{B} - \nabla p = 0$$

$$\vec{B} = \nabla\Phi\times\nabla\psi \qquad \psi = \theta - \iota(\Phi)\phi + \bar{\psi}(\Phi,\theta,\phi)$$

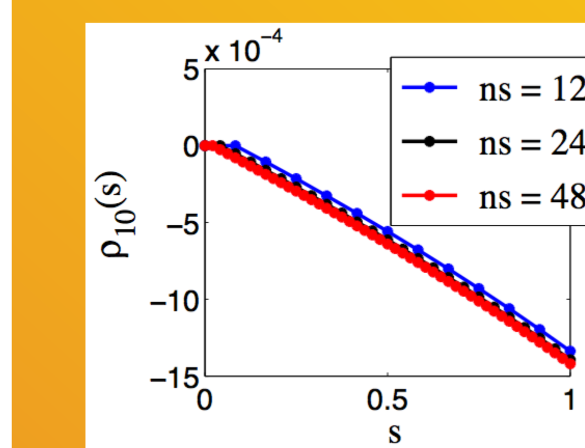
The NSTAB calculation of the benchmark problem showed similar behavior to that of VMEC at low radial resolution.



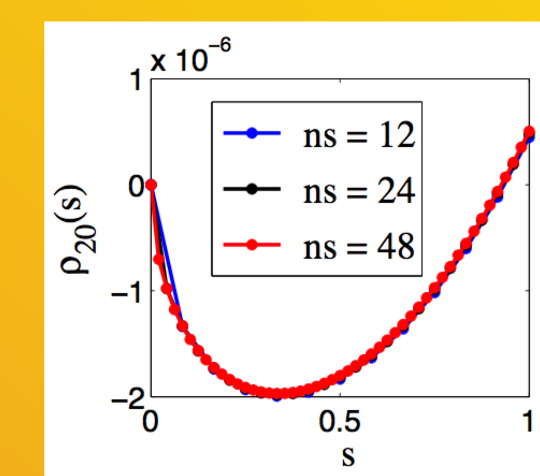
The main axisymmetric coefficient shows a singular behavior near the magnetic axis which is slowly convergent under mesh refinement.



A lower resolution the behavior near the rational surface appears continuous but becomes increasingly sharp at higher resolution.

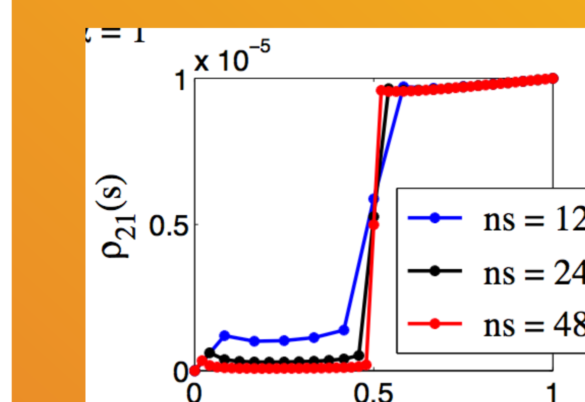


A helical perturbation of the equilibrium is also present.

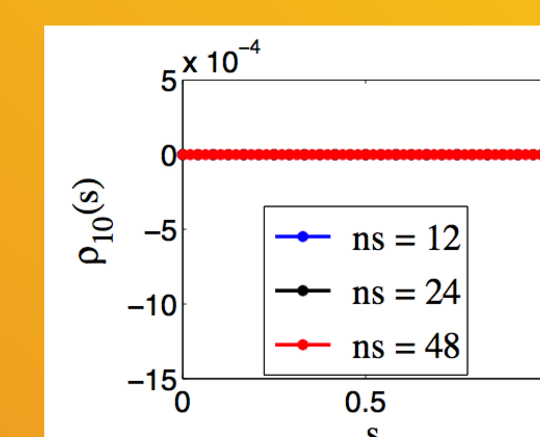


The m=2, n=0 is a result of the non-linearity of the code.

The NSTAB code can be run in purely cylindrical geometry allowing toroidal modes to be decoupled.

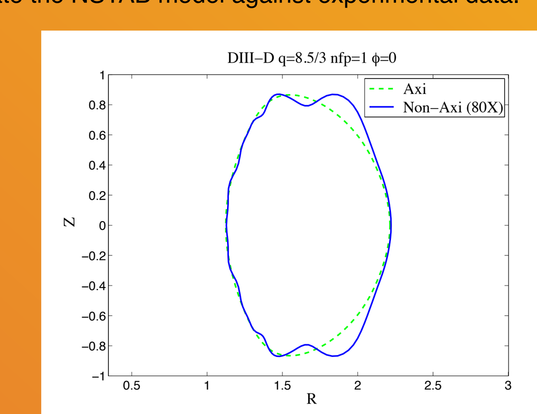


In cylindrical form the behavior of near the rational surface is preserved.



Here the nonlinear coupling between modes is completely suppressed.

Work is underway to validate the NSTAB model against experimental data.



Deformation of a flux surface near a rational surface for DIII-D shot 142603. Inboard ripple is consistent with linear calculations.

