Penetration and amplification of resonant perturbations in 3D ideal-MHD equilibria *

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Abstract:

The nature of ideal MHD equilibria in three-dimensional geometry is profoundly affected by resonant surfaces, and this is particularly true for equilibria with non-zero pressure. If not treated carefully, non-physical currents arise in equilibria with continuously-nested magnetic surfaces. We demonstrate that three-dimensional, ideal-MHD equilibria, with nested surfaces and δ-function current-densities (i.e. sheet currents) that produce a *discontinuous* rotational-transform are well defined and tractable computationally. The results are of direct practical importance: we predict that resonant magnetic perturbations penetrate past the rational surface (i.e. "shielding" is incomplete, even in purely ideal-MHD) and that the perturbation is amplified by plasma pressure, increasingly so as stability limits are approached.

1 Importance of $\nabla p = \mathbf{j} \times \mathbf{B}$

The properties and numerical computation of three-dimensional (3D), ideal-MHD equilibria, as described by the seemingly simple, yet deceptively complicated, ideal force-balance equation $\nabla p = \mathbf{j} \times \mathbf{B}$ is of fundamental importance for understanding the behavior of both magnetically confined fusion and astrophysical plasmas [1, 2]. The instabilities that cause edge-localized modes (ELMs), an important concern for ITER [3], are widely believed to be ideal, peeling-ballooning modes [4]; and a 'hot-topic' of present research is to discover how these modes may be suppressed by applying resonant magnetic perturbations (RMPs), i.e. by 3D effects [5]. Whilst there exist extended-MHD codes [6] with non-ideal physics, these codes come with significant computational cost; and so the plasma response to 3D perturbations is routinely determined using codes such as IPEC [7], the ideal, perturbed equilibrium code.

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However, there are two fundamental difficulties that are frequently over-looked in calculations of ideal-MHD equilibria with nested flux-surfaces and smooth pressure and rotational-transform profiles. The first is the existence of *infinite* currents near resonant, rational rotational-transform surfaces, and the second is that ideal-MHD equilibria are *not* analytic functions of the 3D boundary.

Our new class of solutions for ideal-MHD equilibria avoids these problems. Most importantly, our solutions yield predictions that are in sharp contrast to previous predictions, and which may have direct implications for understanding the effect of RMPs in tokamaks. Even in ideal-MHD, a resonant perturbation penetrates past the rational surface and into the core of the plasma; and the perturbation is magnified by pressure inside the resonant surface, increasingly so as stability limits are approached.

The content and outline of this paper is as follows. As our new class of solutions and their properties have already been described in detail in earlier publications [8, 9], this paper shall review the salient features from an intuitive perspective and raise (and hopefully clarify) some of the problematic issues that remain. To appreciate the new class of solutions, it is required to understand the difficulties with the conventional solutions, so this paper first provides a review of the key problem of equilibria with nested flux surfaces and smooth pressure gradients, namely the formation of *unphysical* currents near rational surfaces. Then, we briefly review the problem of *non-analyticity* of ideal-MHD equilibria considered as functions of the three-dimensional boundary, and how this undermines perturbative approaches; and we provide an illustration in cylindrical geometry. Both of these problems are avoided in equilibria with sheet currents that create *discontinuous* rotational-transform, and the characteristics of RMP penetration in such equilibria are described.

2 Unphysical, infinite currents

The infinite, unphysical currents arise from enforcing charge conservation, $\nabla \cdot \mathbf{j} = 0$, where $\mathbf{j} = \nabla \times \mathbf{B}$. It is convenient to write the current-density as $\mathbf{j} \equiv u\mathbf{B} + \mathbf{j}_\perp$, and by crossing $\nabla p = \mathbf{j} \times \mathbf{B}$ with $\mathbf{B}$ we derive an expression for the perpendicular current-density, $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. Wherever there are pressure-driven, perpendicular current-densities, there must also be parallel current-densities that satisfy $\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$. Such equations, called magnetic differential equations [10], take a simple form in straight-field line coordinates, (ψ, θ, ζ), for which $\mathbf{B} = \nabla\psi \times \nabla\theta + \iota\!\!\text{-}(\psi)\nabla\zeta \times \nabla\psi$, where $2\pi\psi$ is the toroidal flux enclosed by a flux surface, θ and ζ are straight fieldline poloidal and toroidal angles, and $\iota\!\!\text{-}(\psi)$ is the rotational-transform. (Note that $\mathbf{B} \cdot \nabla\zeta = \sqrt{g}^{-1}$, where $\sqrt{g}$ is the coordinate Jacobian.) This is a linear equation, and may be solved in Fourier space by the representation $u = \sum u_{m,n} \exp(im\theta - in\zeta)$. The directional-derivative $\sqrt{g}\,\mathbf{B} \cdot \nabla = \iota\!\!\text{-}\,\partial_\theta + \partial_\zeta$, reduces to $\sqrt{g}\,\mathbf{B} \cdot \nabla = i(\iota\!\!\text{-}\, m - n)$. The solution for each Fourier harmonic of the parallel current-density is

