Penetration and Amplification of Resonant Perturbations in 3D Ideal-MHD Equilibria

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Importance of ideal-MHD. Three-dimensional (3D), ideal-MHD equilibria, as described by $\nabla p = \mathbf{j} \times \mathbf{B}$, are fundamental for understanding the behavior of magnetically-confined plasmas. Edge-localized modes (ELMs), an important concern for ITER, are widely believed to be ideal, peeling-ballooning modes; and a 'hot-topic' of current research is to discover how ELMs may be suppressed by resonant magnetic perturbations (RMPs). The plasma response to 3D perturbations is routinely determined perturbatively using codes such as IPEC, the Ideal, Perturbed Equilibrium Code [*Phys. Plasmas*, 14:052110, 2007]. However, there are two fundamental difficulties that are frequently over-looked: the existence of *infinite* currents near resonant surfaces, and that ideal-MHD equilibria are *not* analytic functions of the 3D boundary.

Unphysical, pressure-driven currents. The infinite currents arise from charge conservation, $\nabla \cdot \mathbf{j} = 0$, in equilibria with smooth profiles. Wherever there are pressure-driven, perpendicular current-densities, $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$, there must also be parallel current-densities that satisfy $\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$, where $\mathbf{j} \equiv u\,\mathbf{B} + \mathbf{j}_\perp$. The solution for each Fourier harmonic in straight-field line coordinates is $u_{m,n} = i(\sqrt{g}\,\nabla \cdot \mathbf{j}_\perp)_{m,n}/x + \Delta_{m,n}\,\delta(x)$, where $x \equiv \iota\!\!\!- - n/m$. MHD is a *macroscopic* model of plasma dynamics with no intrinsic length scale; and the δ-function density is just a mathematical representation of localized currents. Singularities in the current-*density* are allowed, but the total current, $\int \mathbf{j} \cdot d\mathbf{s}$, passing through *each* and *every* surface must be finite for a physically-acceptable equilibrium. The *net* current resulting from the resonant δ-function density between adjacent flux-surfaces, e.g. $x = -\epsilon$ and $x = +\epsilon$ as $\epsilon \to 0$, actually integrates to zero; but the so-called Pfirsch-Schlüter current resulting from the resonant $1/x$ current-density passing through $x = \epsilon$ and $x = \delta$, where $0 < \epsilon < \delta$, and $\theta = 0$ and $\theta = \pi/m$ is proportional to $\int_\epsilon^\delta 1/x\,dx$, which approaches infinity as $\epsilon \to 0$. This is not physical: the ideal-MHD equilibrium model, with nested flux-surfaces, cannot admit pressure gradients in a small neighborhood of each rational surface. Because the rational surfaces are dense, this means that there can be no pressure at all, if the pressure is smooth; or the pressure must be fractal [Grad, *Phys. Fluids*, 10:137, 1967] and not amenable to standard numerical discretization; or the pressure must be discontinuous, as is assumed in the Stepped Pressure Equilibrium Code (SPEC) [*Phys. Plasmas*, 19:112502, 2012].

Breakdown of perturbation theory. The non-analyticity is encountered when computing the 'linear' plasma displacement, $\boldsymbol{\xi}$, given by $L_0[\boldsymbol{\xi}] \equiv \nabla \delta p - \delta \mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta \mathbf{B} = 0$. The operator, $L_0[\boldsymbol{\xi}]$, is singular. To match a finite displacement at the boundary, and to ensure that magnetic islands do not form, the solution for $\boldsymbol{\xi}$ is discontinuous at each resonant surface. This, however, is inconsistent with nested flux-surfaces. The breakdown of perturbation theory was known to Rosenbluth *et al.* [*Phys. Fluids*, 16:1894, 1973], who wrote "we must abandon the perturbation theory approach" when computing ideal-MHD equilibria in 3D.

A new class of self-consistent solutions. These difficulties are not fundamental flaws in ideal-MHD, which remains perhaps the most successful, relevant yet simplest

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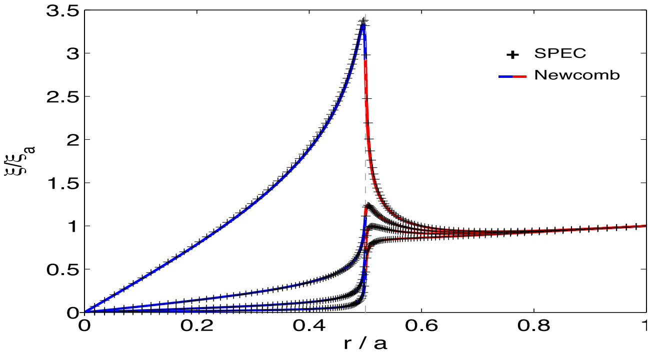

