COMPUTATION OF MULTI-REGION, RELAXED MHD EQUILIBRIA

SOLUTIONS TO $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ IN 3D GEOMETRY

Dr. S.R. Hudson

With Prof. M.J. Hole & Em. Prof. R.L. Dewar

The Australian National University
Princeton Plasma Physics Laboratory

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MHD determines the plasma equilibrium, and the linear and non-linear stability.

"MHD represents the simplest self-consistent model describing the macroscopic equilibrium and stability properties of a plasma."

"The model describes how magnetic, inertial and pressure forces interact within an ideal perfectly conducting plasma in an arbitrary magnetic geometry."

J.P. Freidberg, **Ideal Magnetohydrodynamics,** Plenum Press, New York, 1987

Equilibrium

- Grad-Shafranov, VMEC, NSTAB, IPEC, . . .
- Reconstruction, e.g. EFIT, V3FIT, STELLOPT
- Experimental design, e.g. ITER, W7-X

Stability

- Kink
- Ballooning
- Peeling ballooning, e.g. ELMs, ELITE, . . .

Transport

- Neoclassical
- Turbulent

Require solution to $\nabla p = \mathbf{j} \times \mathbf{B}$ with
1) nested magnetic surfaces, and
2) smooth profiles.

$\boxed{\text{This talk: given } p \text{ and e.g. } \iota\!\!\!-, \text{ find } \mathbf{B}.}$

Problem: solutions to force balance with nested surfaces have a non-analytic dependence on 3D boundary.

Breakdown of perturbation theory:

Following Rosenbluth, Dagazian & Rutherford, [Phys. Fluids **16**, 1894 (1973)]

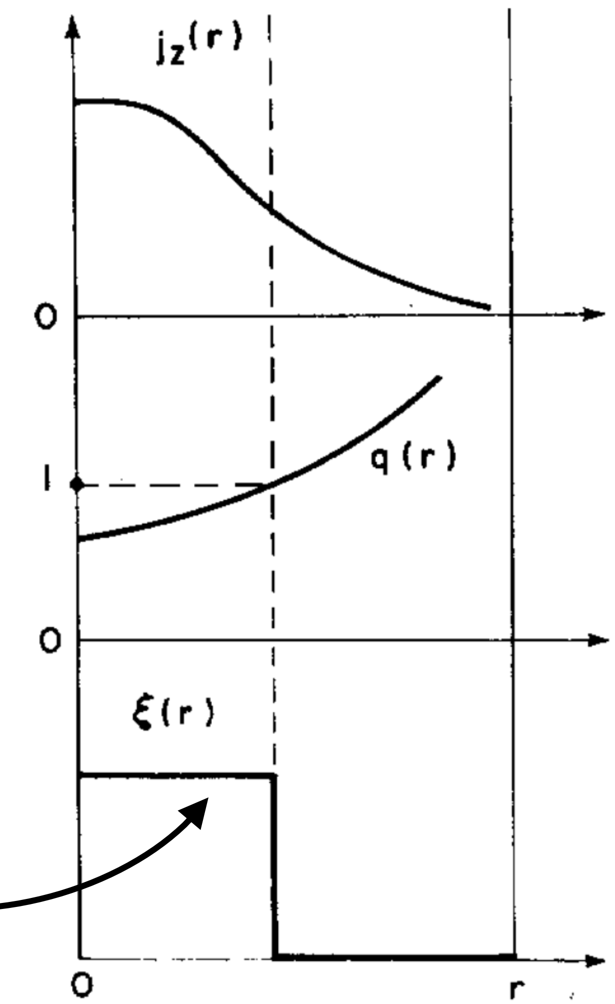
" .. the standard perturbation theory approach .. is not applicable here due to the singular nature of the lowest order step function solution for $\boldsymbol{\xi}$"

$$\boldsymbol{\xi} = \epsilon\,\boldsymbol{\xi}_1 + \epsilon^2\,\boldsymbol{\xi}_2 + \epsilon^3\,\boldsymbol{\xi}_3 + \ldots$$

$$\delta\mathbf{B}[\boldsymbol{\xi}] \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B}),$$
$$\delta p[\boldsymbol{\xi}] \equiv (\gamma - 1)\boldsymbol{\xi} \cdot \nabla p - \gamma \nabla \cdot (p\boldsymbol{\xi})$$

Equilibrium and perturbed equations:

$$\begin{array}{rcl}
\mathbf{F}[\mathbf{x}] & \equiv & \nabla\, p_0 - \mathbf{j}_0 \times \mathbf{B}_0 = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_1] & \equiv & \nabla \delta p[\boldsymbol{\xi}_1] - \delta\mathbf{j}[\boldsymbol{\xi}_1] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}_1] = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_2] & = & \ldots
\end{array}$$



"However, since $\mathcal{L}_0$ is a singular operator .. this equation cannot, in general, be solved, .."
"we must abandon the perturbation theory approach.."

The singularity also affects Newton iterative solvers: $\mathbf{x}_{i+1} \equiv \mathbf{x}_i - \nabla\mathbf{F}^{-1} \cdot \mathbf{F}[\mathbf{x}_i]$

Problem: solutions to force balance with nested surfaces have singularities in the parallel current-density.

$\nabla p = \mathbf{j} \times \mathbf{B}$ yields $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$.

$\mathbf{j}$ is current-*density*, current $= \int_{\mathcal{S}} \mathbf{j} \cdot ds$.

Write $\mathbf{j} = \sigma \mathbf{B} + \mathbf{j}_\perp$, $\nabla \cdot \mathbf{j} = 0$ yields $\boxed{\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp}$ (1)

Nested flux surfaces allows (ψ, θ, ζ) s.t.

$$\begin{array}{rcl} \mathbf{B} & = & \nabla\psi \times \nabla\theta + \iota\, \nabla\zeta \times \nabla\psi \\ \sqrt{g}\,\mathbf{B} \cdot \nabla & = & \partial_\zeta + \iota\, \partial_\theta \\ \sqrt{g}\,\mathbf{B} \cdot \nabla\zeta & = & 1 \end{array}$$

Fourier, $\sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) e^{i(m\theta - n\zeta)}$, Eqn(1) becomes $\boxed{(\iota m - n)\sigma_{m,n} = i(\sqrt{g}\nabla \cdot \mathbf{j}_\perp)_{m,n}}$ (2)

Resonant, parallel current-density : $\sigma_{m,n} = \underbrace{\frac{g_{m,n}(x)\, p'(x)}{x}}_{\text{Pfirsch-Schlüter}} + \Delta_{m,n} \underbrace{\delta_{m,n}(x)}_{\delta\text{-function}}$, where $x \equiv \iota - n/m$.

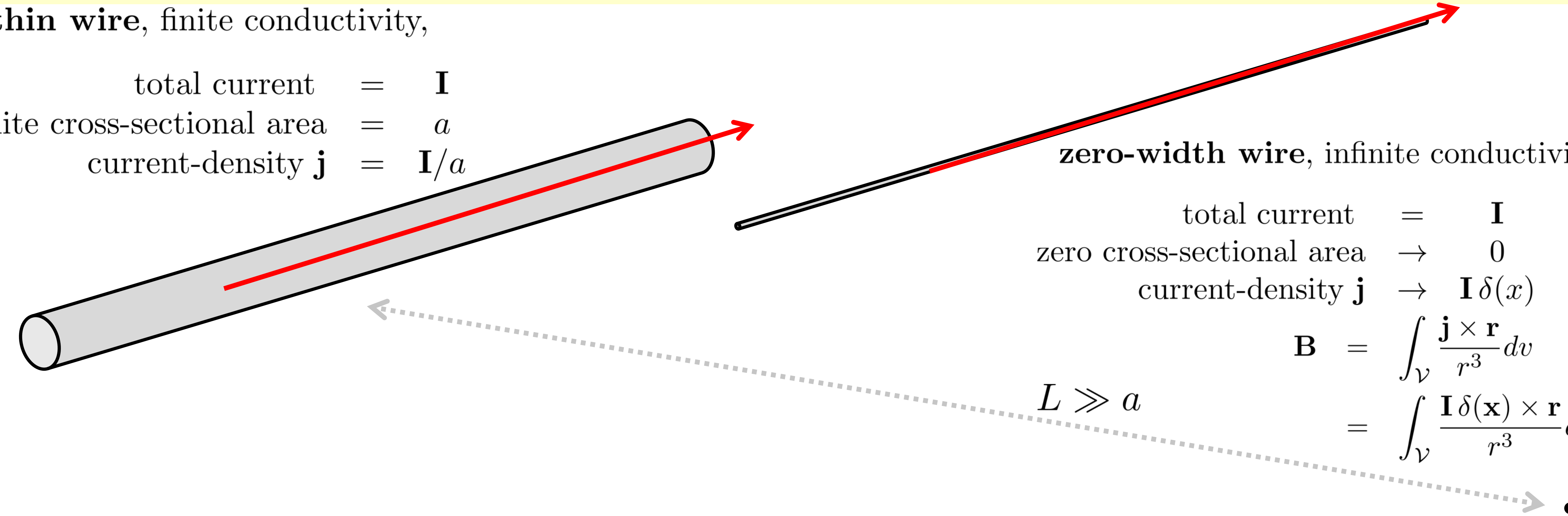
The δ-function current-density is integrable, e.g.

$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0), \int_{-\infty}^{\bar{x}} \delta(x)dx = H(\bar{x}) = \textit{Heaviside step function}, \; xH' = 0,$$

and is an acceptable mathematical idealization of localized currents.

thin wire, finite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{finite cross-sectional area} &= a \\ \text{current-density } \mathbf{j} &= \mathbf{I}/a \end{aligned}$$



$L \gg a$

zero-width wire, infinite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{zero cross-sectional area} &\rightarrow 0 \\ \text{current-density } \mathbf{j} &\rightarrow \mathbf{I}\,\delta(x) \\ \mathbf{B} &= \int_{\mathcal{V}} \frac{\mathbf{j} \times \mathbf{r}}{r^3} dv \\ &= \int_{\mathcal{V}} \frac{\mathbf{I}\,\delta(\mathbf{x}) \times \mathbf{r}}{r^3} dv \end{aligned}$$

Approximating a localized current-density by a δ-function current density

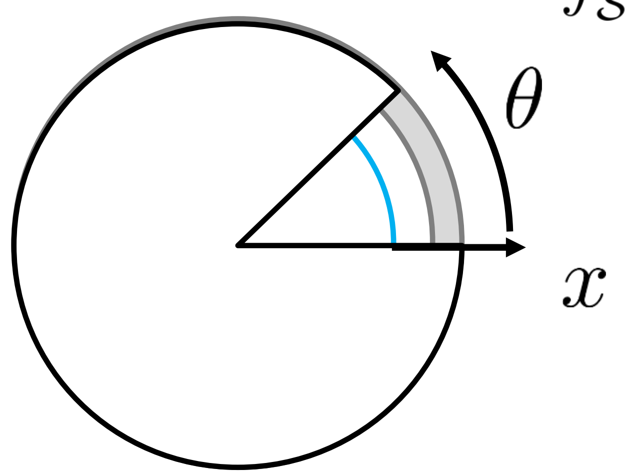
1. is acceptable for a **macroscopic** physical model that assumes **infinite conductivity**, and
2. is mathematically-tractable (one just needs to accommodate discontinuous solutions).

$$\begin{aligned} \text{Net current through cross-section} \quad \int_{\mathcal{S}} \mathbf{j} \cdot d\mathbf{s} &= \int d\psi \int d\theta \, \sqrt{g}\, \mathbf{j} \cdot \nabla\zeta \\ &= \int_{-\epsilon}^{+\epsilon} dx \int_0^{2\pi} d\theta \;\; \Delta_{m,n} \, \delta_{m,n}(x) \, e^{i(m\theta - n\zeta)} \sqrt{g}\, \mathbf{B} \cdot \nabla\zeta \\ \sqrt{g}\,\mathbf{B}\cdot\nabla\zeta = 1 \qquad &= 0 \\ i.e. \quad & \text{no discontinuity in rotational-transform} \end{aligned}$$

Problem: the pressure-driven 1/x current density gives infinite parallel currents through certain surfaces.

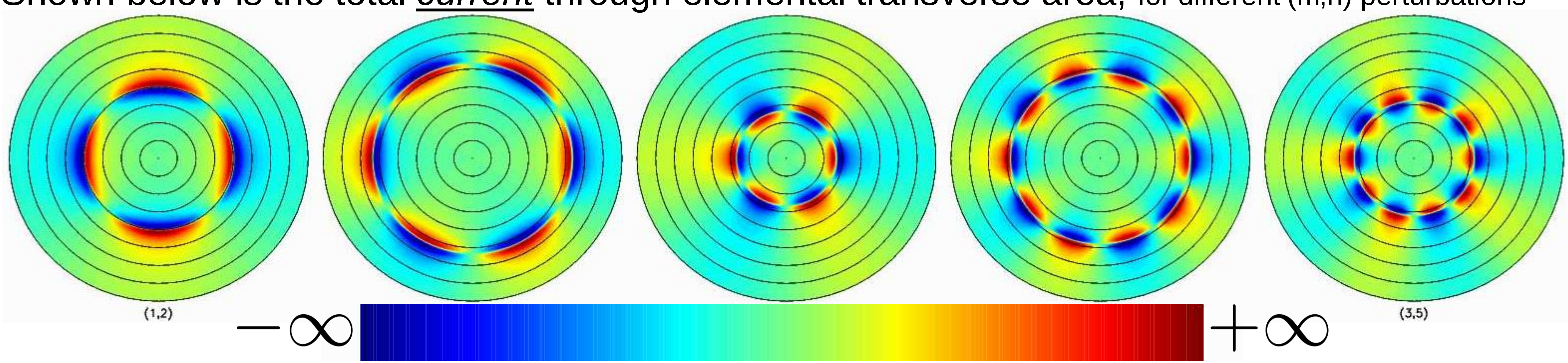
Parallel current-density
$$\mathbf{j}_{\parallel} = \sum_{m,n}\left[\frac{g_{m,n}\,p'}{x} + \Delta_{m,n}\delta_{m,n}(x)\right] e^{(im\theta - in\zeta)}\mathbf{B}.$$

Parallel current through cross-section
$$\begin{aligned}
\int_{\mathcal{S}} \mathbf{j}_{\parallel}\cdot d\mathbf{s} &= \int d\psi \int d\theta \;\; \sqrt{g}\,\mathbf{j}_{\parallel}\cdot\nabla\zeta \\
&= \int_{\epsilon}^{\delta} dx \int_{0}^{\pi/m} d\theta \;\; \frac{g_{m,n}\,p'}{x} e^{i(m\theta - n\zeta)} \sqrt{g}\,\mathbf{B}\cdot\nabla\zeta \\
&= g_{m,n,0}\,p_0' \frac{2}{m}\int_{\epsilon}^{\delta} dx \frac{1}{x} \\
&= g_{m,n,0}\,p_0' \frac{2}{m}(\ln\delta - \ln\epsilon) \to \infty \text{ as } \epsilon \to 0.
\end{aligned}$$



The problem is *NOT* a lack of numerical resolution.
Is a dense collection of alternating infinite currents physical?

Shown below is the total <u>*current*</u> through elemental transverse area, for different (m,n) perturbations



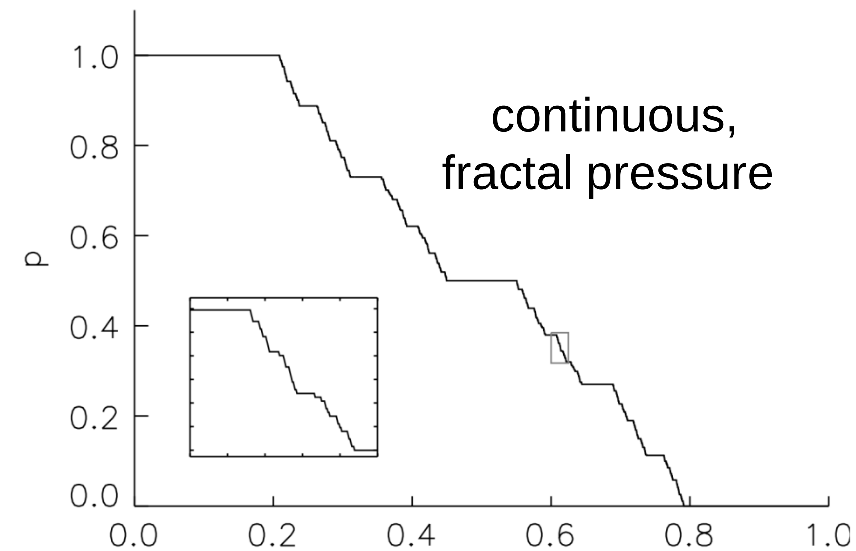
If there are rational surfaces, then we must choose:

1. flatten pressure near rationals, smooth pressure; ✗
2. flatten pressure near rationals, fractal pressure; ✗
3. flatten pressure near rationals, discontinuous pressure; ✓
4. restrict attention to “healed” configurations [Weitzner, PoP **21**, 022515 (2014); Zakharov, JPP **81**, 515810609, (2015)]

1. Locally-flattened, smooth pressure:

if (i.) $p'(x) = 0$ if $|x - n/m| < \epsilon_{m,n}$, $\forall (n,m)$,

and (ii.) $p'(x)$ is continuous, then $p'(x) = 0$, $\forall x$. **No pressure!**



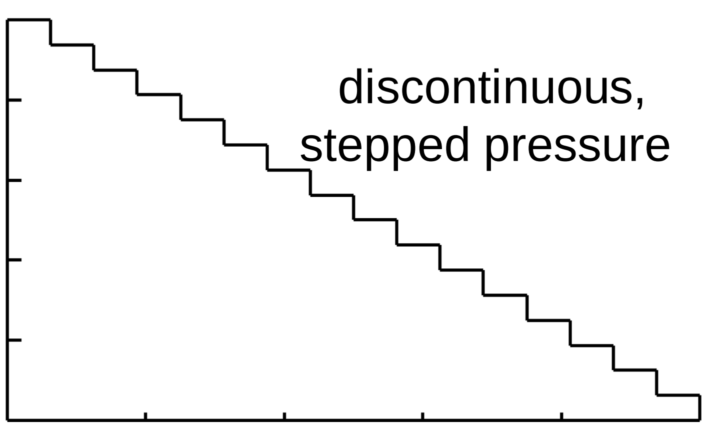
2. “Diophantine” pressure profile: e.g. from KAM theory

$$p'(x) = \begin{cases} 1, & \text{if } |x - n/m| > r/m^k, \quad \forall (n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |x - n/m| < r/m^k, \quad \exists (n,m), \end{cases}$$

$p'(x)$ is discontinuous on an uncountable infinity of points,

“The function p is continuous but its derivative is pathological.” Grad, Phys. Fluids 10, 137 (1967)]

3. “Stepped” pressure profile: ✓



Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure
[Bruno & Laurence, Commun. Pure Appl. Math. **49**, 717 (1996)]