$$u_{m,n} = \frac{i(\sqrt{g}\,\nabla \cdot \mathbf{j}_\perp)_{m,n}}{x} + \Delta_{m,n}\,\delta(x), \tag{1}$$

where $x \equiv \iota\!\!\!{}^{-} m - n$ is the "distance" in rotational-transform from the rational surface, and $\Delta_{m,n}$ is an as-yet undetermined constant. Two singularities are present: a pressure-driven, so-called "Pfirsch-Schlüter", $1/x$ current-density that arises *around* rational surfaces, and a δ-function current-density that develops *at* rational surfaces.

Singularities in the current-*density* are acceptable. As a simple example, consider a finite, non-zero current passing along a thin wire: if the cross-sectional area of the wire is very thin, then representing the current-density as a δ-function is an excellent mathematical approximation to the physical reality. In the limit that the the conductivity becomes infinite, a wire of zero cross-sectional area can support a finite current, and the δ-function approximation becomes exact. The existence of δ-function current-densities, which are also sometimes called "sheet" currents or "surface" currents, is perfectly compatible with the physical model of ideal-MHD.

Singularities in the *current*, however, are not acceptable: the total current, $\int \mathbf{j} \cdot d\mathbf{s}$, passing through *each* and *every* surface must remain finite for any physically acceptable equilibrium. The current resulting from the pressure-driven Pfirsch-Schlüter density passing through the cross-sectional area enclosed by the flux-surfaces defined by $x = \epsilon$ and $x = \delta$ between $\theta = 0$ and $\theta = \pi/m$ is proportional to $\int_\epsilon^\delta 1/x\, dx$, which logarithmically approaches infinity as ϵ approaches zero.

This is not physical. Furthermore, the presence of an equal-and-opposite infinite current, in a sense, through a similar cross-sectional surface on the opposing side of the rational surface (i.e., the surface bounded by $x = -\epsilon$ and $x = -\delta$) should not be thought of as a panacea. It would seem that the ideal-MHD equilibrium model, with nested flux-surfaces, has the fatal flaw of not allowing for pressure gradients in a small-but-*finite* neighborhood of each rational surface! Considering that rational surfaces are dense in any continuous magnetic field with shear, and therefore the regions in which the pressure gradient must be zero will overlap, this means that there can be no pressure at all in equilibria with nested flux surfaces and smooth pressure (unless the rotational-transform is everywhere irrational, as we discuss in more detail below).

There are other equilibrium models that should be mentioned, but which will not be discussed at length in this article. If the pressure gradient is allowed to be *discontinuous*, a pressure profile with finite gradients where the rotational-transform is sufficiently *irrational* will provide a non-trivial, finite-pressure equilibriuum. For example: choose $p'(\psi) = 1$ where the rotational-transform satisfies a Diophantine condition, e.g. $|\iota\!\!\!{}^{-}(\psi) - n/m| > k/m^2$ for all rationals n/m and some $k > 0$, and $p'(\psi) = 0$ elsewhere. Such a profile would avoid the pressure-driven infinite-currents near the rational surfaces and would yield a non-trivial pressure profile (by virtue of the fact that a finite measure of irrationals satisfy the Diophantine condition). In this case, however, the equilibrium must display fractal properties, described by Grad as "pathological" [1], and would not be amenable to standard numerical discretization (it is almost impossible to constrain the topology of a non-integrable field to be consistent with a given, *fractal* pressure). Other possibilities include equilibria in which the pressure is discontinuous, as is assumed in the Stepped Pressure Equilibrium Code (SPEC) [11]; and equilibria for which the boundary is appropriately constrained to ensure that the resonant harmonic of the geometry, which is related to the numerator in Eq.(1), vanishes in the vicinity of the rational surfaces

[12, 13, 14].