MARS-F

The MARS-F code solves for the perturbed linear ideal MHD plasma response given an axisymmetric equilibrium as the unperturbed state. It solves the following set of equations:

$$i(\omega+n\Omega)\vec{\xi} = \vec{v} + \left(\vec{\xi}\cdot\nabla\Omega\right)R^2\nabla\phi$$

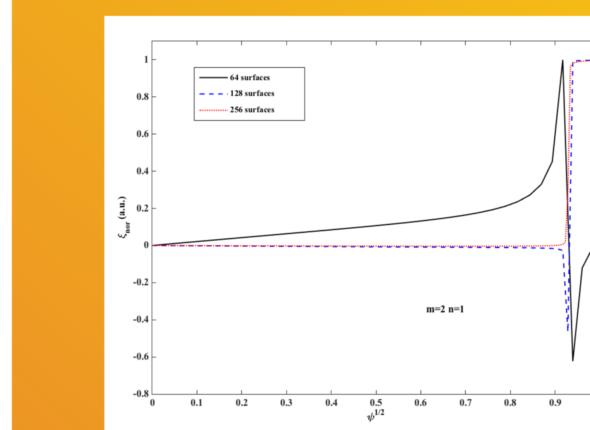
$$i\rho(\omega+n\Omega)\vec{v} = -\nabla p + \vec{j}\times\vec{B}_0 + \vec{j}_0\times\vec{B} + \rho\left[2\Omega\hat{Z}\times\vec{v} - (\vec{v}\cdot\nabla\Omega)R^2\nabla\phi\right] - \nabla\cdot\left(\rho\vec{\xi}\right)\Omega\hat{Z}\times\vec{v}_0$$

$$i(\omega+n\Omega)\vec{B} = \nabla\times\left(\vec{v}\times\vec{B}_0\right) + \left(\vec{B}\cdot\nabla\Omega\right)R^2\nabla\phi$$

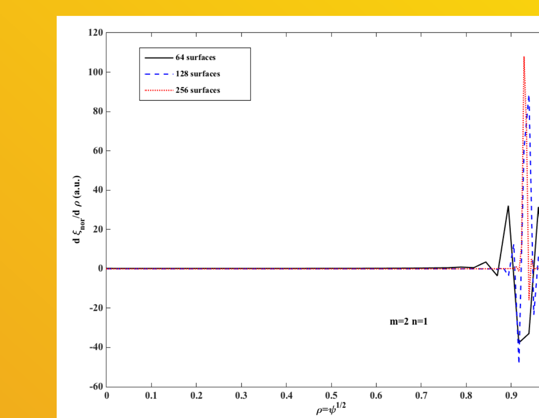
$$i(\omega+n\Omega)p = -\vec{v}\cdot\nabla p_0 - \Gamma p_0\nabla\cdot\vec{v}$$

$$\vec{j} = \nabla\times\vec{B}$$

Because MARS-F is a perturbative in nature, flux surfaces may overlap at each rational surface. The verification problem was examined using grid spacing equidistant in flux. This allows direct comparison with the NSTAB and VMEC results.

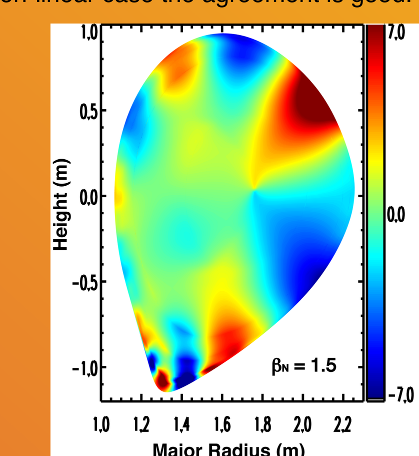


The MARS code predicts a strong delta current response at the q=2 surface.

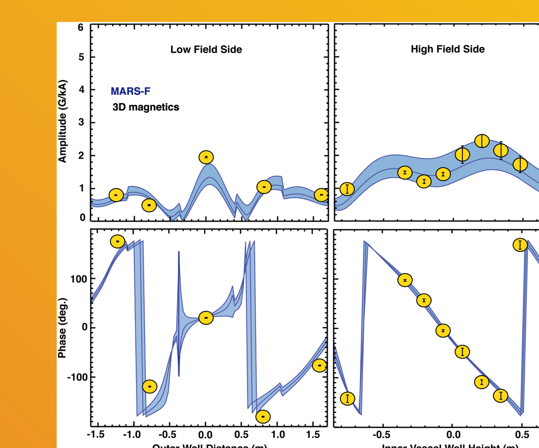


The overlap criterion is clearly violated for this mode.

The response of the ideal perturbed MHD model has been validated against DIII-D shot 153485. As in the non-linear case the agreement is good.



For experimental tokamak geometries the normal components of the magnetic field indicate strong responses near the edge.



Validation of the MARS-F plasma response model against DIII-D shot 153485 shows good agreement. Lines correspond to variations +/-5% in bootstrap current profile amplitude.

Conclusions

In this work a perturbed circular cross section large aspect ratio tokamak equilibrium was investigated by both linear and non-linear ideal MHD equilibrium codes

- Exact solutions to this problem indicate that either the constraint of continuous flux surfaces or continuous rotational transform profiles may be violated
- The nested flux surface constraint prevents the non-linear codes (VMEC, NSTAB) from achieving a true Dirac-δ response at the rational surface.
- Lack of a nested flux surface constraint allows the linear code (MARS-F) to realize a Dirac-δ response.

Validation of the VMEC and MARS-F codes indicate that both are capable of reproducing experimental measures of 3D fields. Inclusion of a Δiota model in the various codes would allow for an exact benchmark.

References

[1] Oberkampf, William L., and Christopher J. Roy. Verification and validation in scientific computing. Cambridge University Press (2010).
[2] Hirshman, Steven P., and J. C. Whitson. "Steepest-descent moment method for three-dimensional magnetohydrodynamic equilibria." Physics of Fluids 26, 3553 (1983)
[3] Taylor, Mark. "A high performance spectral code for nonlinear MHD stability." Journal of Computational Physics 110, (1994)
[4] Liu, Y.Q., Bondeson, A., Fransson, C. M., Lennartson, B., and Breitholtz, C. "Feedback stabilization of nonaxisymmetric resistive wall modes in tokamaks. I. Electromagnetic Model." Physics of Plasmas 7, 3681 (2000)