Culmination of long history of “waterbag” or “sharp-boundary” equilibria:
[Potter, Methods Comp. Phys., **16**, 43 (1976); Berk et al., Phys. Fluids, **29**, 3281 (1986); Kaiser & Salat Phys. Plasmas **1**, 281 (1994)]

Relaxed MHD ← Multi-Region relaxed MHD → Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ **Relaxed MHD**

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$

$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$

$+$ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

$N_V = \infty$ **Ideal MHD**

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$

(fluxes & helicity conserved)

$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

Relaxed MHD ← Multi-Region relaxed MHD → Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B}\, dv}_{helicity},$$

$\delta\mathbf{B} \equiv \nabla \times \delta\mathbf{A}$ is arbitrary in $\mathcal{R}$
($\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial\mathcal{R}$)
+ constrained flux

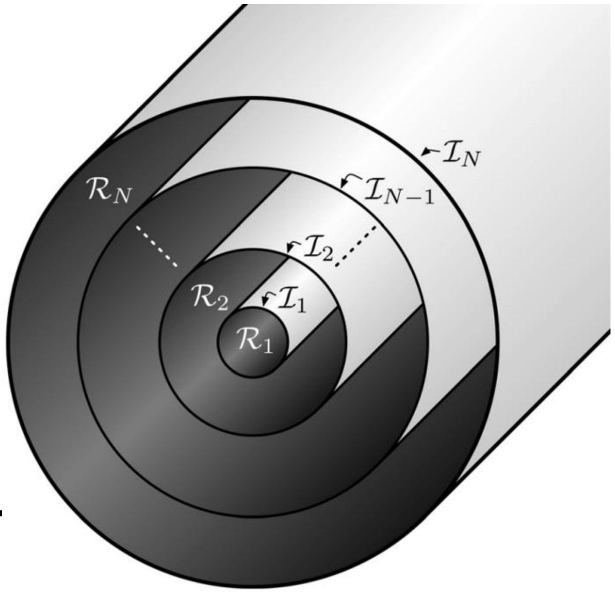
$$\delta\mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu\, \mathbf{B} \text{ in } \mathcal{R};$$

$N_V < \infty$ MRx MHD

$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B}\, dv \right\},$$

$\delta\mathbf{B}_i \equiv \nabla \times \delta\mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$
$\delta\mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial\mathcal{R}_i$
+ constrained fluxes in $\mathcal{R}_i$

$$\delta\mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i\, \mathbf{B} \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial\mathcal{R}_i;$$



$\boxed{\text{Stepped Pressure Equilibrium Code}}$

[Hudson, Dewar et al., Phys. Plasmas **19**, 112502 (2012)]

$N_V = \infty$ Ideal MHD

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta\mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$
(fluxes & helicity conserved)

$$\delta\mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

Relaxed MHD $\leftarrow$ Multi-Region relaxed MHD $\rightarrow$ Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$
$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$
+ constrained flux

$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \text{ in } \mathcal{R};$$

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$\delta \mathbf{B}_i \equiv \nabla \times \delta \mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$
$\delta \mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial \mathcal{R}_i$
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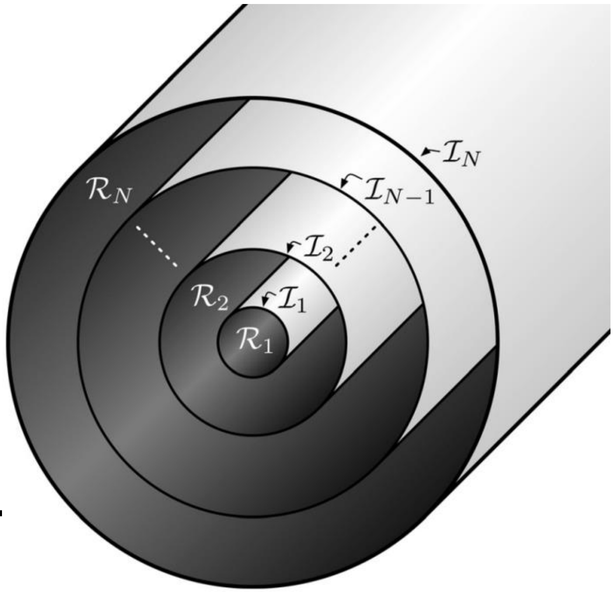
$$\delta \mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i \mathbf{B} \text{ in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial \mathcal{R}_i;$$



$$\rightarrow \quad p(\psi), \nabla p = \mathbf{j} \times \mathbf{B} \quad \text{as} \quad N_V \rightarrow \infty,$$

[Dennis, Hudson et al., Phys. Plasmas **20**, 032509, 2013]

$N_V = \infty$ Ideal MHD

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

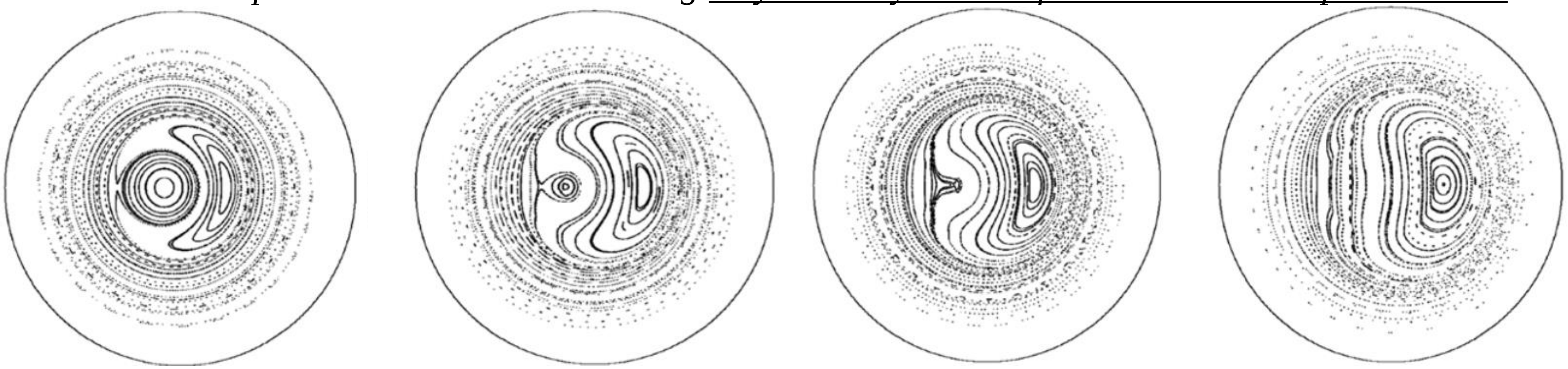
$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$
(fluxes & helicity conserved)

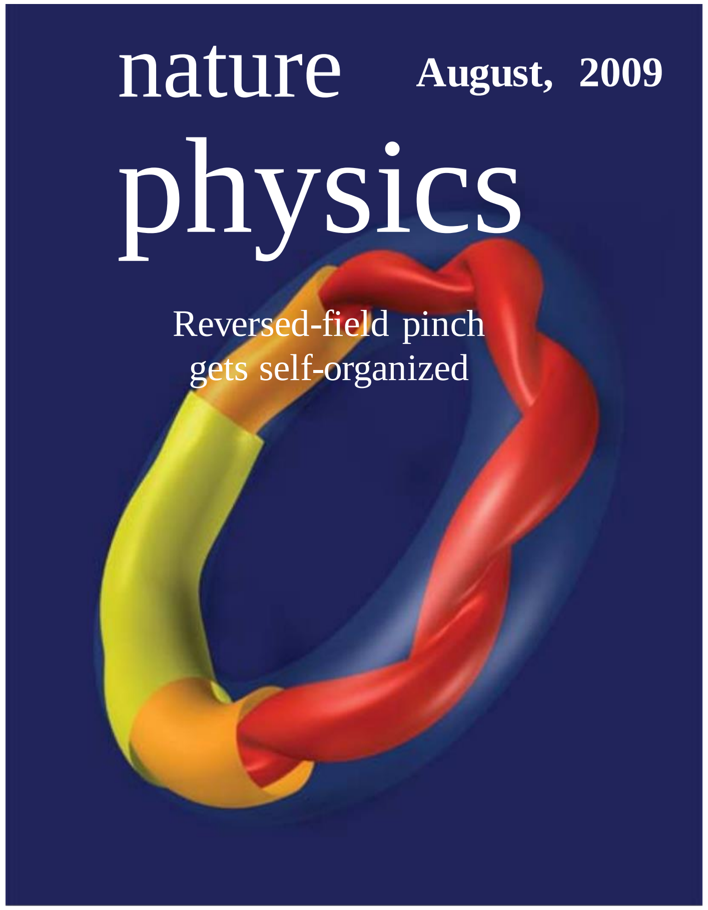
$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \text{ in } \mathcal{R}.$$

N_v=2, “double-Taylor” state with transport barrier; MRxMHD explains self-organization of RFP into helix.

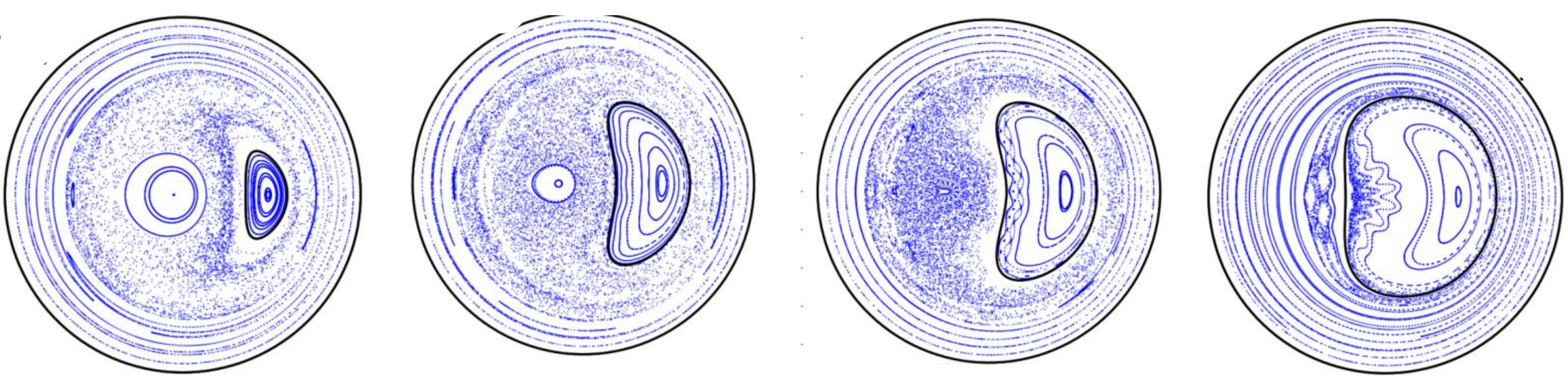
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nucl. Fusion*, ***49*** *104019 (2009)*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation”





NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
Minimally Constrained Model of Self-Organized Helical States in Reversed-Field Pinches
G. Dennis, S. Hudson et al. Phys. Rev. Lett. **111**, 055003 (2013)



Excellent qualitative agreement between numerical calculation and experiment.

Nv=3, compute the δ-function current density, 3D ideal-MHD solutions require ***infinite shear!***

Cartesian, slab geometry with an $(m, n) = (1, 0)$ resonantly-perturbed boundary

i. $N_V = 3$ MRxMHD calculation, no pressure, $\iota(\psi)$ given discretely,

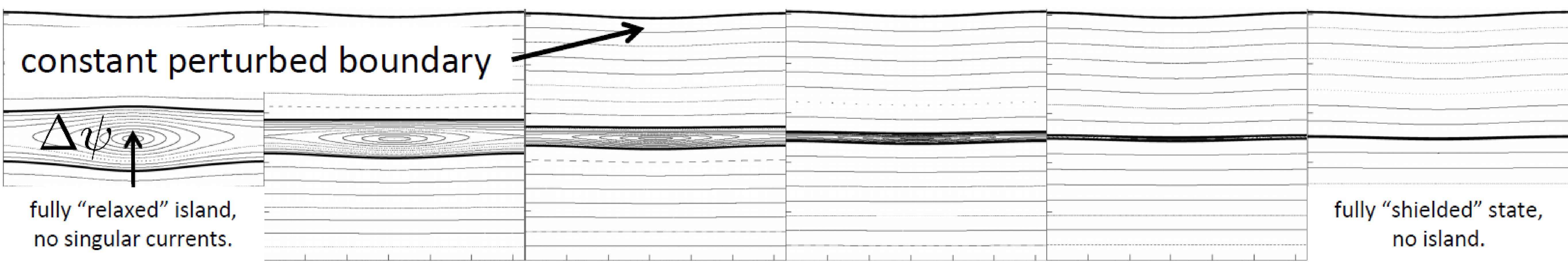
ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$,

iii. shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\text{MUST HAVE } \beta > \alpha, \text{ i.e. infinite-shear}}$

iv. island forced to vanish $\implies$ **island-shielding**

v. resonant $\delta_{m,n}$-function current-density $\equiv$ tangential discontinuity in $\mathbf{B}$.

Analytic verification with SPEC



[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

Nv=∞, compute the 1/x current-density, 3D ideal-MHD solutions require ***infinite shear!***

Cartesian, slab geometry with an $(m, n) = (1, 0)$ resonantly-perturbed boundary

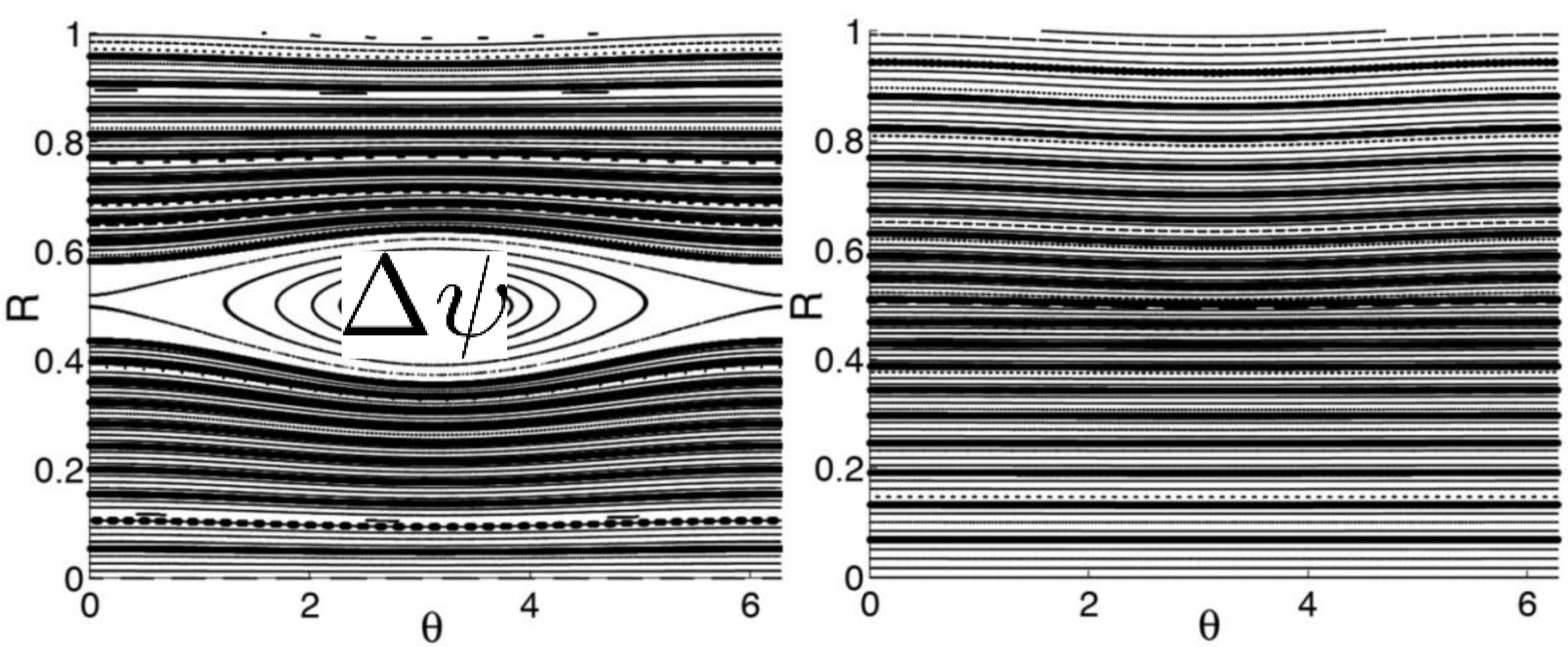
i. N_V = large MRxMHD calculation, stepped pressure ≈ smooth pressure,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$,

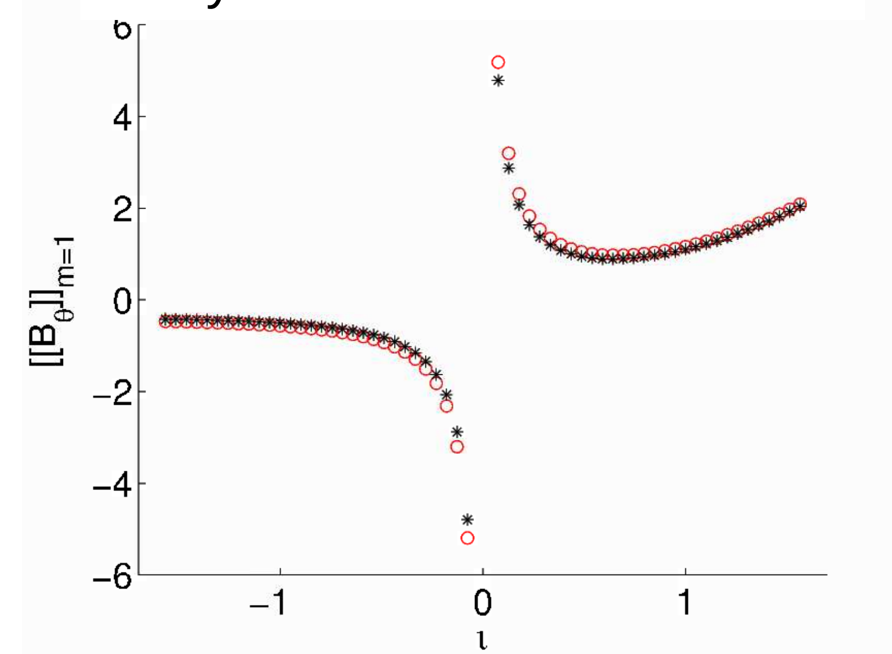
iii. shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\text{MUST HAVE } \beta > \alpha\text{, i.e. infinite-shear}}$

iv. island forced to vanish $\Longrightarrow$ **island-shielding**

v. resonant p'/x current-density ≡ tangential discontinuity in $\mathbf{B}$.