Figure 1: Comparison of the linear solution and the SPEC solution for a perturbed cylindrical equilibrium, with $\Delta\iota\!\!\!- > \Delta\iota\!\!\!-_{min}$, and for increasing pressure. The RMP penetrates past the resonant surface at $r = a/2$, and is spectacularly amplified by plasma pressure. The nonlinear SPEC solution converges to the linear solution with error $\sim \mathcal{O}(\epsilon^2)$, as it should when the equations and solution are analytic.

model of plasma dynamics. It is just that, until recently, self-consistent solutions to the ideal-MHD equilibrium equation for arbitrary 3D geometry had not been discovered. There is, surprisingly, a class of solutions that eliminates both the infinite currents and the non-analyticity, even for smooth pressure profiles. Recently, Loizu *et al.* [*Phys. Plasmas*, 22:022501, 2015], for the first time, computed the $1/x$ and δ-function current-densities in 3D equilibria. Self-consistent solutions demand locally-infinite shear at the resonant surfaces. We then introduced [*Phys. Plasmas*, 22:090704, 2015] a new class of solutions that admit additional δ-function current-densities that *do* produce finite net currents between adjacent flux-surfaces, with a commensurate *discontinuity* in the rotational-transform that removes the singularities. *Most importantly,* we will present new predictions that are in sharp contrast to previous understanding, with direct implications for the penetration of RMPs in tokamaks: in ideal-MHD, a resonant perturbation penetrates past the rational surface and into the core of the plasma; and the perturbation is magnified by pressure inside the resonant surface, increasingly so as stability limits are approached!

Verification with analytic solution. For illustration and verification, we consider the linear and nonlinear, ideal plasma response to an RMP in cylindrical geometry. The equilibrium is defined by an arbitrary, smooth pressure profile, $p(r)$, and a rotational-transform profile, $\iota\!\!\!-(r) = \iota\!\!\!-_0(r) + \Delta\iota\!\!\!-/2$ for $r < r_s$ and $\iota\!\!\!-(r) = \iota\!\!\!-_0(r) - \Delta\iota\!\!\!-/2$ for $r > r_s$, with $\iota\!\!\!-_0(r)$ chosen so that $\iota\!\!\!-(r)$ jumps across the rational $\iota\!\!\!-_s \equiv n/m$, namely $\iota\!\!\!-(r_s) = \iota\!\!\!-_s \pm \Delta\iota\!\!\!-/2$. The linearized equation, $L_0[\boldsymbol{\xi}] = 0$, reduces to Newcomb's equation, $d(f\, d\xi/dr)/dr - g\,\xi = 0$, where ξ is the radial component of the resonant plasma displacement, and $f(r)$ and $g(r)$ are determined by the equilibrium. For $\Delta\iota\!\!\!- = 0$, Newcomb's equation is singular where $\iota\!\!\!-(r_s) = n/m$. For $\Delta\iota\!\!\!- > 0$ the singularity is removed; and for $\Delta\iota\!\!\!- > \Delta\iota\!\!\!-_{min}$ (the minimum required to ensure that the perturbed flux-surfaces do not overlap) the perturbation expansion accurately approximates the true nonlinear solution. The value of $\Delta\iota\!\!\!-_{min}$ may be estimated analytically. For continuous transform, the linear solution demands complete 'shielding' (and non-physical overlapping flux surfaces). For $\Delta\iota\!\!\!- > 0$, the self-consistent displacement penetrates into the origin, as shown in Fig.1.

Benchmarking linear & nonlinear codes. Our solution has resolved a confusion in efforts [*Nucl. Fus.*, 55:063026, 2015] to benchmark the linearly-perturbed solutions with the nonlinear solutions provided by the 3D equilibrium codes VMEC and NSTAB, both of which enforce nested flux-surfaces; whereas, for $\Delta\iota\!\!\!- = 0$, the 'linear' solution gives overlapping flux-surfaces. Of course these solutions will disagree near the rational surfaces! For $\Delta\iota\!\!\!- > \Delta\iota\!\!\!-_{min}$, the perturbation expansion converges: the linear and nonlinear codes should agree. However, VMEC and NSTAB are restricted to work with smooth profiles and cannot formally compute equilibria with discontinuous rotational-transform. SPEC *does* allow for discontinuities, and Fig.1 shows excellent agreement between SPEC and the linear approximation.