Australian National University

PPPL
PRINCETON PLASMA PHYSICS LABORATORY

Spectrum of multi-region-relaxed magneto-hydrodynamic modes in slab geometry

or

Putting the D in MRxMHD

a prescription for all that ails ideal MHD!

Robert L. Dewar,* L.H. (Alexis) Tuen, Matthew J. Hole

Plasma Theory & Modelling,

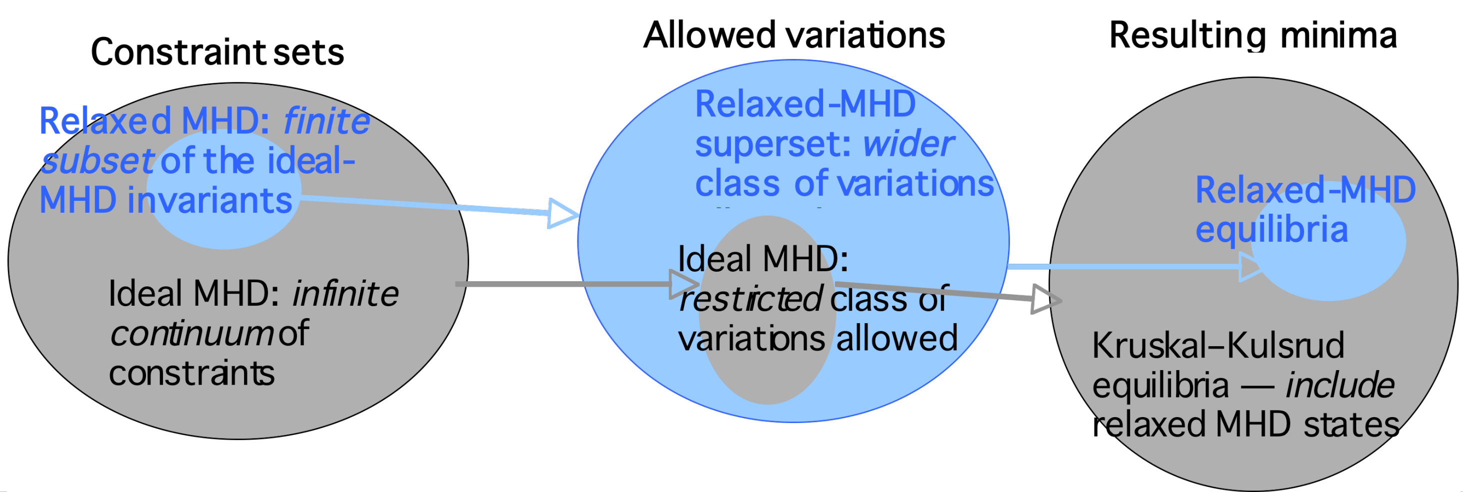
Centre for Plasmas and Fluids,

Australian National University

**Visiting Princeton Plasma Physics Laboratory*

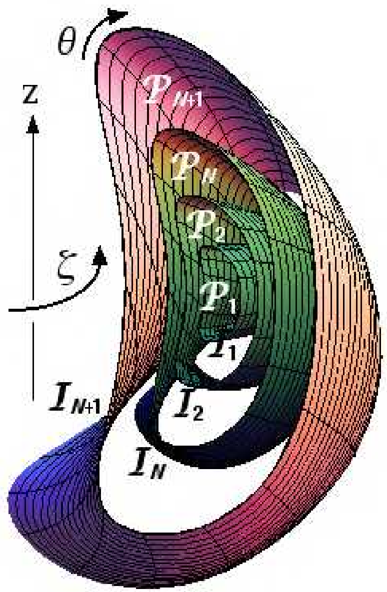
If it ain't broke, why fix it?

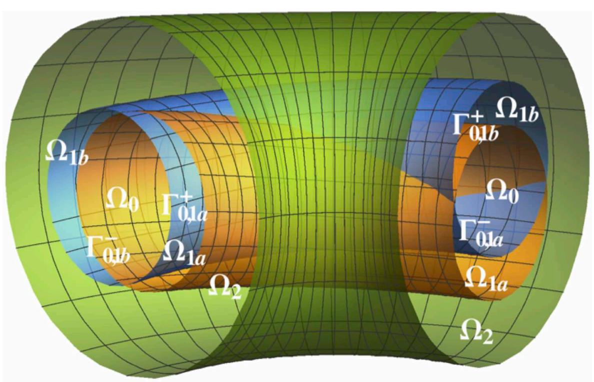
- Ideal MHD is overconstrained
  - No heat transport along field lines
  - No reconnection so islands or chaos cannot form
  - Thus *inapplicable* to hot and 3D plasmas!
- Fix by removing the bad constraints and keeping the good, doing more with less!