Analytic verification with SPEC



[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

Infinite shear ≡ discontinuous rotational-transform: introduce new class of solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$

1. Cylindrical geometry with an $(m,n) = (2,1)$ resonantly-perturbed boundary

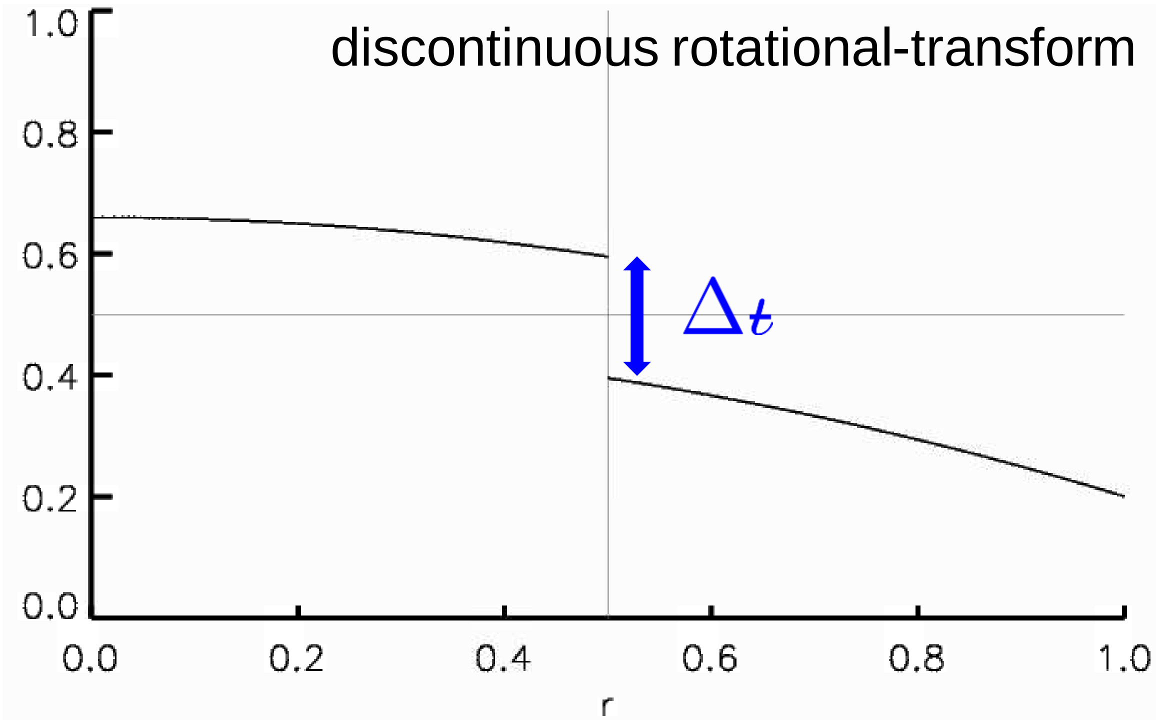
   i. $\boxed{p = 0,}$ $\quad \iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota,$

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1+\iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \quad - \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
   for $\Delta\iota \to 0$, recover step-function solution



RMP penetrates into the core, even for ideal-MHD.

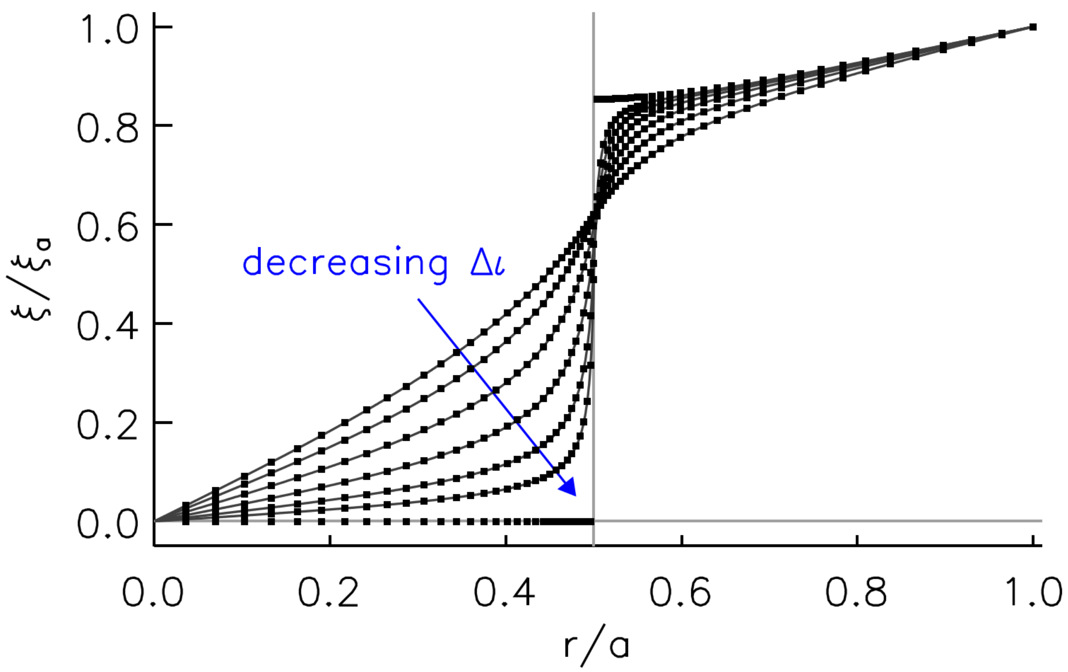
2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$

Analytic verification with SPEC



[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

Infinite shear ≡ discontinuous rotational-transform: introduce new class of solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$

1. Cylindrical geometry with an $(m, n) = (2, 1)$ resonantly-perturbed boundary

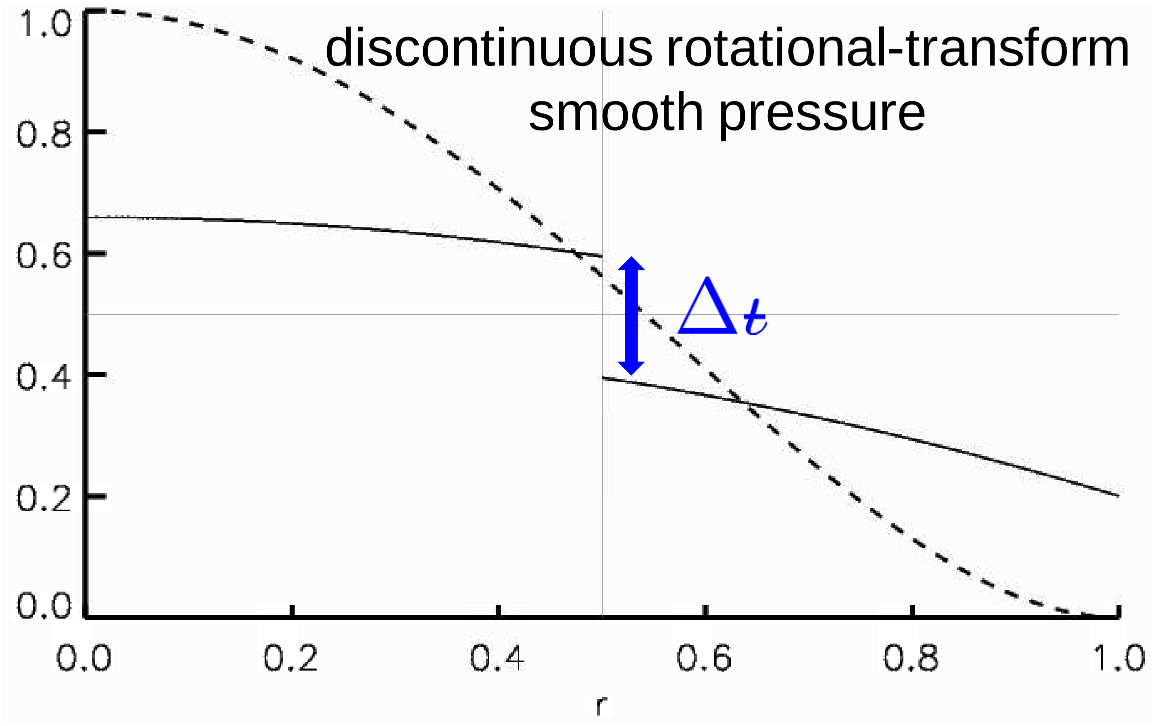
   i. $\boxed{p = p_0(1 - 2r^2 + r^4)}$, $\iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota$,

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1 + \iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \boxed{\nabla \delta p} - \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
   for $\Delta\iota \to 0$, recover step-function solution



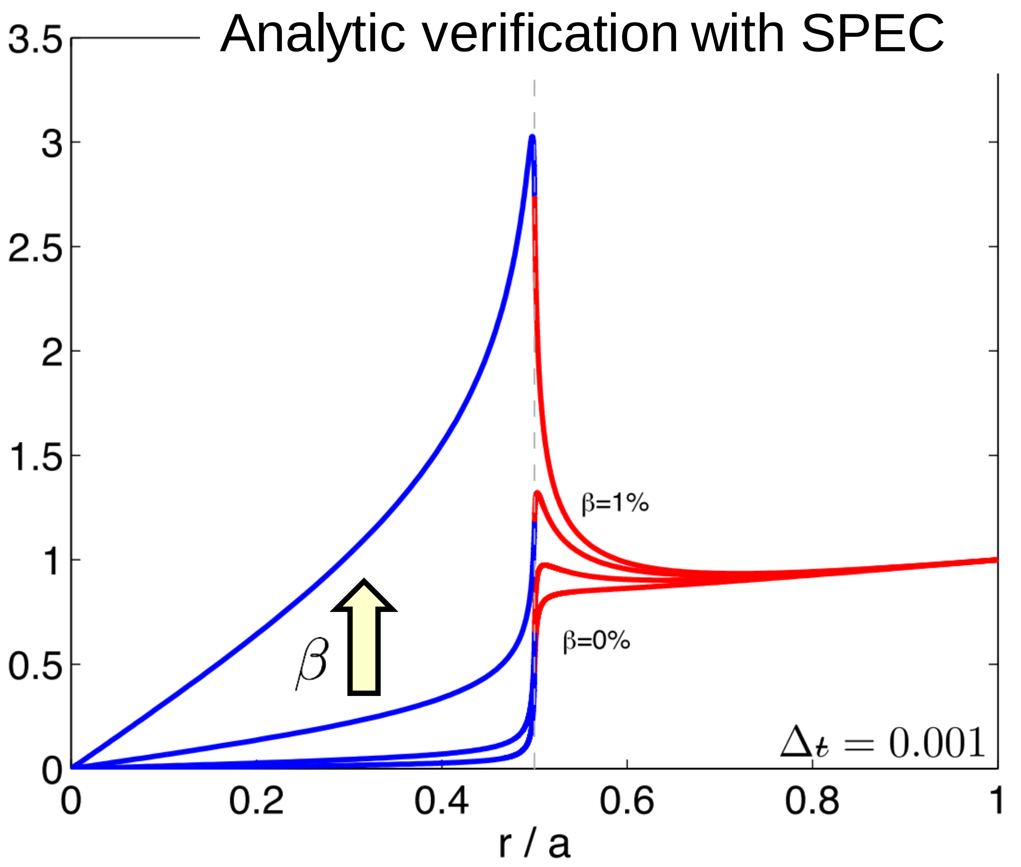
Perturbation amplified by pressure near and inside "resonant" surface

2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$



[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

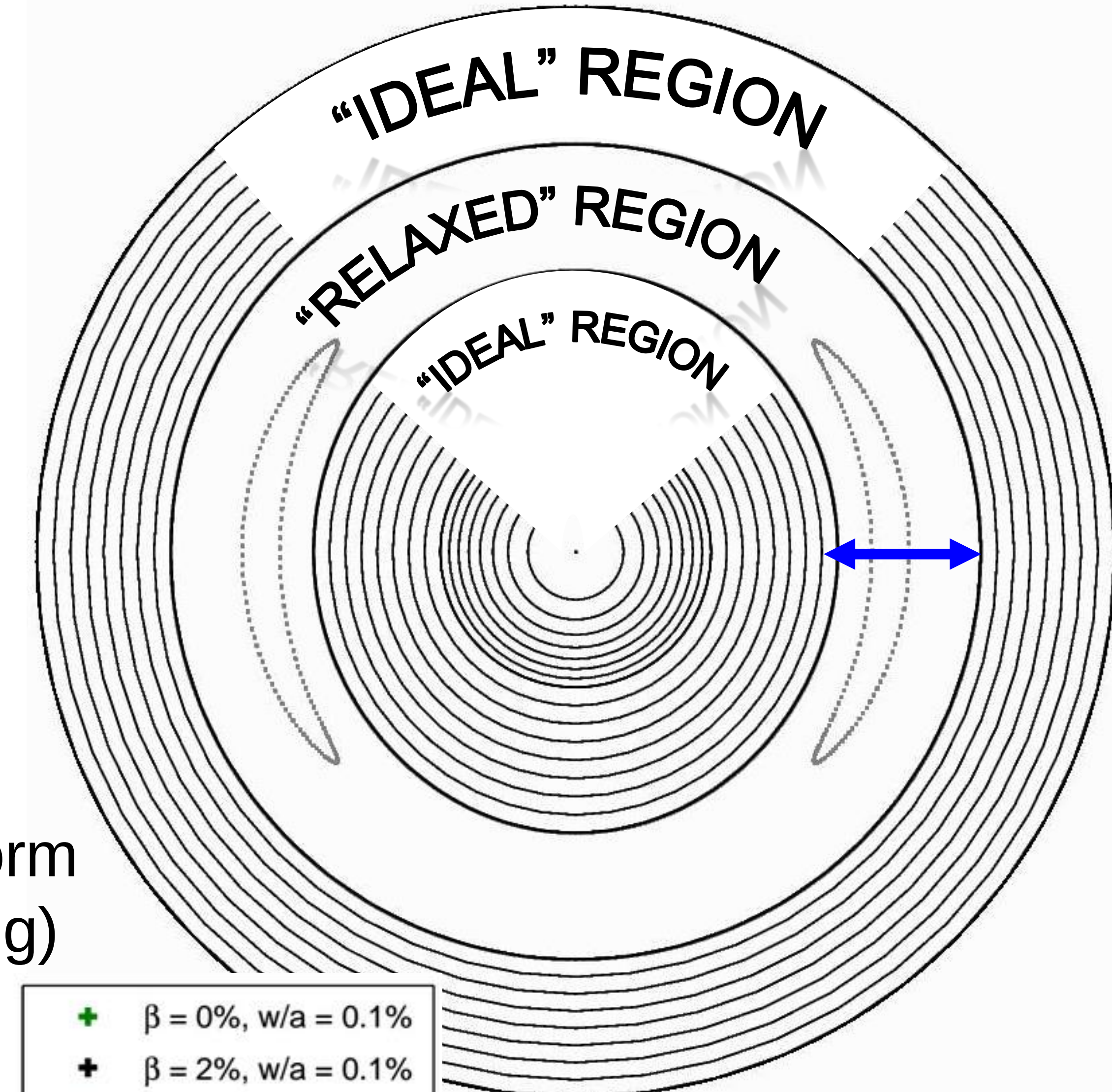
Now, including pressure and an island . . . amplification and penetration of the RMP is still present.

1. Now, include a "relaxed" region,

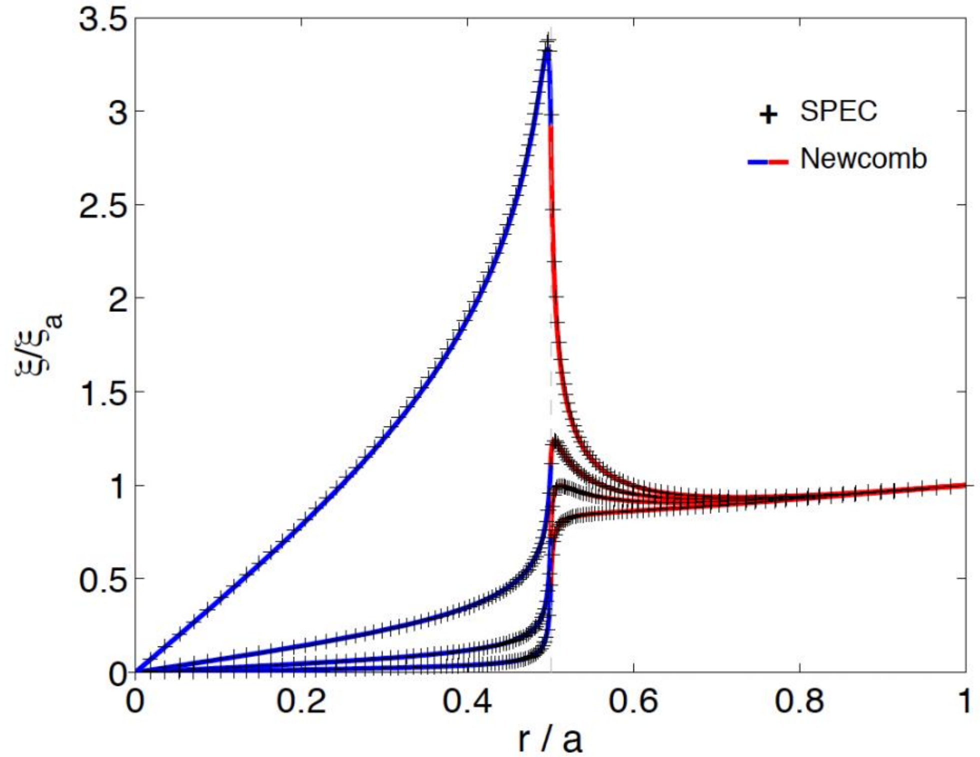
   i. $\Delta\psi_t \equiv$ toroidal flux in relaxed region.

   ii. $\Delta\iota\!\!\!{-} \equiv$ jump in transform across relaxed region.

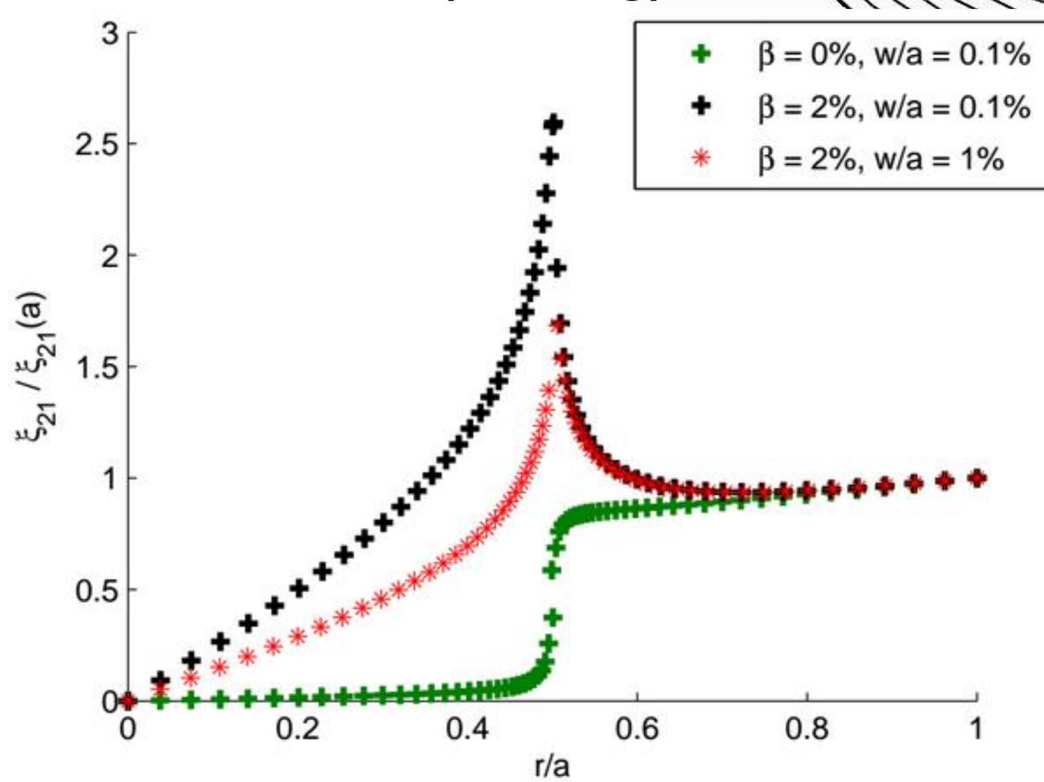
   so that an island is allowed to form.