3 Non-analyticity

The second problem, the non-analyticity of ideal-MHD equilibria, is encountered when computing the plasma displacement, $\boldsymbol{\xi} = \xi^\psi \mathbf{e}_\psi + \xi^\theta \mathbf{e}_\theta + \xi^\zeta \mathbf{e}_\zeta$, induced by an $\mathcal{O}(\epsilon)$ perturbation to the boundary of an equilibrium state. Assuming an expansion

$$\boldsymbol{\xi} \equiv \epsilon\, \boldsymbol{\xi}_1 + \epsilon^2\, \boldsymbol{\xi}_2 + \ldots, \tag{2}$$

the first-order term satisfies $L_0[\boldsymbol{\xi}_1] = 0$, where

$$L_0[\boldsymbol{\xi}] \equiv \nabla \delta p - \delta \mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta \mathbf{B}, \tag{3}$$

where the ideal variations in the magnetic field and pressure are $\delta \mathbf{B}[\boldsymbol{\xi}] \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ and $\delta p[\boldsymbol{\xi}] = (\gamma - 1)\, \boldsymbol{\xi} \cdot \nabla p - \gamma \nabla \cdot p \boldsymbol{\xi}$.

The linear operator, $L_0[\boldsymbol{\xi}]$, is singular. To simultaneously match a non-trivial condition at the boundary and the condition that $\delta \mathbf{B} \cdot \nabla \psi = 0$ at the rational surface (so that a magnetic island does not form), the solution for $\boldsymbol{\xi}_1$ must, generally, be discontinuous [8].

A discontinuous plasma displacement is, however, inconsistent with the assumption of nested flux-surfaces: in fact, magnetic surfaces overlap if the displacement anywhere violates $|d\xi/d\psi| < 1$. The second order term satisfies $L_0[\boldsymbol{\xi}_2] = -\delta \mathbf{j}[\boldsymbol{\xi}_1] \times \delta \mathbf{B}[\boldsymbol{\xi}_1]$, and $\boldsymbol{\xi}_2$ is even more singular than $\boldsymbol{\xi}_1$. That perturbation theory is not a valid approach for computing ideal equilibria with nested surfaces was known by Rosenbluth *et al.* [15], who wrote "we must abandon the perturbation theory approach" when computing ideal-MHD equilibria in 3D. These problems are partly associated with what might be considered fundamental flaws in ideal-MHD [16], namely that ideal-MHD dynamical evolutions do not allow the topology of the magnetic field to "tear".

Despite such flaws in ideal-MHD, the equation of force balance, $\nabla p = \mathbf{j} \times \mathbf{B}$, which this paper is primarily concerned with, remains an excellent description of the macroscopic forces that determine plasma equilibrium properties. Considering first the presumably simpler task, that of only seeking *static* solutions, and ignoring the more-complicated questions regarding ideal *dynamical* evolution, the main obstruction to computing mathematically self-consistent solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ with nested flux surfaces and smooth pressure profiles was that well-defined solutions had not, until quite recently, been discovered. There is, surprisingly, a class of ideal-MHD equilibria with nested magnetic surfaces and arbitrary pressure profiles that eliminates both the infinite currents and the non-analyticity.

In recent work [17] we, for the first time, computed the $1/x$ and δ-function singular current-densities in 3D equilibria; and we realized that self-consistent solutions demand locally-infinite shear at the resonant surfaces. We then [8] introduced a new class of solution that admits additional δ-function current-densities that produce finite sheet currents, with a commensurate *discontinuity* in the rotational-transform. Removing the rational surfaces removes both the non-physical currents and the non-analyticity.

3.1 Illustration in cylindrical geometry

For illustration and verification, we construct such an equilibrium in cylindrical geometry, with "major" radius R and minor radius a, and calculate the linear, ideal response to a resonant boundary perturbation. (Note, as mentioned above, we do not explicitly follow the ideal dynamical response, but instead simply solve $\nabla p = \mathbf{j} \times \mathbf{B}$ with the perturbed boundary, i.e. we seek the static solution.) The cylindrically symmetric solution to $\nabla p = \mathbf{j} \times \mathbf{B}$ satisfies

$$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z^2\left(1 + ι^2\frac{r^2}{R^2}\right)\right] + \frac{r\,ι^2 B_z^2}{R^2} = 0, \tag{4}$$

where $ι \equiv RB_\theta/rB_z$, and is uniquely determined by the value of the axial field at the origin, e.g. $B_z(0) = 1$, by an arbitrary pressure profile, $p(r)$, and a rotational-transform profile, $ι(r)$. This equation may be cast as an ordinary differential equation for B_z; and can be integrated radially (even for discontinuous rotational-transform profiles) from $r = 0$ to $r = a$,