Discontinuous transform with no island (ideal)

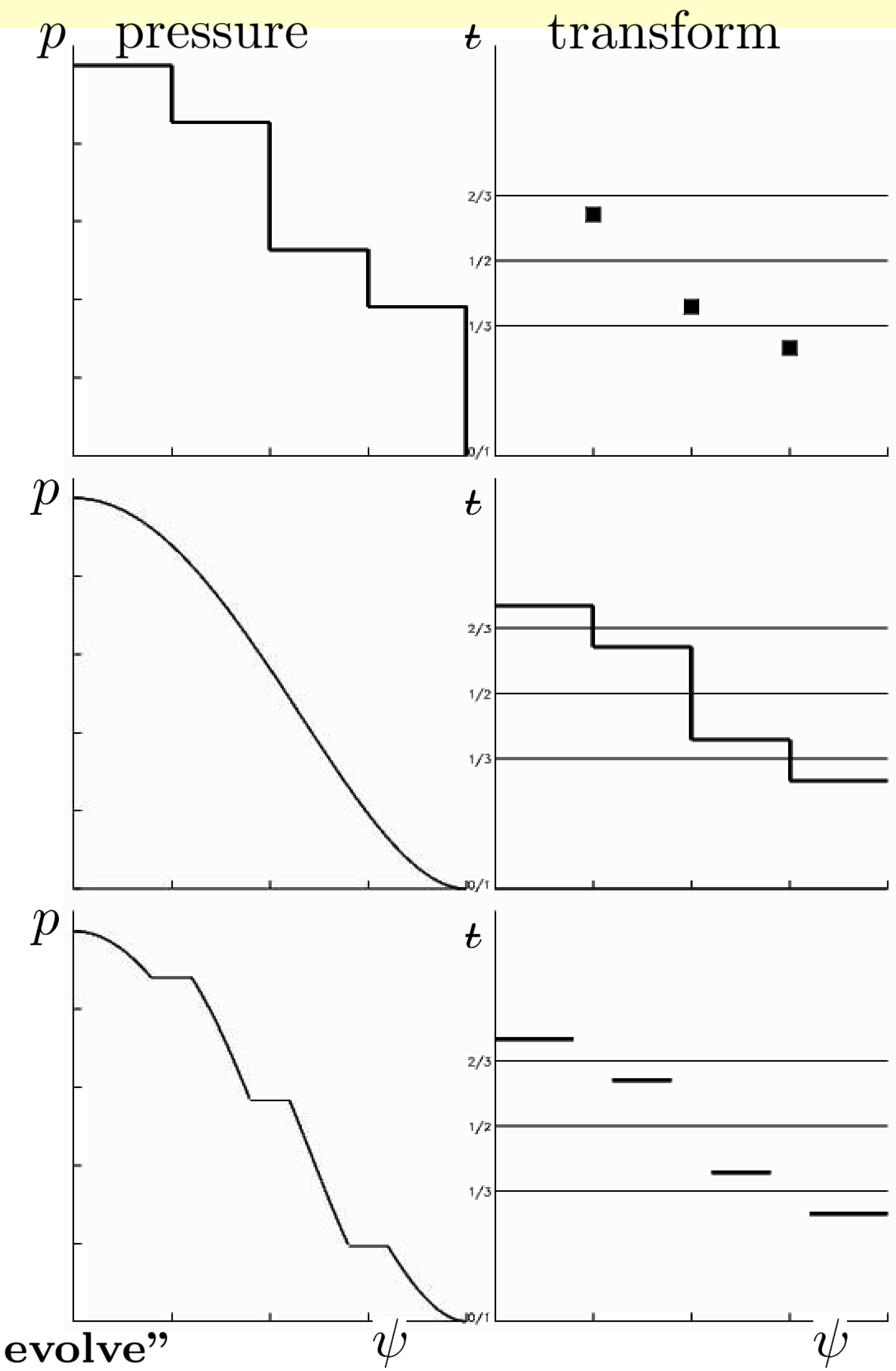


Continuous transform with island (tearing)



The two classes of general, relevant, tractable 3D MHD equilibria are:

1. **Stepped-pressure equilibria,**
   1. Bruno & Laurence states
   2. extrema of MRxMHD energy functional
   3. transform constrained discretely
   4. pressure discontinuity at $\iota\!\!\!-$ = irrational
   5. allows for islands, magnetic fieldline chaos
2. **Stepped-transform equilibria,**
   1. introduced by Loizu, Hudson et al.
   2. extrema of ideal MHD energy functional
   3. transform (almost) everywhere irrational
   4. arbitrary, smooth pressure
   5. continuously-nested flux surfaces
3. **Or, a combination of the above.**
   1. **each can be computed using SPEC**
   2. suggests VMEC, NSTAB, should be modified to allow for discontinuous transform



Q. **How does a state with continuous transform "ideally evolve" into a 3D state with discontinuous transform?**
implications for ideal stability if no accessible 3D state exists?

Ongoing development/applications of SPEC = Stepped Pressure Equilibrium Code

1. RECENT code improvements:

   i. finite-elements replaced by Chebshev polynomials

   e.g. $\mathbf{A} \equiv \sum_{l,m,n}^{L,M,N} [\alpha_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\theta + \beta_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\zeta]$

   ii. linearized equations

   iii. Cartesian, cylindrical, toroidal geometry

   iv. detailed online documentation,
   `http://w3.pppl.gov/~shudson/Spec/spec.html`

   v. easy-to-use, easy-to-edit, graphical user interface

2. ONGOING physics applications

   i. W7-X vacuum verification calculations, OP1.1 [completed]

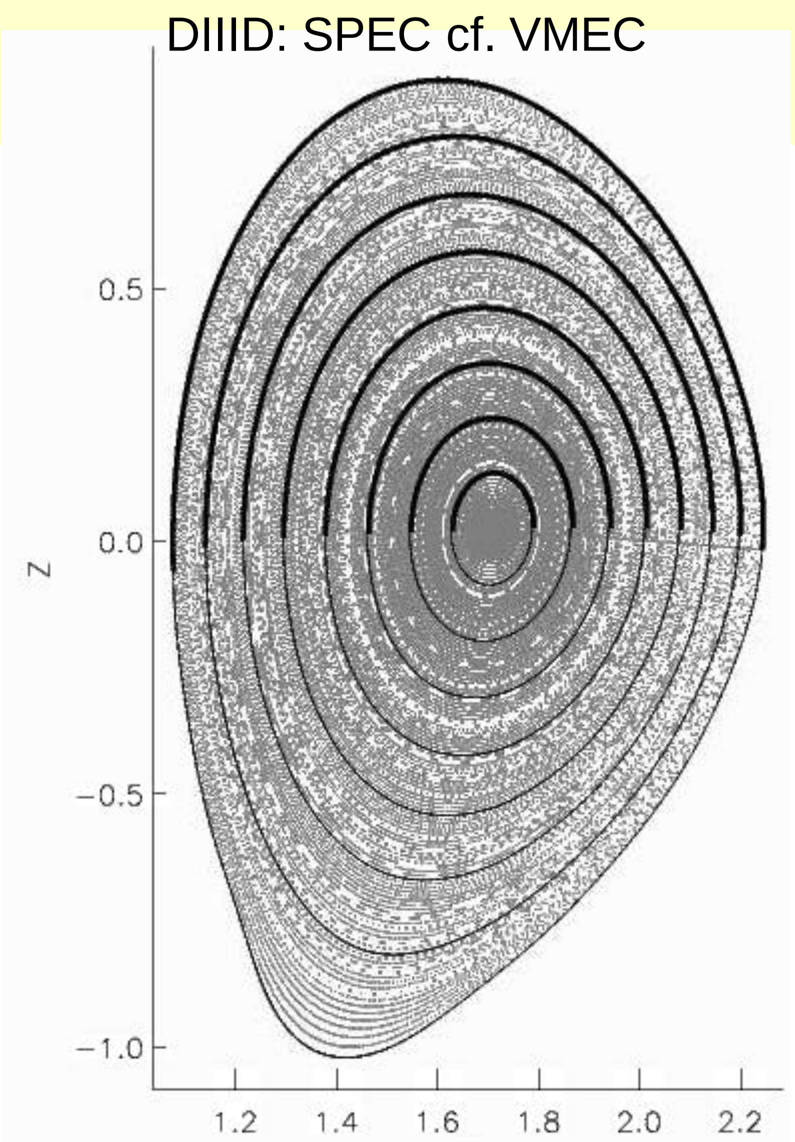
   ii. non-stellarator symmetric, e.g. DIIID, [completed]

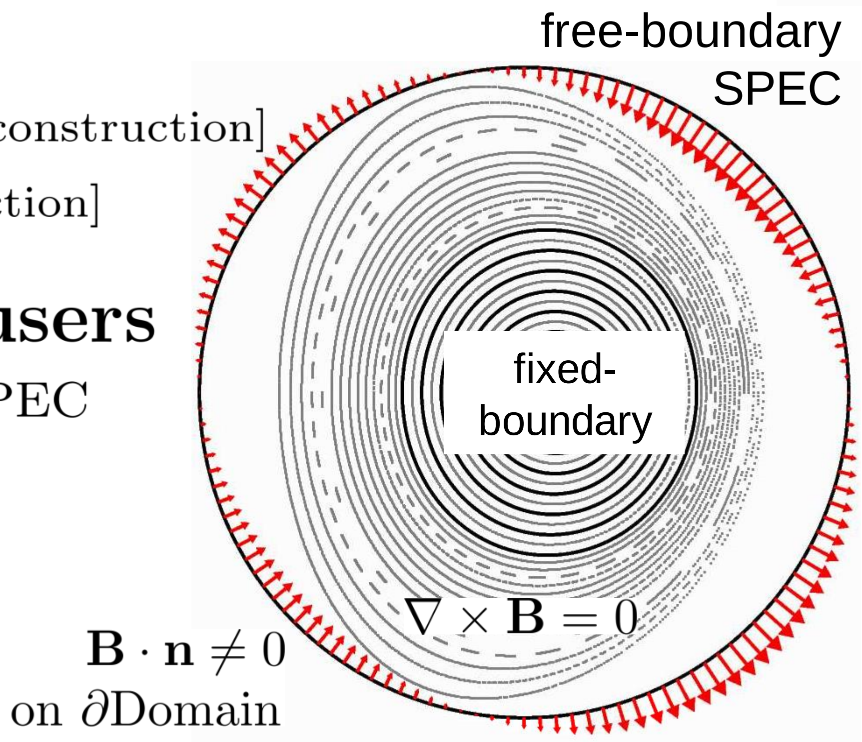
   iii. free-boundary, [completed]

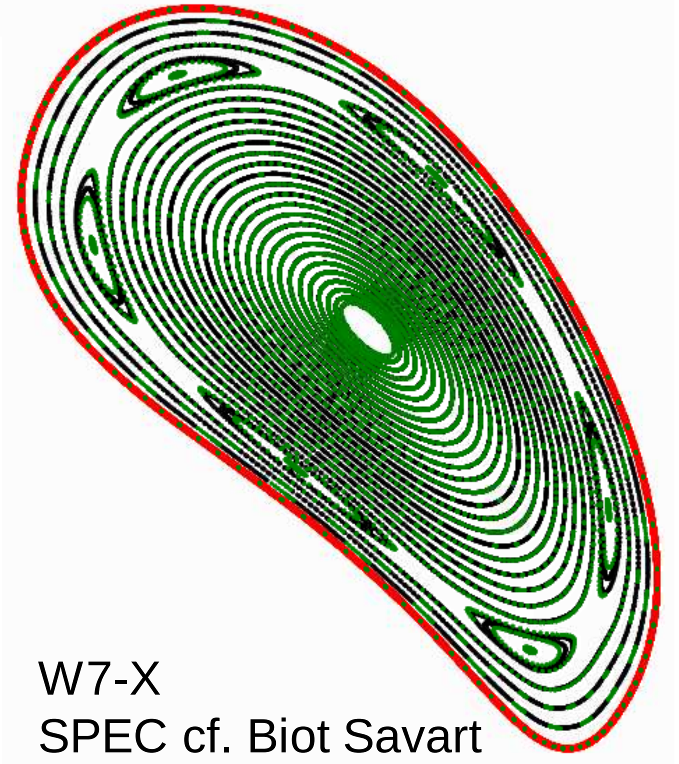
   iv. including flow, anisotrophy, . . [under construction]

   v. MRxMHD linear stability, [under construction]

3. **Seeking collaborators, code-users**
   please email `shudson@pppl.gov`, re: SPEC







Back up slides

Now, including pressure and an island . . . amplification and penetration of the RMP is still present.

1. Now, include a "relaxed" region,

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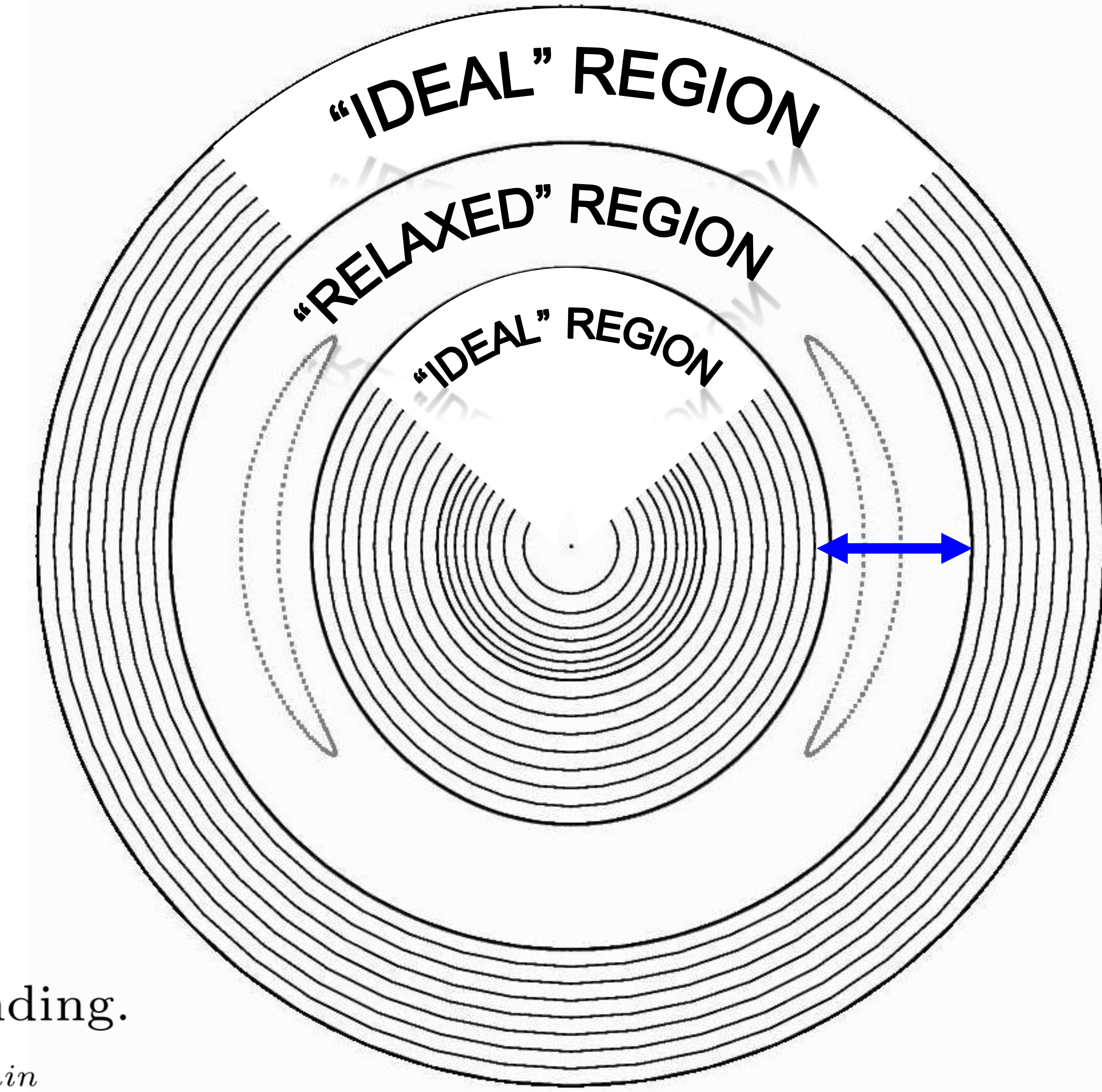
   so that an island is allowed to form.



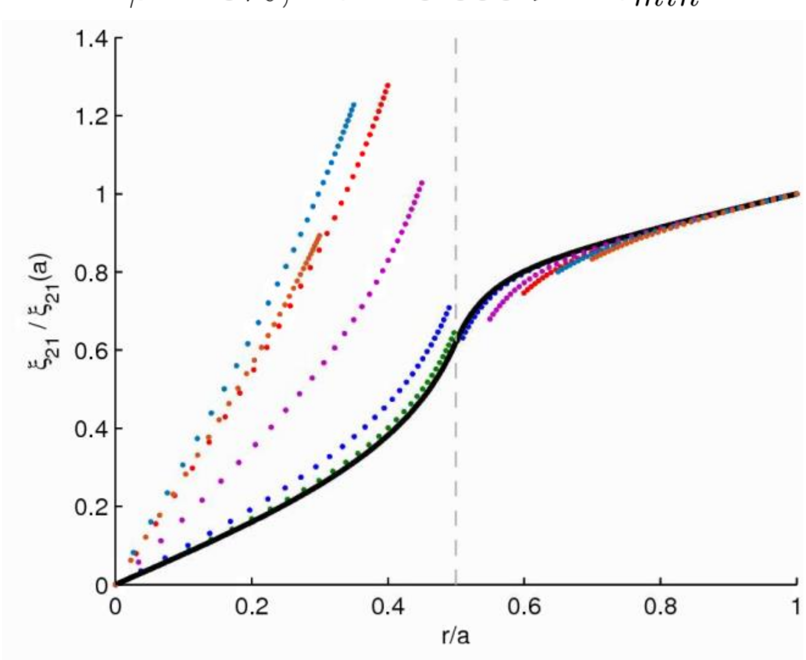
2. SPEC calculations indicate that

   i. The perturbation still penetrates.

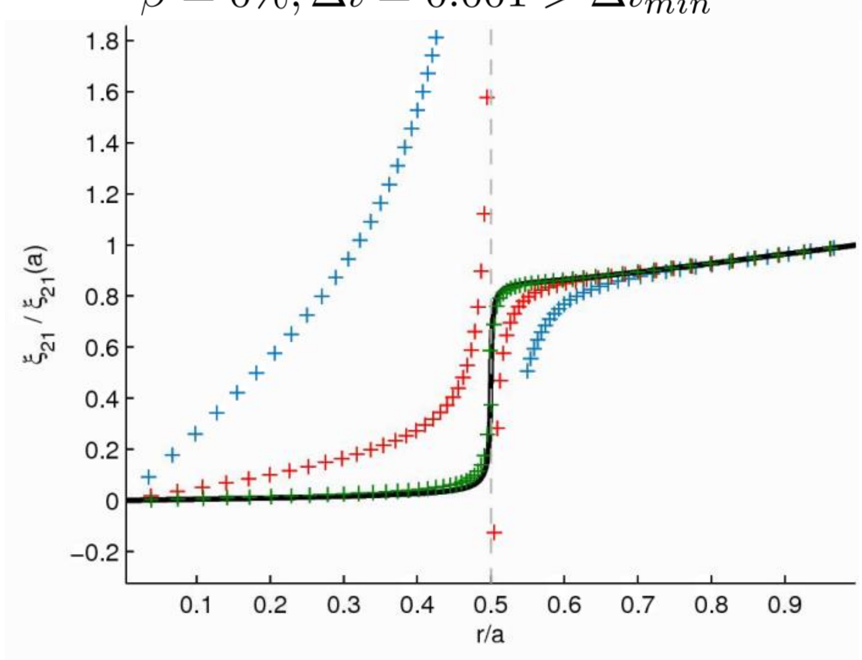
   ii. The perturbation is still amplified by pressure.

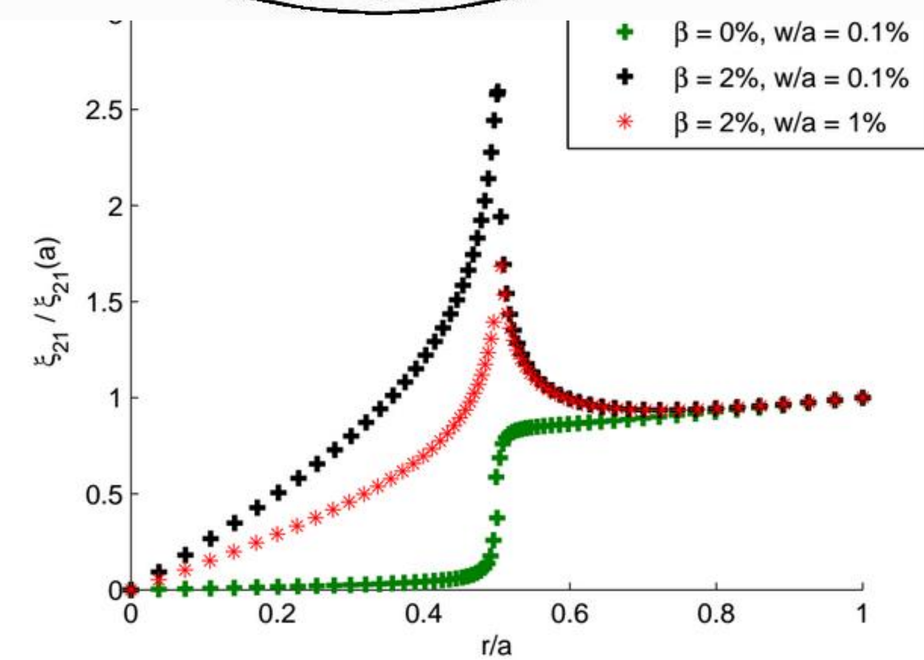
3. Precise comparison of SPEC cf. tearing mode theory pending.

$\beta = 0\%, \Delta\iota = 0.050 > \Delta\iota_{min}$



$\beta = 0\%, \Delta\iota = 0.001 > \Delta\iota_{min}$

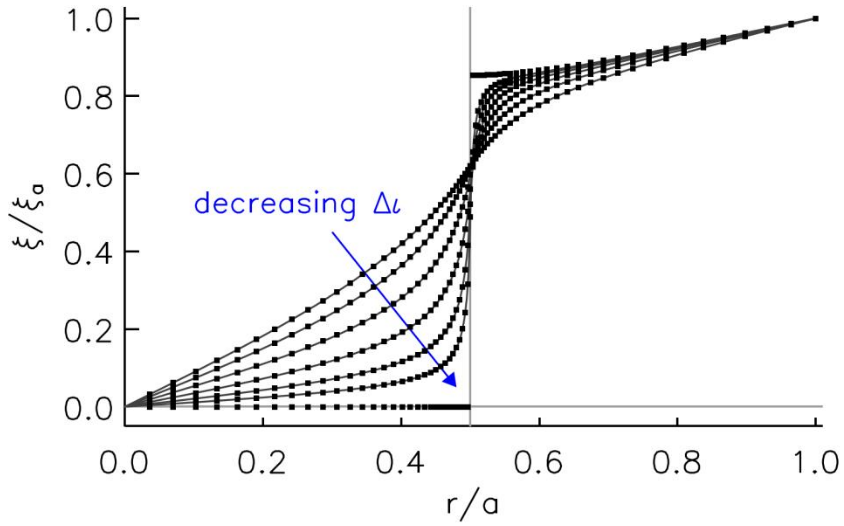




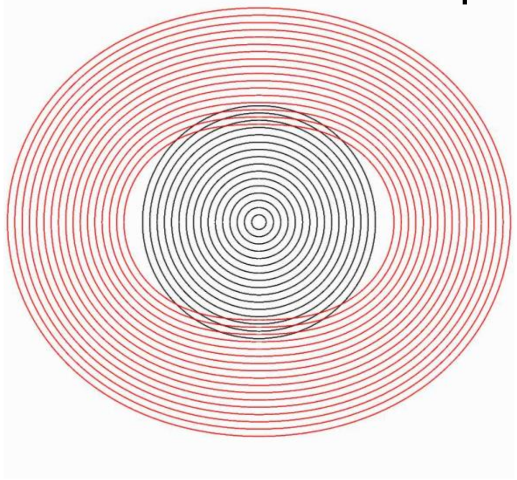
Necessary condition for non-overlapping of perturbed surfaces
Existence of non-linear solutions

1. **Condition for non-overlapping perturbed surfaces**

$$\left|\frac{\partial \xi}{\partial r}\right|_{max} < 1$$



Discontinuously-perturbed flux surfaces overlap!



2. **An asymptotic analysis near the rational surface**
   gives the *sine-qua-non* condition *(an indispensable condition, element, or factor; something essential)*

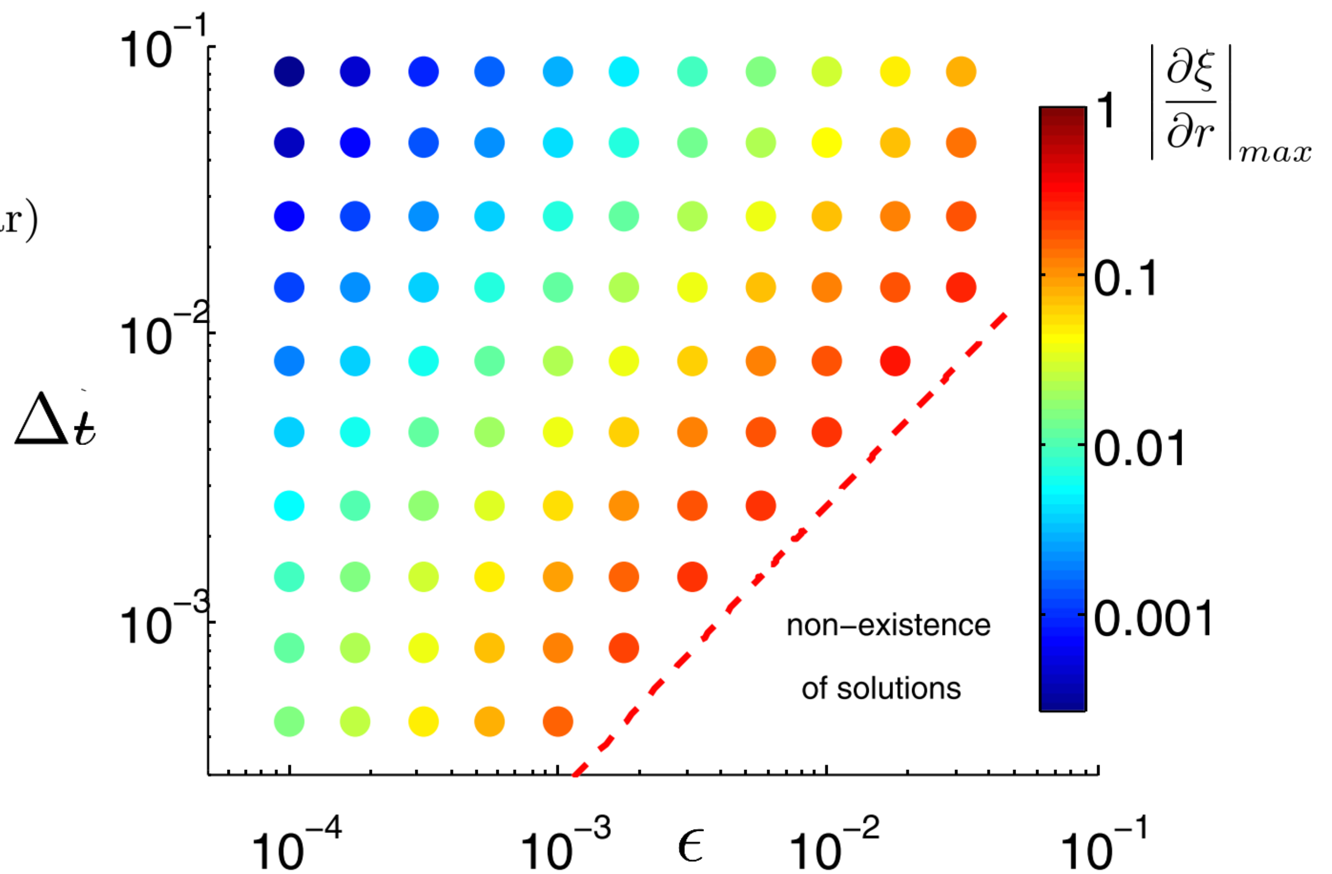
$$\Delta\iota > \Delta\iota_{min}, \quad \text{where } \Delta\iota_{min} \equiv 2\iota'_s \xi_s$$

(analysis for cylindrical, zero-β; general result probably similar)

3. **If this condition is violated, non-linear solutions do not exist.**

   i. Shown is ξ', as computed using non-linear SPEC calculations, as a function of $(\epsilon, \Delta\iota)$

   ii. SPEC fails in ideal-limit, i.e. $N_V \to \infty$, when $\Delta\iota < \Delta\iota_{min}$

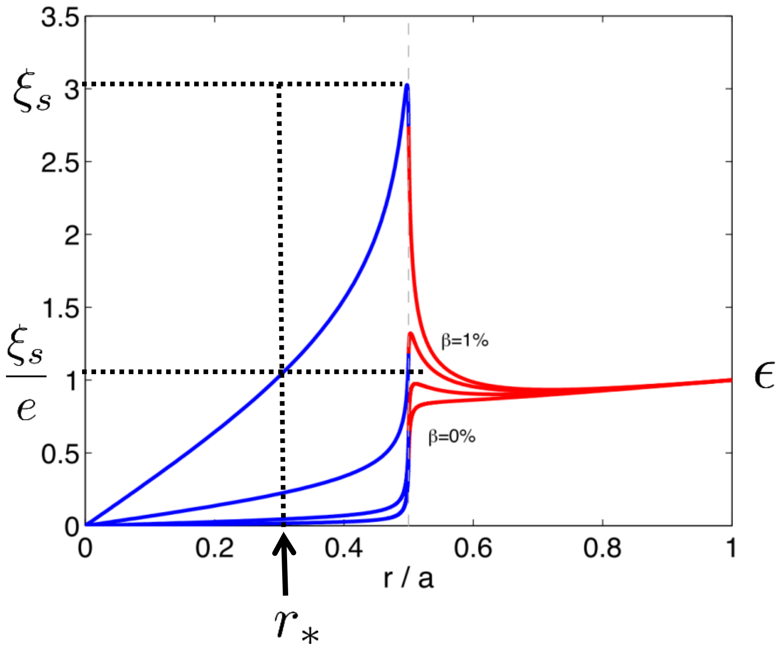


[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

Amplification and penetration as stability boundary is approached

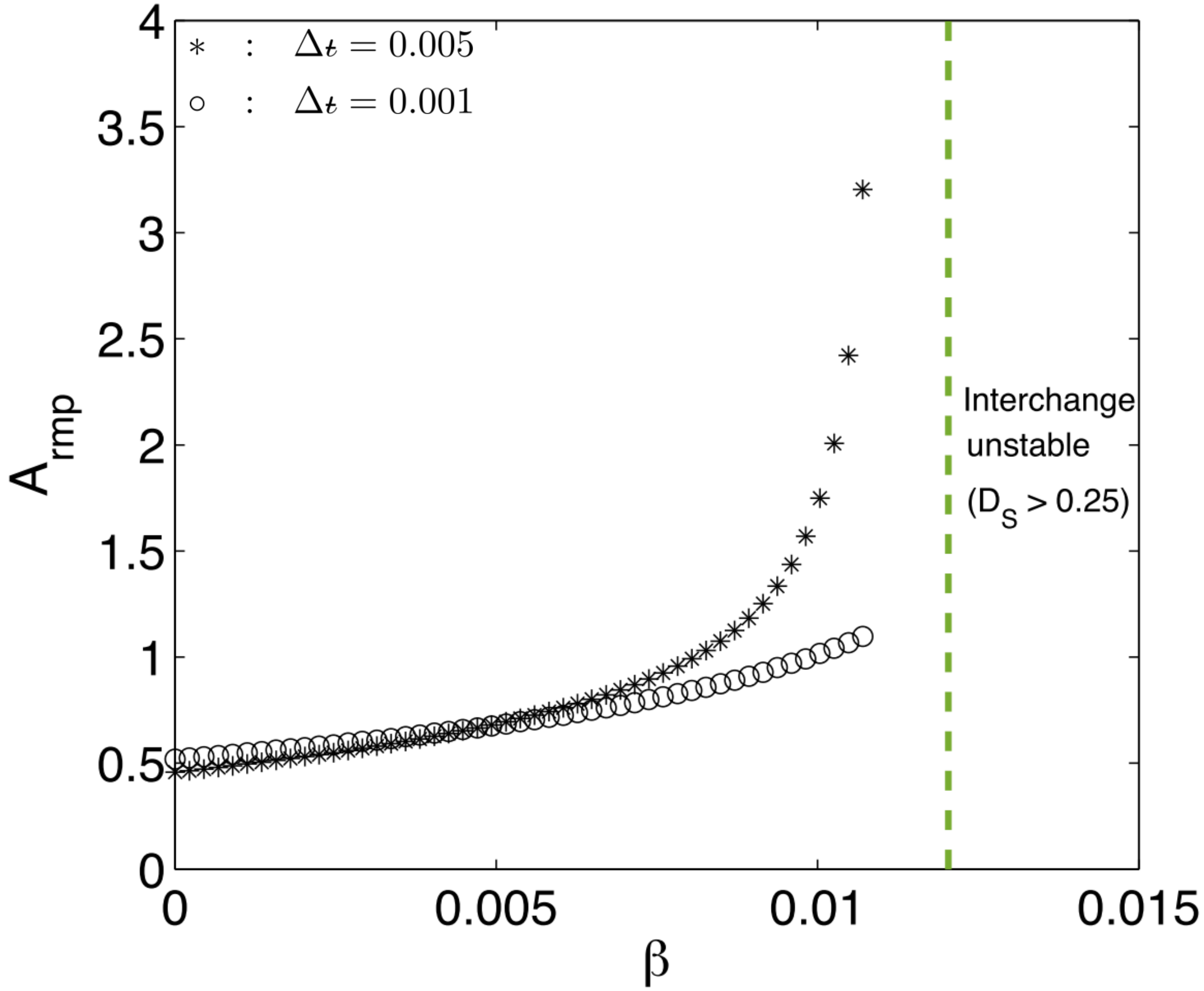
1. Can define a measure of

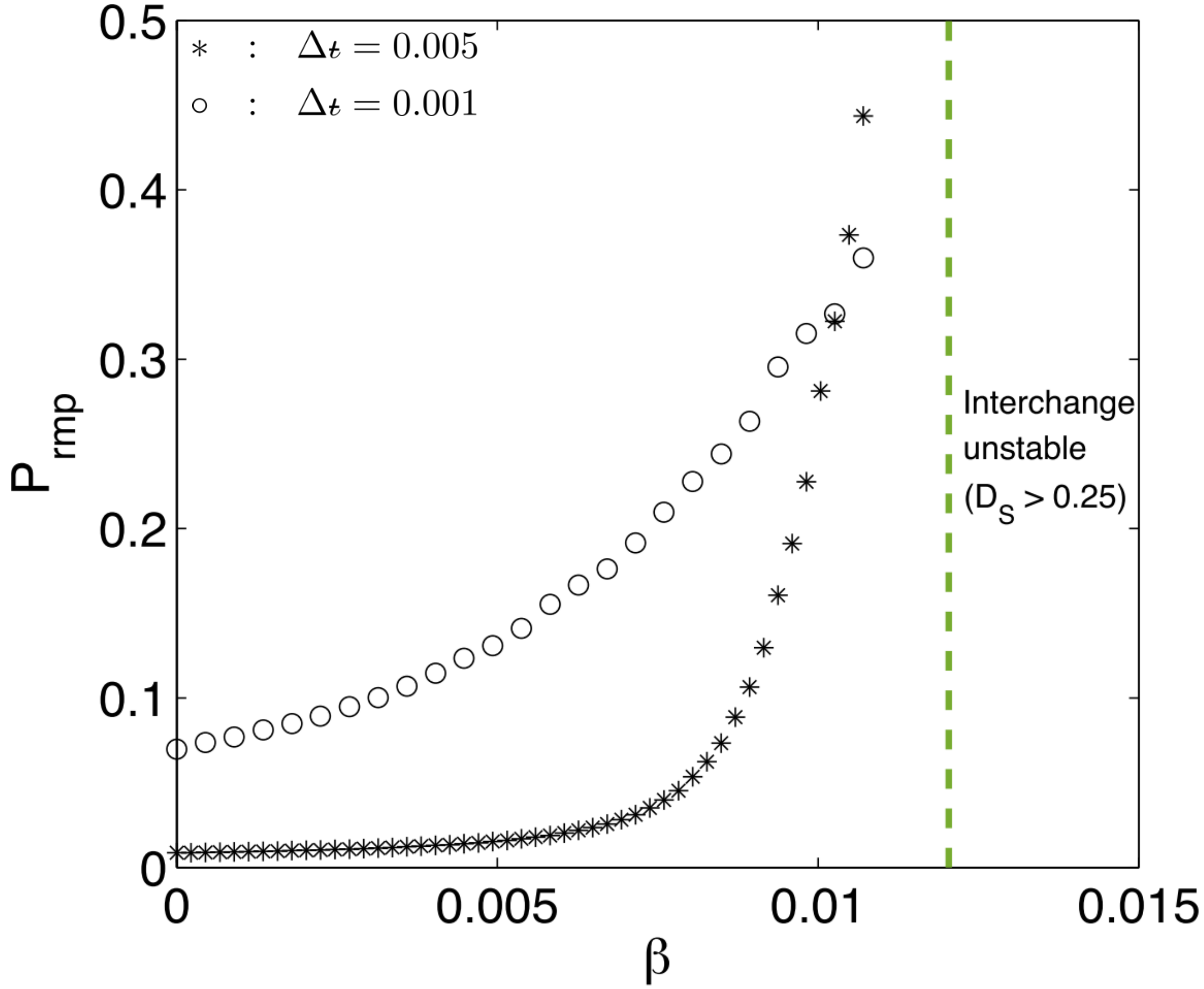
$$\text{“Amplification”} \quad A_{rmp} = \xi_s/\epsilon, \quad \text{where } \epsilon \equiv \text{boundary deformation}$$
$$\text{“Penetration”} \quad P_{rmp} = 1 - r_*/r_s, \quad \text{where } \xi(r_*) \equiv \xi_s/e$$