In cylindrical geometry there is no coupling, to lowest order, between perturbations of different helicities; and we may for simplicity of illustration imagine that there is a single resonant surface of concern. (In a fully three-dimensional configuration, all rational surfaces within the equilibrium will potentially create problems.) We therefore choose a rotational-transform profile

$$ι(r) = \begin{cases} ι_0 - ι_1(r/a)^2 + \Delta ι/2, & \text{for } r < r_s, \\ ι_0 - ι_1(r/a)^2 - \Delta ι/2, & \text{for } r > r_s, \end{cases} \tag{5}$$

with $ι_0$ and $ι_1$ chosen so that $ι(r)$ jumps across the rational $ι_s \equiv n/m$, namely $ι(r_s) = ι_s \pm \Delta ι/2$. The pressure can be an arbitrary smooth function, e.g.

$$p(r) \;=\; p_0[1 - 2(r/a)^2 + (r/a)^4], \tag{6}$$

and for later reference we define $\beta \equiv 2p(0)/B_z^2(0)$.

In cylindrical geometry, the linearized equation, $L_0[\boldsymbol{\xi}] = 0$, reduces to Newcomb's equation [18],

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0, \tag{7}$$

where $\xi^r \equiv \xi(r)\cos(m\theta - n\zeta)$, and f and g are determined by the equilibrium [8]. For $\Delta ι = 0$, Newcomb's equation is singular where $ι(r_s) = n/m$, because there is a order-two zero in $f(r)$ and an order-one zero in $g(r)$. For $\Delta ι \neq 0$, the singularity is removed; and the equilibrium equations comprise an analytic function of the 3D boundary and the perturbation expansion is valid.

Fig.1 shows the result of numerical integration of Eq. (7), with different values of p_0 and for $\Delta ι = 10^{-3}$. The linear radial displacement is continuous and smooth provided $\Delta ι \neq 0$, and the condition $|d\xi/d\psi| < 1$ is satisfied if $\Delta ι > \Delta ι_{min}$, which can be estimated theoretically [8]. Even for a small, local change in the transform profile the global solution

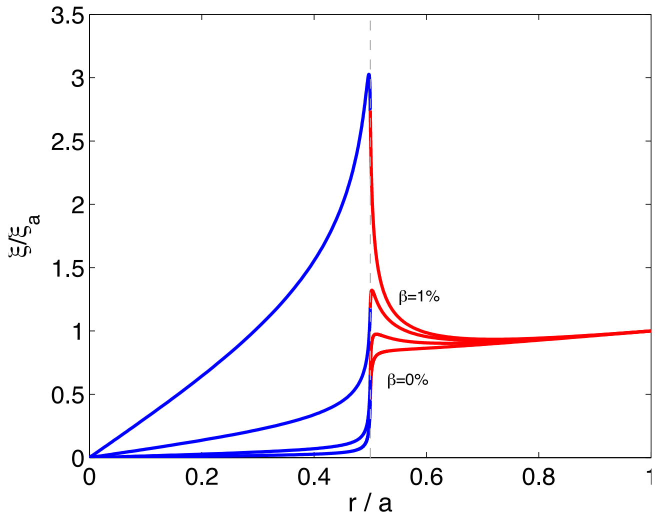

Fig.1. Solutions for the perturbed displacement for $\beta = 0\%$ (lower curve) to $\beta = 1\%$ (upper curve); and for $\Delta\iota\!\!\!\iota = 10^{-3}$. Reproduced from [9].

is significantly different and the displacement penetrates inside the resonant surface and into the origin. In fact, for non-zero pressure, the perturbation to the displacement inside the resonant surface is magnified, extremely so as ideal-stability limits are approached. To quantify the amplification and penetration of the perturbation, we define

$$A_{rmp} \equiv \xi_s/\xi_a \ , \qquad P_{rmp} \equiv 1 - r_*/r_s \ , \tag{8}$$

where r_* is the radius at which $\xi(R_*) = \xi_s/e$. In Fig.2 we show these quantities as a function of β. These figures show that as the equilibrium approaches the Suydam limit for interchange stability,

$$D_S = \left(\frac{2p' \iota\!\!\!\iota^2}{r B_z^2 \iota\!\!\!\iota'^2} \right)_s < \frac{1}{4}, \tag{9}$$

there is a fantastic increase in both the amplification and penetration of the resonant magnetic perturbation.