2. A necessary condition for interchange stability in a screw pinch is given by the Suydam criterion, $\boxed{D_S \equiv -\left(\frac{2p'\iota^2}{rB_z^2\iota'^2}\right)_s < \frac{1}{4}.}$

3. Amplification and penetration of RMP **fantastically increased** as stability limit approached.

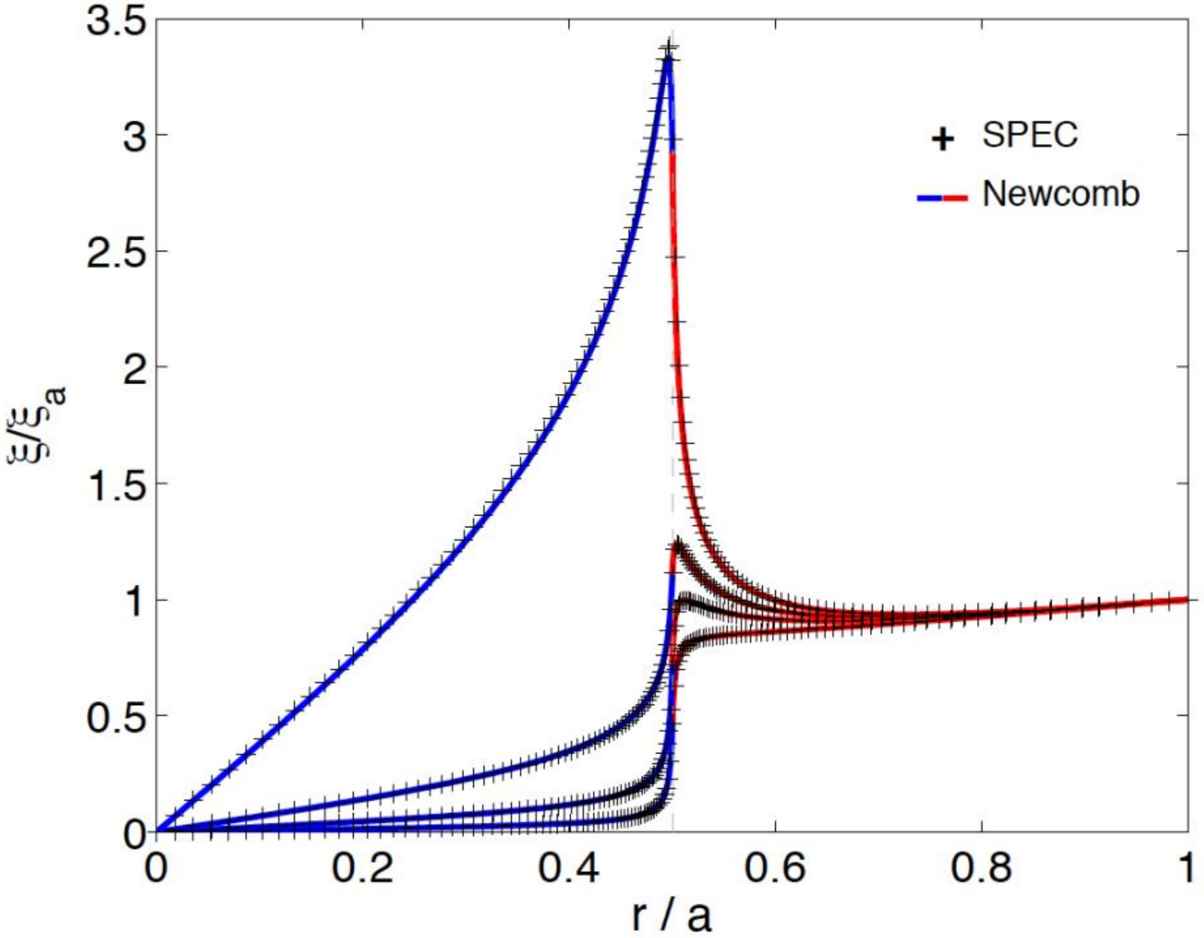




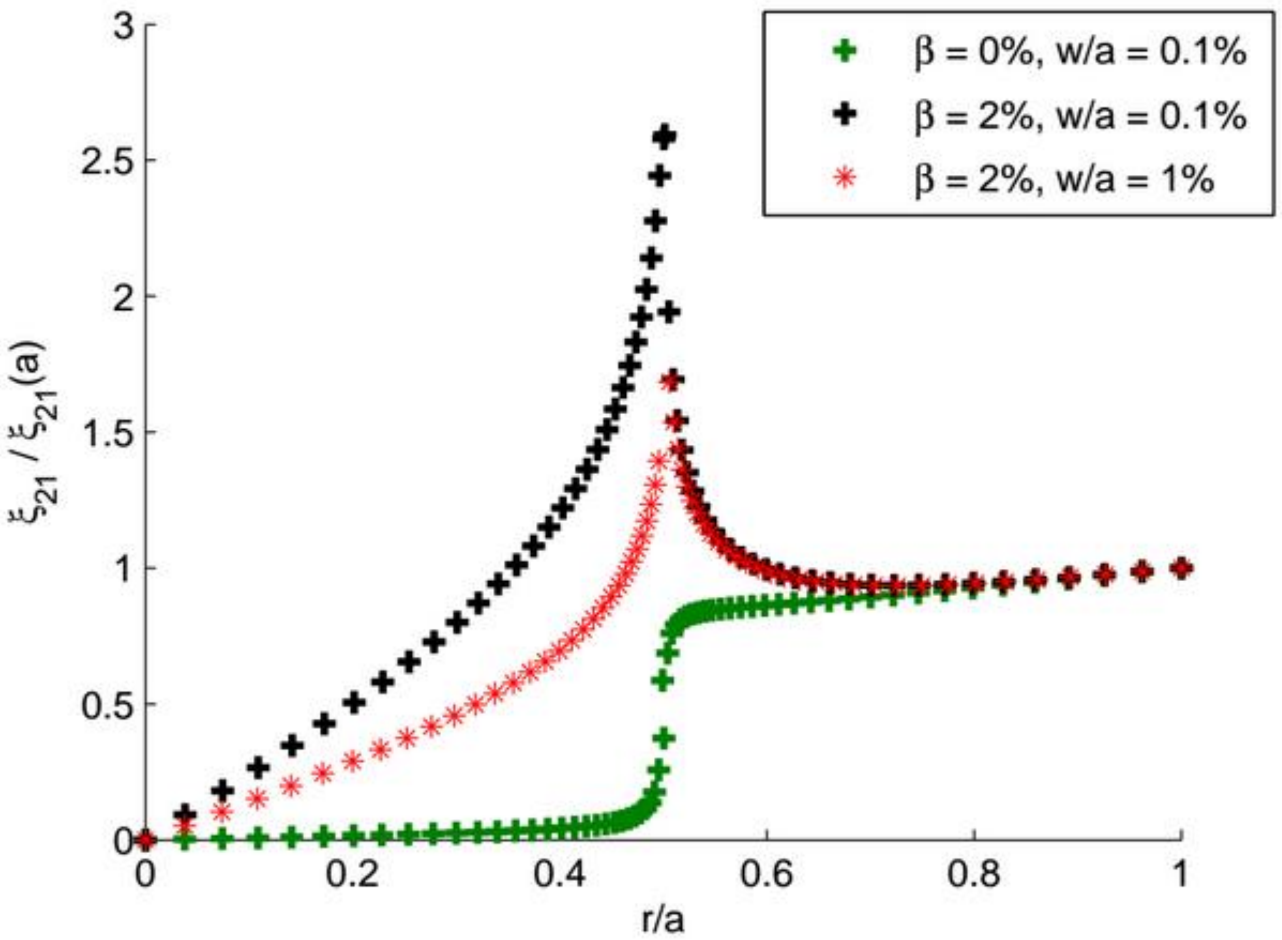
[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

Discontinuous transform solution cf. “Tearing” solution

Discontinuous transform with no island (ideal)



Continuous transform with island (tearing)



SPEC	allows	discontinuous	profiles: exact	agreement
VMEC	assumes	smooth	profiles: approximate	agreement

1. VMEC assumes smooth profiles

   and smooth profiles imply discontinuous displacement

2. but, VMEC enforces nested flux surfaces

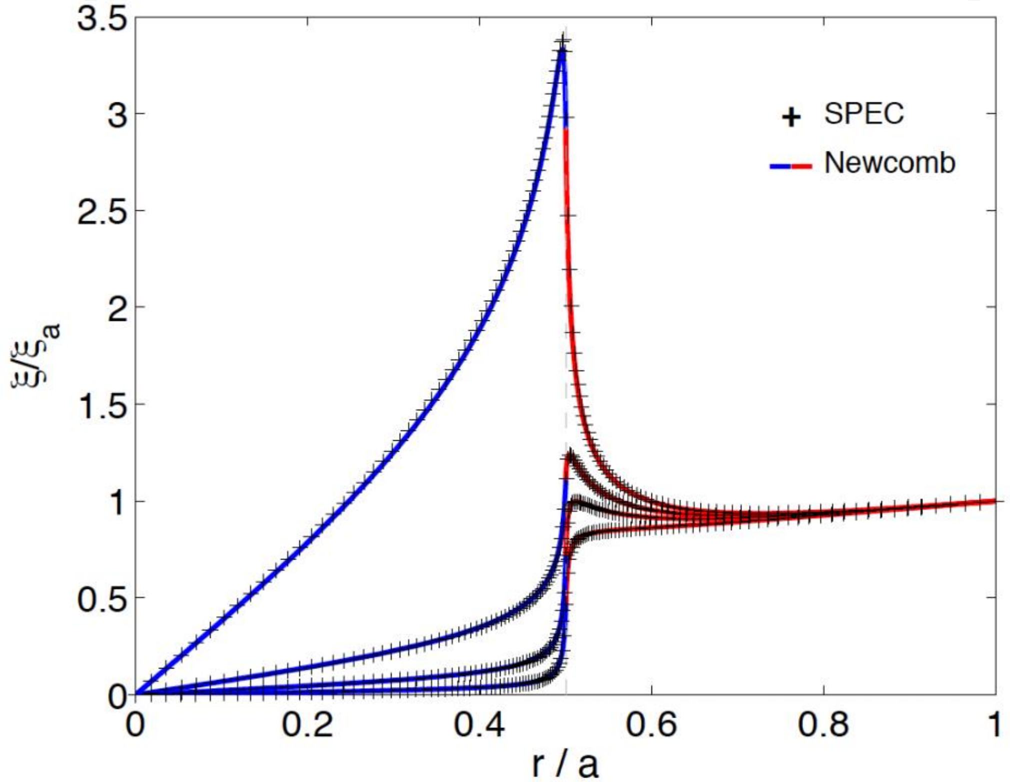
   nested flux surfaces in 3D imply $\dfrac{\partial \xi}{dr} < 1$ displacement from 2D

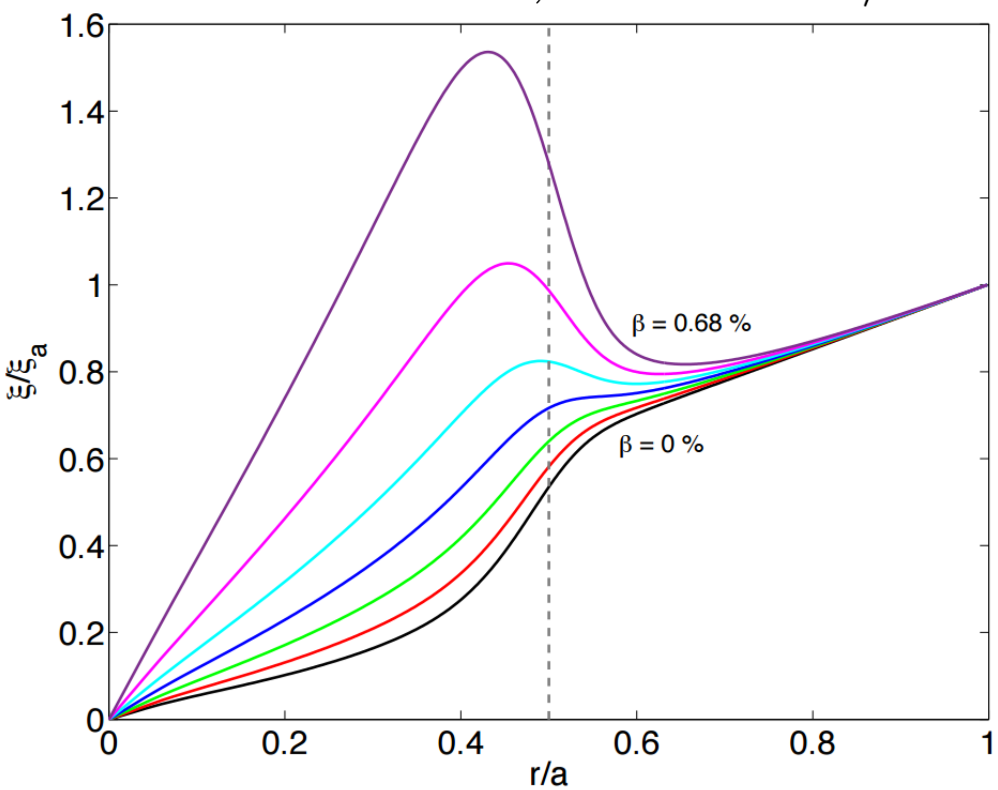
   and this is consistent only with discontinuous transform with $\Delta \iota > \Delta \iota_{min}$

3. Empirical study (i.e. radial convergence) shows that

   VMEC qualitatively reproduces self-consistent, perturbed solution

   interpretation: finite radial resolution implies an "effective" $\Delta \iota \sim \iota' h$, where $h \equiv 1/N$?





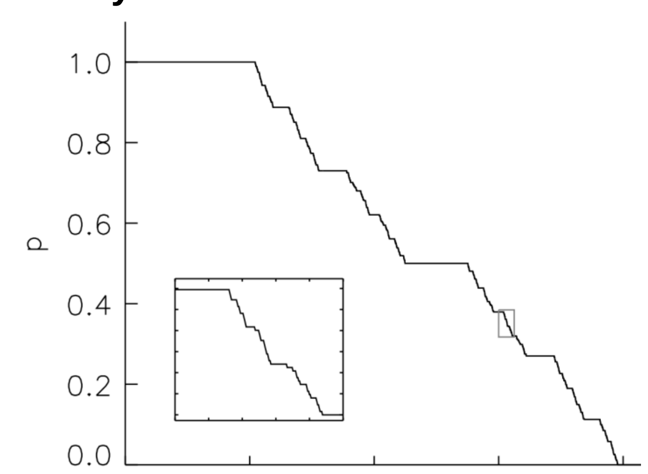
[Loizu, Hudson, …, Lazerson, ..., Phys. Plasmas **23**, 055703 (2016)]

Given continuous, non-integrable $\boldsymbol{B}$, $\boldsymbol{B}.\nabla p = 0$ implies p is fractal. Given fractal p, what is continuous, non-integrable $\boldsymbol{B}$?

- **Defn.** An equilibrium code computes the magnetic field consistent with a given p and e.g. given ι.
- **Theorem.** The topology of $\mathbf{B}$ is partially dictated by p.
  - $\hookrightarrow$ Where $p' \neq 0$, $\mathbf{B} \cdot \nabla p = 0$ implies $\mathbf{B}$ must have flux surfaces.
  - $\hookrightarrow$ Where $p' = 0$, $\mathbf{B}$ can have islands, chaos and/or flux surfaces.

TRANSPORT: given $\mathbf{B}$, solve for p.

1. Given general, non-integrable magnetic field, $\mathbf{B} = \nabla \times [\psi \nabla \theta - \chi(\psi, \theta, \zeta) \nabla \zeta]$
   i. fieldline Hamiltonian: $\chi(\psi, \theta, \zeta) = \chi_0(\psi) + \sum_{m,n} \chi_{m,n}(\psi) e^{i(m\theta - n\zeta)}$
2. KAM theorem: for suff. small perturbation, "sufficiently irrational" flux surfaces survive
   i. if ι satisfies a "Diophantine" condition, $|\iota - n/m| > r/m^k, \forall(n,m)$, **excluded interval about every rational**
   ii. need e.g. Greene's residue criterion to determine if flux-surface$_\iota$ exists; lot's of work;
3. With $\mathbf{B} \cdot \nabla p = 0$, i.e. infinite parallel transport, pressure profile must be fractal:



$$p'(\iota) = \begin{cases} 1, & \text{if } |\iota - n/m| > r/m^k, \quad \forall(n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |\iota - n/m| < r/m^k, \quad \exists(n,m), \end{cases}$$

$p'(x)$ is discontinuous on an uncountable infinity of points; impossible to discretize accurately;

EQUILIBRIUM: given p, solve for $\mathbf{B}$.

Q. <u>*Given*</u> a fractal p', how can the topology of $\mathbf{B}$ be constrained to enforce $\mathbf{B} \cdot \nabla p = 0$?

i. e.g. if $p(\psi)$ is continuous and smooth, nowhere zero, then $\mathbf{B}$ *must* be integrable, i.e. $\chi_{m,n}(\psi) = 0$

ii. if $p'(\psi)$ is fractal, then what are $\chi_{m,n}(\psi) = ?$

Convergence studies using VMEC

[Lazerson, Loizu et al., Phys. Plasmas **23**, 012507 (2016)]

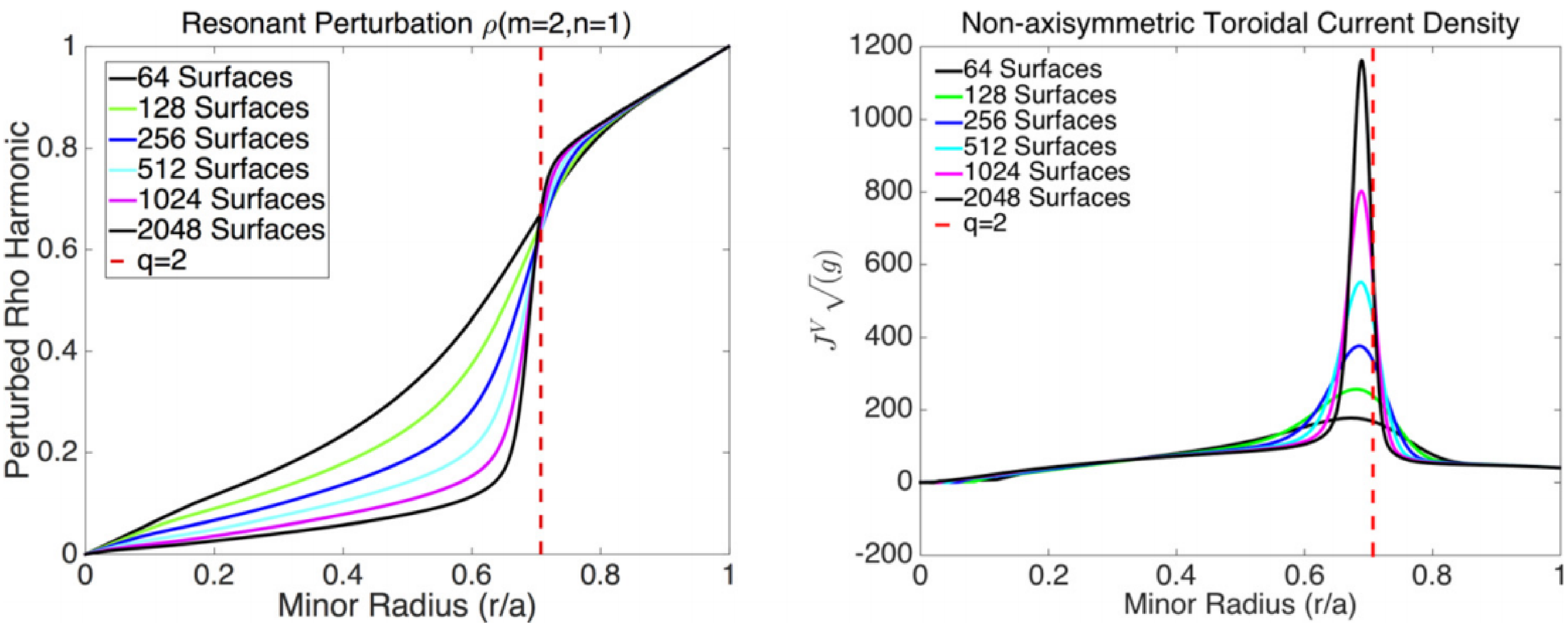


FIG. 2. Profile of the perturbed ρ harmonic (left) and the $m=2$ $n=1$ component of the toroidal current density (right) showing dependence on radial resolution at fixed shear. Boundary perturbation 1×10^{-4} of minor radius. The $q=2$ surface is located at $s=0.5$ ($r/a \sim 0.7$) in this plot. Note that the toroidal current density includes a Jacobian factor.

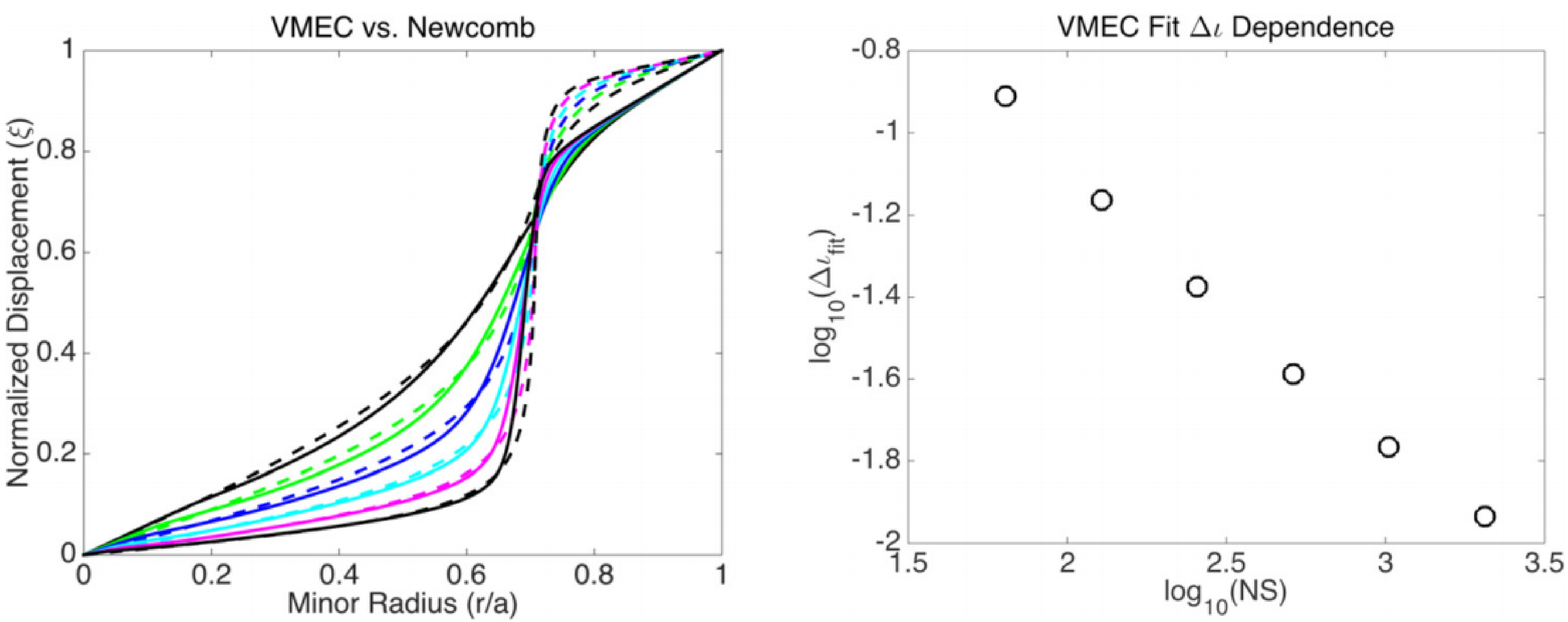


FIG. 5. Comparison of VMEC response (solid) to Loizu's solution to Newcomb's equation (dotted) (left) and the effective $\Delta\iota$ necessary to fit each curve (right). The colors are the same as those in Figure 2, and NS refers to the number of radial grid points.

Published SPEC convergence / verification calculations

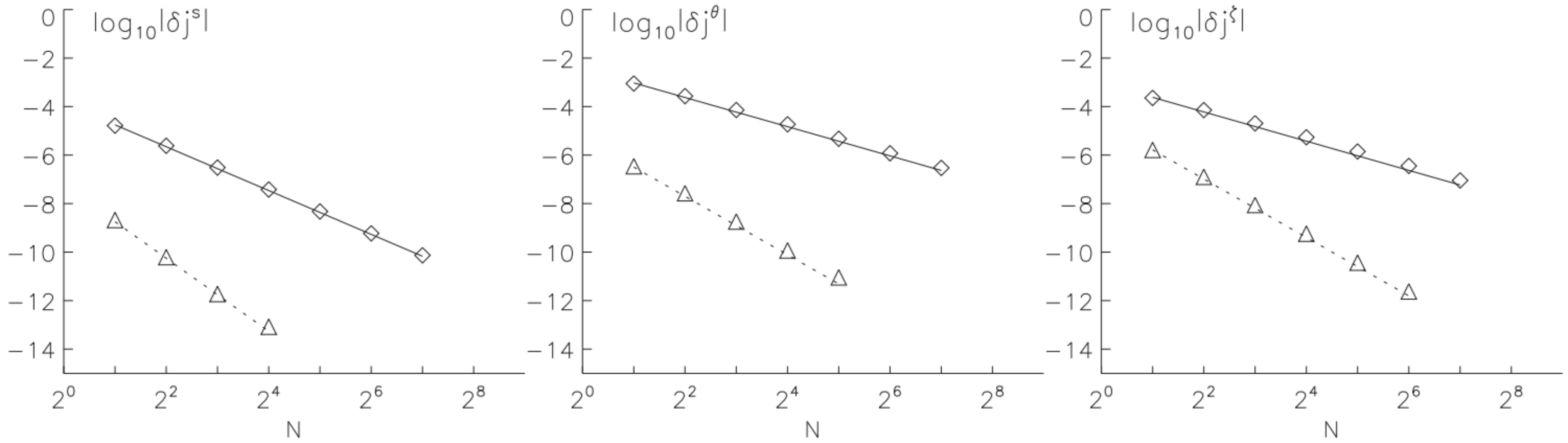


FIG. 2. Scaling of components of error, $\delta\mathbf{j} \equiv \mathbf{j} - \mu\mathbf{B}$, with respect to radial resolution. The diamonds are for the $n=3$ (cubic) basis functions, the triangles are for the $n=5$ (quintic) basis functions. The solid lines have gradient -3, -2, and -2, and the dotted lines have gradient -5, -4, and -4.

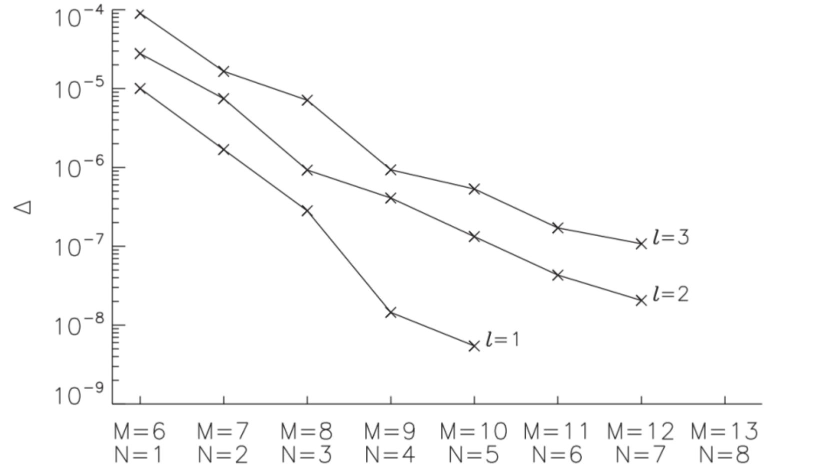


FIG. 6. Difference between finite M, N approximation to interface geometry, and a high-resolution reference approximation (with $M=13$ and $N=8$), plotted against Fourier resolution.

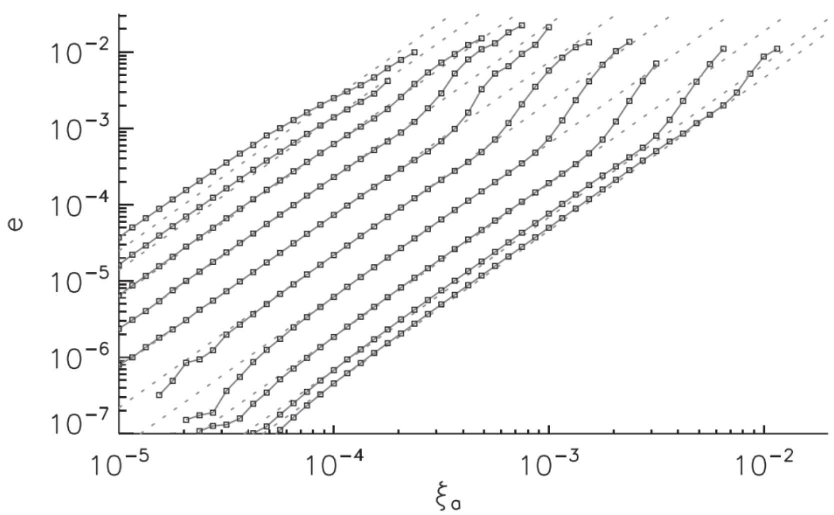


FIG. 2. Convergence of the error between linear and nonlinear SPEC equilibria as ξ_a is decreased, and for different values of $\Delta\text{ł}$, ranging from 10^{-4} (upper curve) to 10^{-1} (lower curve).

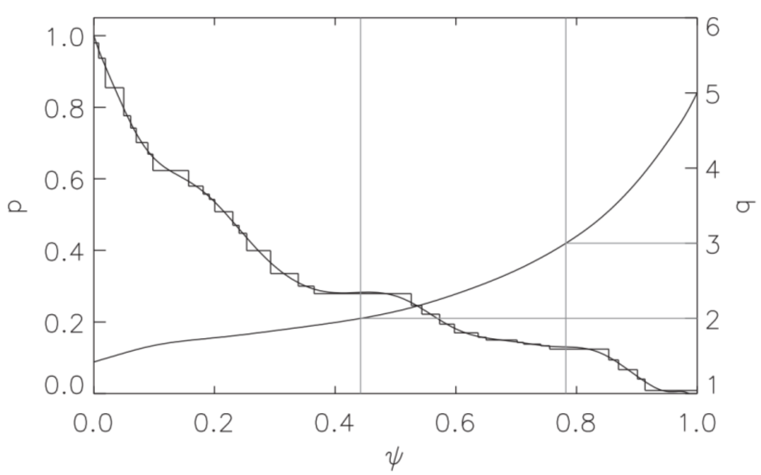


FIG. 7. Pressure profile (smooth) from a DIIID reconstruction using STELLOPT and stepped-pressure approximation. Also, shown is the inverse rotational transform ≡ safety factor.

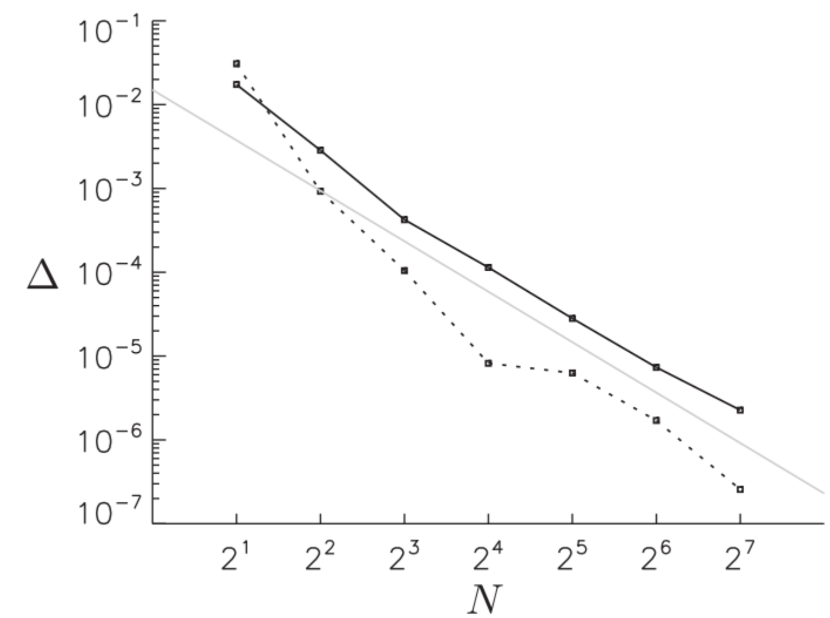


FIG. 5. Convergence: the error (Δ) between the continuous pressure (VMEC) and stepped pressure (SPEC) solutions are shown as a function of the number of plasma regions N for the $s=1/4$ SPEC interface. The dotted line shows the zero-beta case ($p_0=0$), and the solid line shows the high-beta case ($p_0=16$). The grey line has a slope -2, the expected rate of convergence. These simulations were run on a single 3 GHz Intel Xeon 5450 CPU with the longest (the $N=128$ case) taking 10.1 min using 20 poloidal Fourier harmonics and 768 fifth-order polynomial finite elements in the radial direction.