3.2 Verification of a nonlinear equilibrium code

Removing the singularity associated with the rational surface (achieved by allowing for a sheet current that results in a discontinuous rotational-transform) means that ideal-MHD equilibria become analytic functions of the three-dimensional boundary, and this in turn allows nonlinear 3D ideal-MHD equilibrium codes to be verified against the linear codes. Presently, the widely-used, 3D, nonlinear ideal-MHD equilibrium codes VMEC [19] and NSTAB [20] are restricted to work with smooth functions and cannot, formally, compute equilibria with discontinuous rotational-transform (though, finite radial resolution seems to imply an 'effective' discontinuity [21]). The SPEC [11] code *does* allow for

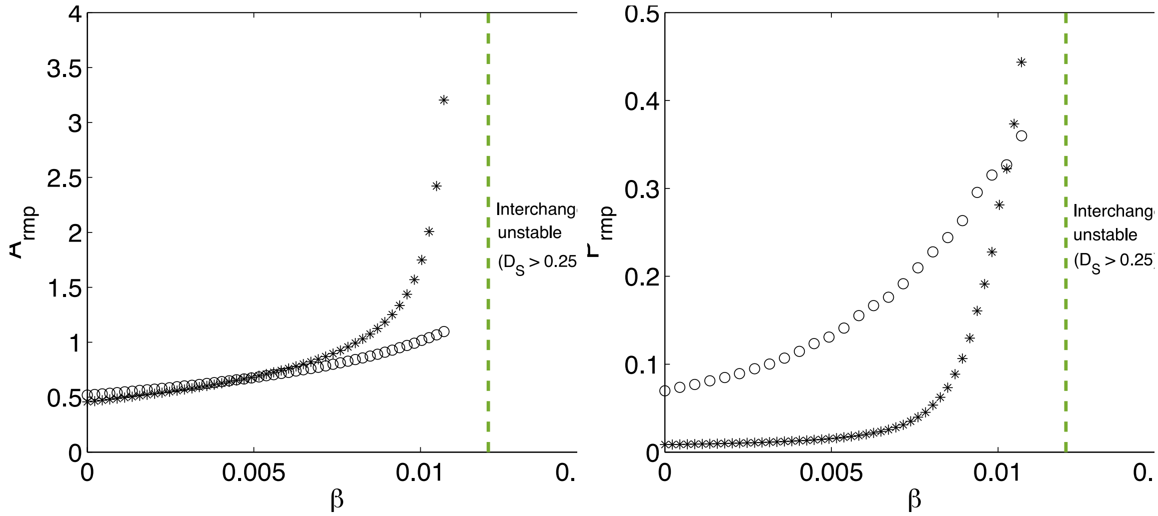

Fig.2. Amplification (left) and penetration (right) of the perturbation on the resonant surface as a function of β, for $\Delta\iota = 0.005$ (circles) and $\Delta\iota = 0.001$ (stars). The vertical dashed line indicates where the Suydam condition is not satisfied. Reproduced from [9].

discontinuities. In Fig.3, we compare the linear plasma displacement computed using a "linearized" version of SPEC to the analytic solution, with excellent agreement. Futhermore, fully *nonlinear* SPEC solutions were compared to the linear solutions [8], obtaining the as-expected agreement.

We make two final comments. Our work has introduced a new class of solution to $\nabla p = \mathbf{j} \times \mathbf{B}$ with discontinuous transform. In this paper, we have considered a single rational surface. To ensure the removal of all non-physical currents in generally perturbed configurations, i.e. in arbitrary, three-dimensional geometry, we may take a *stepped* rotational-transform profile that is piecewise irrational. Second, in this paper we have not addressed the ideal dynamical evolution from a state with continuous transform to a state with discontinuous transfrom; this is a matter of ongoing investigation [16].

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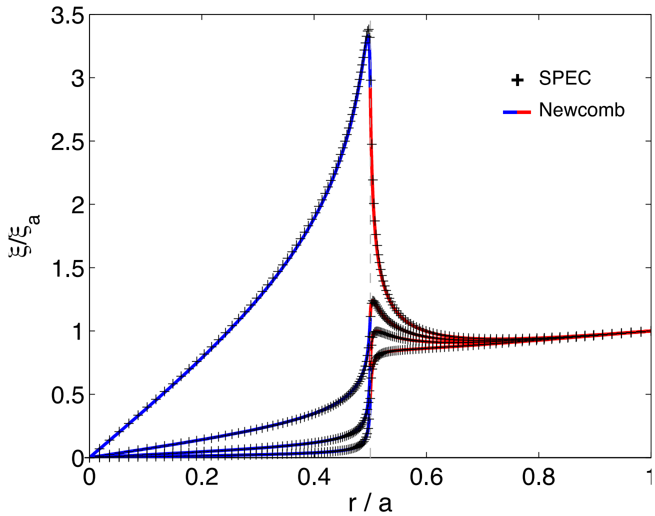

Fig.3. SPEC linear solutions (crosses) and Newcomb solutions (solid lines) for $\beta = 0\%$ (lower curve) to $\beta = 1.1\%$ (upper curve), and for $\Delta ɨ = 0.0014$. Reproduced from [9].

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