Early and recent publications

Hole, Hudson & Dewar,	PoP,	2006	*(theoretical model)*
Hudson, Hole & Dewar,	PoP,	2007	*(theoretical model)*
Dewar, Hole et al.,	Entropy,	2008	*(theoretical model)*
Hudson, Dewar et al.,	PoP,	2012	*(SPEC)*
Dennis, Hudson et al.,	PoP,	2013	*(MRxMHD $\rightarrow$ ideal as $N_R \rightarrow \infty$)*
Dennis, Hudson et al.,	PRL,	2013	*(helical states in RFP = double Taylor state)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow)*
Dennis, Hudson et al.,	PoP,	2014	*(MRxMHD+flow+pressure anisotrophy)*
Loizu, Hudson et al.,	PoP,	2015	*(first ever computation of 1/x & δ current-densities in ideal-MHD)*
Loizu, Hudson et al.,	PoP,	2015	*(well-defined, 3D MHD with discontinuous transform)*
Dewar, Yoshida et al.,	JPP,	2015	*(variational formulation of MRxMHD dynamics)*
Loizu, Hudson et al.,	PoP,	2016	*(pressure amplification of RMPs)*

Recent and upcoming invited talks

Hudson, Dewar, et al.,	2012	International Sherwood Fusion Theory Conference
Dennis, Hudson, et al.,	2013	International Sherwood Fusion Theory Conference
Dennis, Hudson, et. al.,	2013	International Stellarator Heliotron Workshop
Hole, Dewar, et al.,	2014	International Congress on Plasma Physics
Loizu, Hudson, et al.,	2015	International Sherwood Fusion Theory Conference
Loizu, Hudson, et al.,	2015	APS-DPP
Hudson, Loizu et al.,	2016	International Sherwood Fusion Theory Conference
Hudson, Loizu, et al.,	2016	Asia Pacific Plasma Theory Conference, 2016
Loizu, Hudson, et al.,	2016	Varenna Fusion Theory Conference

Compute the 1/x and δ-function current densities in perturbed geometry
Self-consistent solutions require **infinite shear**

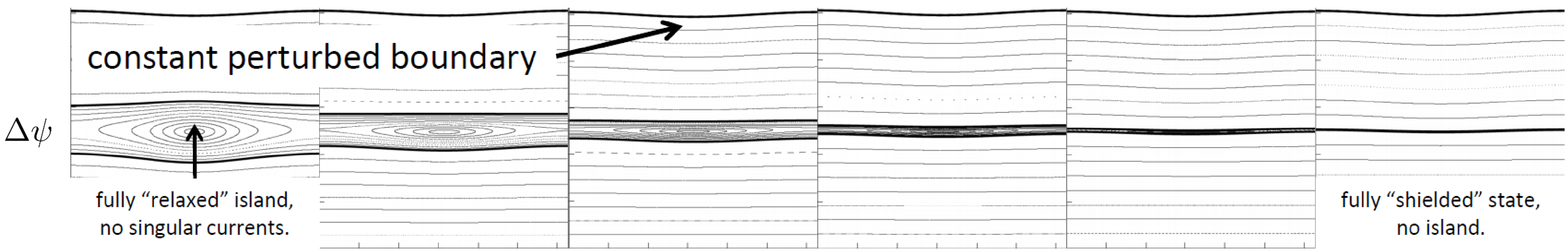
Cartesian, slab geometry with an $(m,n)=(1,0)$ resonantly-perturbed boundary

i. $N_V=3$ MRxMHD calculation, no pressure, $\iota(\psi)$ given discretely,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i=-x^\alpha/2$, $\iota_{i+1}=+x^\alpha/2$, shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\beta>\alpha}$.

iii. island forced to vanish,

iv. resonant $\delta_{m,n}$-function current-density appears as tangential discontinuity in $\mathbf{B}$.



i. $N_V=$ large MRxMHD calculation, stepped pressure $\approx$ smooth pressure,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i=-x^\alpha/2$, $\iota_{i+1}=+x^\alpha/2$,

iii. island forced to vanish,

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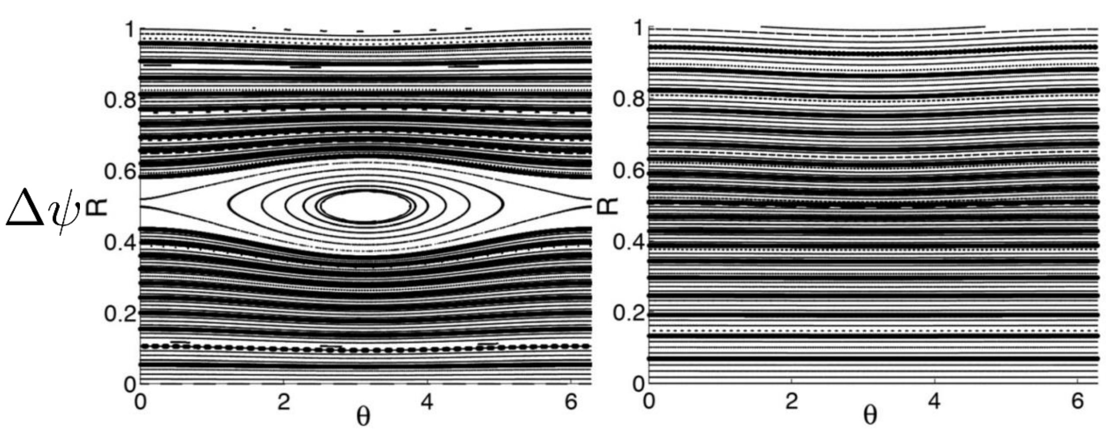
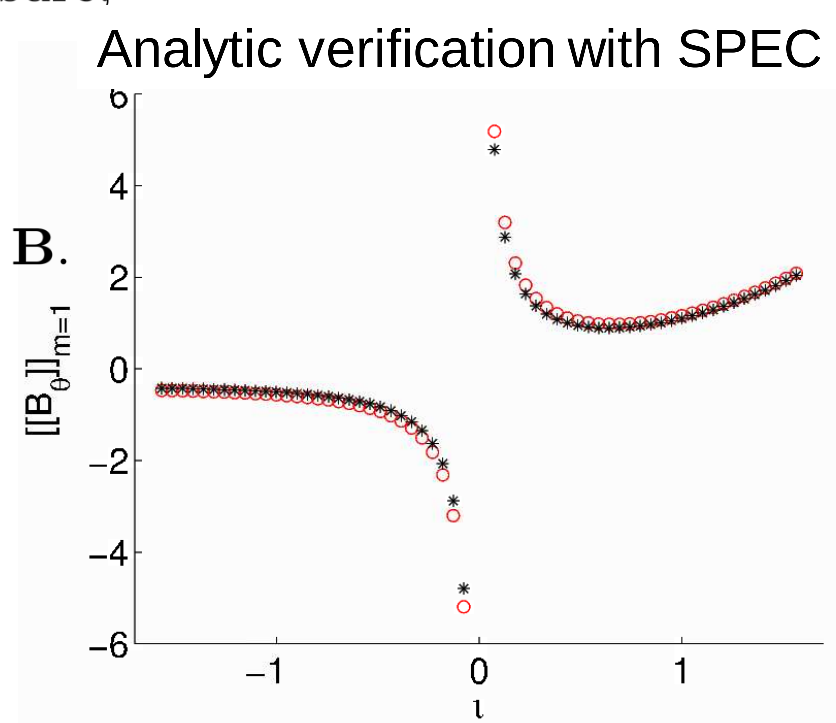


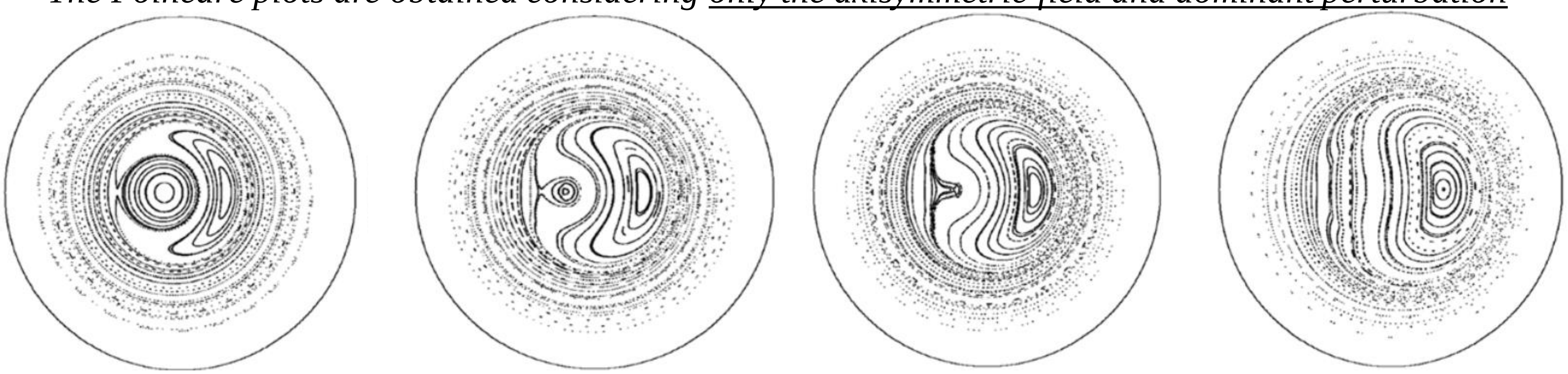


[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

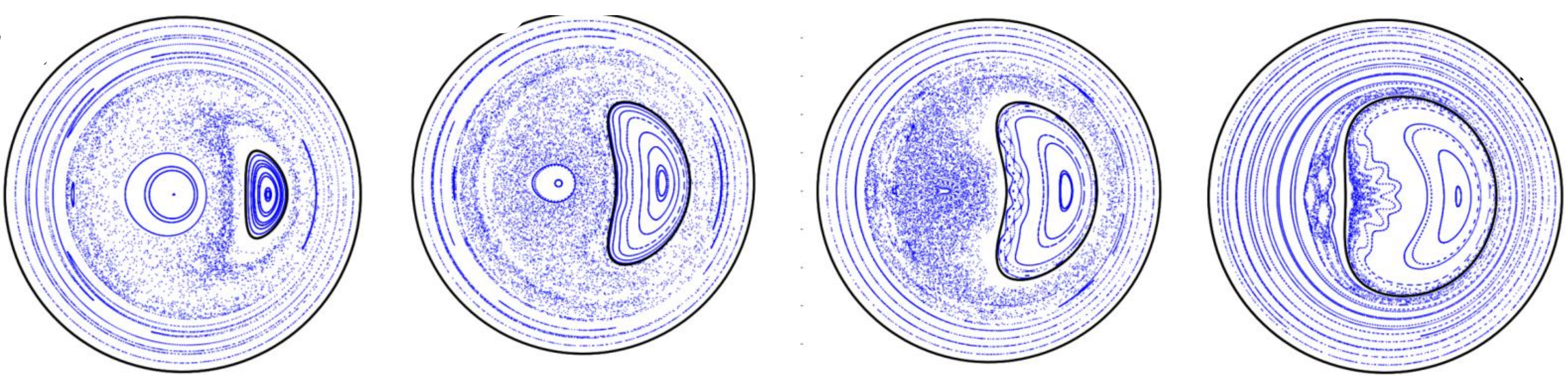
N_V=2 : “Double-Taylor” state with transport barrier; MRxMHD explains self-organization of RFP into helix

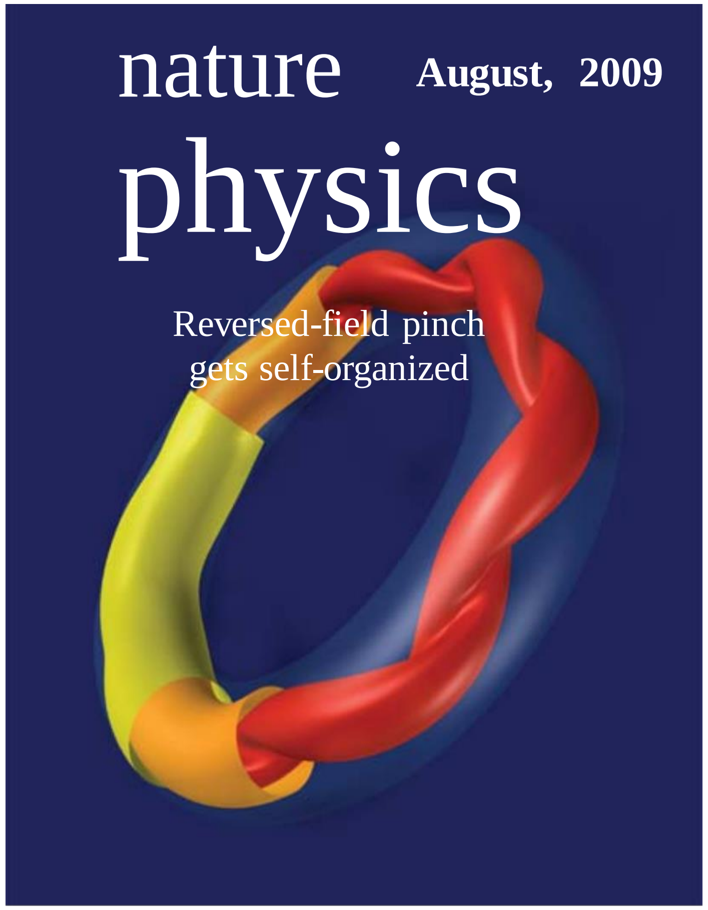
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nucl. Fusion*, ***49*** *104019 (2009)*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering only the axisymmetric field and dominant perturbation”



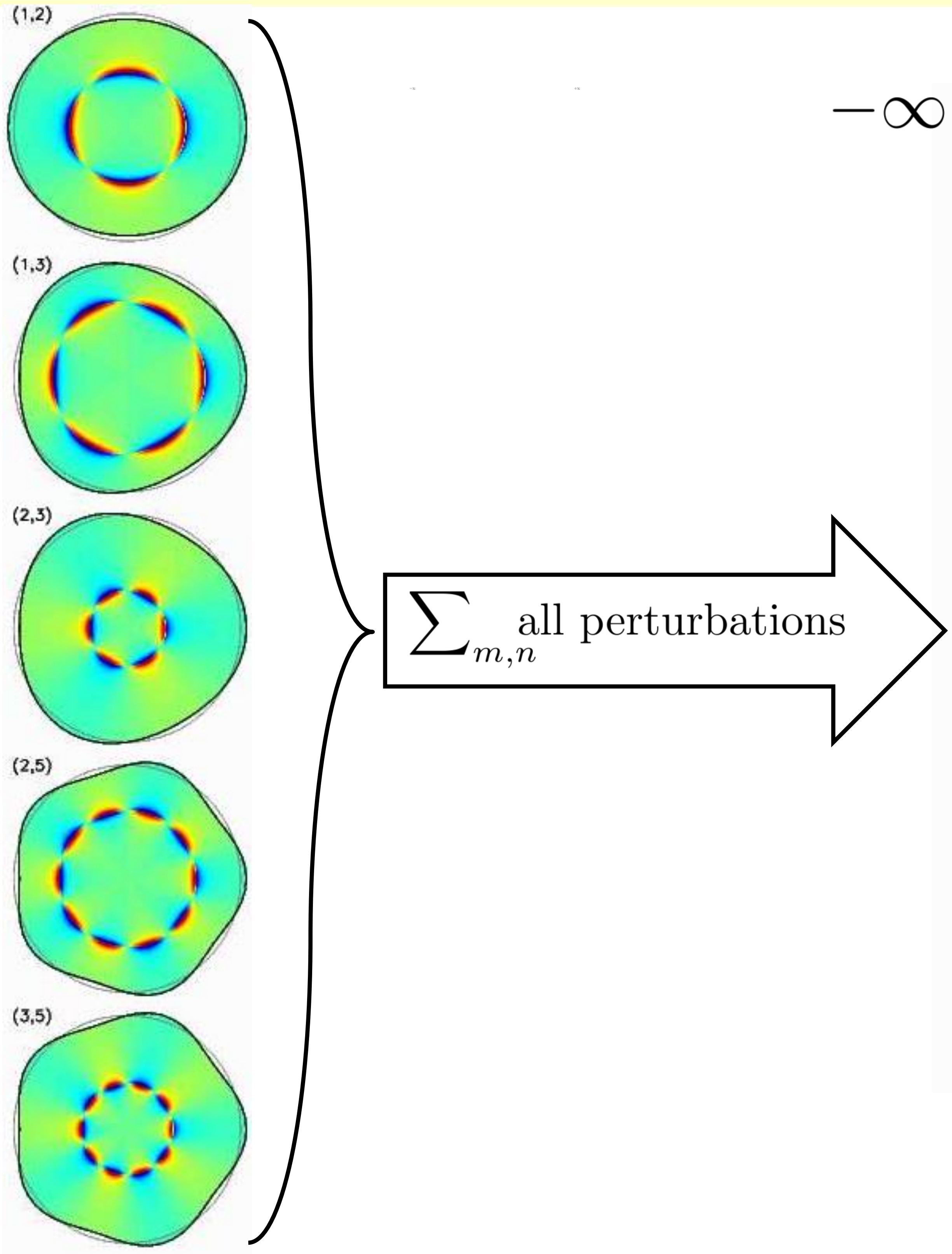
NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
Minimally Constrained Model of Self-Organized Helical States in Reversed-Field Pinches
G. Dennis, S. Hudson,et al. Phys. Rev. Lett. **111**, 055003 (2013)

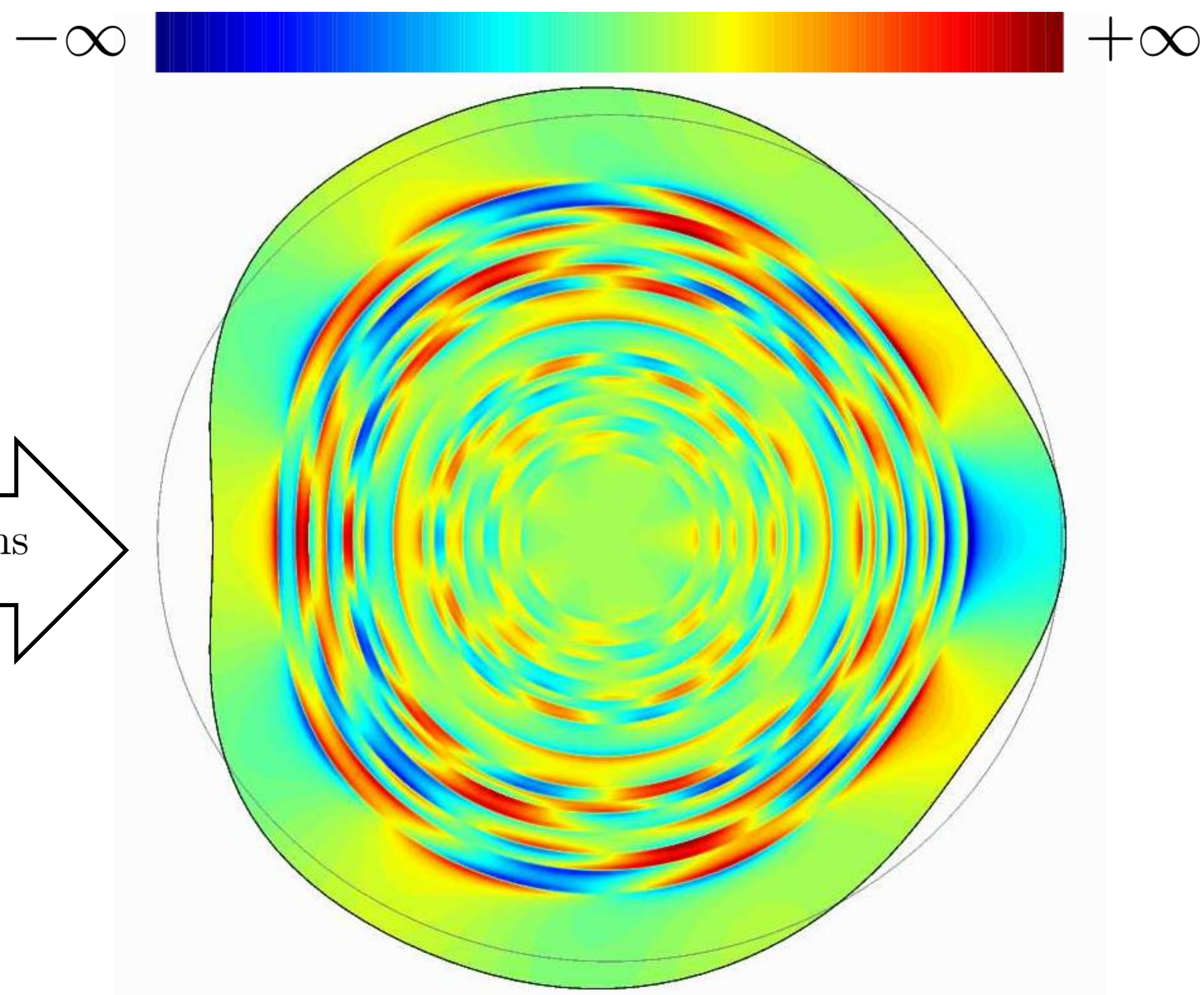




Excellent Qualitative agreement between numerical calculation and experiment
→ this is first (and perhaps only?) equilibrium model able to explain internal helical state with two magnetic axes

In arbitrary, three-dimensional geometry, "solutions" to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with smooth profiles and nested surfaces are nonsense.





Dense collection of alternating infinite currents is not an acceptable solution.