MRxMHD: M stands for **Multi-region** (aka waterbag)

Rx stands for **Relaxed**; ..D stands for **Dynamics**





Fundamental postulates of new **general reformulation of MHD:**

- ∃ transport interfaces, $\mathcal{I}_i$ or $\Gamma_{i,j}$, or $\partial\Omega_{i,j}$ (e.g. nested tori or island separatrices), that act like sheets of ideal-MHD plasma
- Plasma relaxes (in some generalized Taylor sense) in regions $\mathcal{P}_i$ (or Ω_i) bounded by the interfaces
- Only a *subset* of ideal-MHD invariants apply

SPEC (currently) uses MRxMHS, not MRxMHD:

MRxMHS = Multi-region Relaxed Magnetohydro*Statics* (i.e. equilibrium theory)

Ref. Stuart Hudson's talk yesterday

- Taylor relaxation *energy* principle
- constant *pressure* in each region

MRxMHD = Multi-region Relaxed Magnetohydro*Dynamics*

New approach:: use *Hamilton's Principle* — stationarity of time-integrated *Lagrangian*

➪ constant *temperature* in each region

➪ supports sound waves within relaxation regions as well as radially compressible and Alfvén modes + *tearing*

➪ can treat *development* of resonant current sheets

➪ can add equilibrium flow to SPEC and will be basis for a new time-evolution waterbag code

MRxMHD Lagrangian is *kinetic energy* minus MHD *potential energy* + constraint terms:

- MHD Lagrangian density in region i

$$\mathcal{L}^{\mathrm{MHD}} = \rho\frac{v^2}{2} - \frac{p}{\gamma - 1} - \frac{B^2}{2\mu_0}$$

- Constrained Lagrangian in region i

$$L_i = \int_{\Omega_i} \mathcal{L}^{\mathrm{MHD}} dV + \tau_i (S_i - S_{i0}) + \mu_i (K_i - K_{i0})$$

- Helicity and entropy *macroscopic* invariants

$$K_i \equiv \int_{\Omega_i} \frac{\mathbf{A}\cdot\mathbf{B}}{2\mu_0} dV \qquad S_i \equiv \int_{\Omega_i} \frac{\rho}{\gamma - 1} \ln\left(\kappa \frac{p}{\rho^\gamma}\right) dV$$

In varying action, ρ is constrained *holonomically* to the displacement ξ of each fluid element:

- Mass conserved *micro*scopically, i.e. pointwise

$$\delta\rho = -\boldsymbol{\nabla}\cdot(\rho\boldsymbol{\xi}) \text{ in } \Omega_i$$

- Helicity and entropy constrained *macro*scopically, throughout Ω_i, using Lagrange multipliers μ_i and τ_i, while *p and* **A** are free fields
- Including vacuum field energy, total Lagrangian is

$$L = \sum_i L_i - \int_{\Omega_\mathrm{v}} \frac{\mathbf{B}\cdot\mathbf{B}}{2\mu_0}\, dV$$

- Setting variation of action to 0 gives EL equations:

$$\delta \int L dt = 0$$

Equations within Ω_i

- Mass conservation (microscopic constraint)

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla}\cdot(\rho \mathbf{v})$$

- δ*p* ➪ Isothermal equation of state

$$p = \tau_i \rho \quad (\text{N.B. } \tau_i = C_{\mathrm{s}i}^2)$$

- δ**A** ➪ Beltrami equation

$$\boldsymbol{\nabla}\times\mathbf{B} = \mu_i \mathbf{B} \quad (\text{N.B. } \Rightarrow \mathbf{j}\times\mathbf{B} = 0)$$

- $\boldsymbol{\xi}$ ➪ Momentum equation (Euler fluid)

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v}\right) = -\boldsymbol{\nabla} p$$

Equations on interface $\Gamma_{i,j}$

- $\boldsymbol{\xi}$ ⇨ Force balance

$$\left[\!\left[p + \frac{B^2}{2\mu_0} \right]\!\right]_{i,j} = 0$$

- Surface constraints

$$\mathbf{n}_i \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega_i$$

$$\mathbf{n}_i \cdot [\![\mathbf{v}]\!]_{i,j} = 0 \quad \text{on } \partial\Omega_{i,j}$$

- Complete set of equations, consistent because derived from single scalar function *L*

Proving the MRxMHD pudding:

- *Q1) What is the MRxMHD spectrum and what are the effects of field-line curvature and <u>equilibrium mass flow</u> on stability?*
- *Q2) When are the current sheets topologically stable towards internal plasmoid formation (reconnection)?*
- *Q3) When do unstable modes saturate at a low level or develop nonlinearly into explosive events?*

What happens in static limit?

- $\partial_t \to 0$ ⇨ $\nabla\cdot(\rho\mathbf{v}) = 0 \qquad \mathbf{v}\cdot\nabla\mathbf{v} = -\tau_i \nabla \ln\rho$

⇨ only solutions valid for *any* flowline configuration, from nested surfaces to arbitrarily chaotic, are

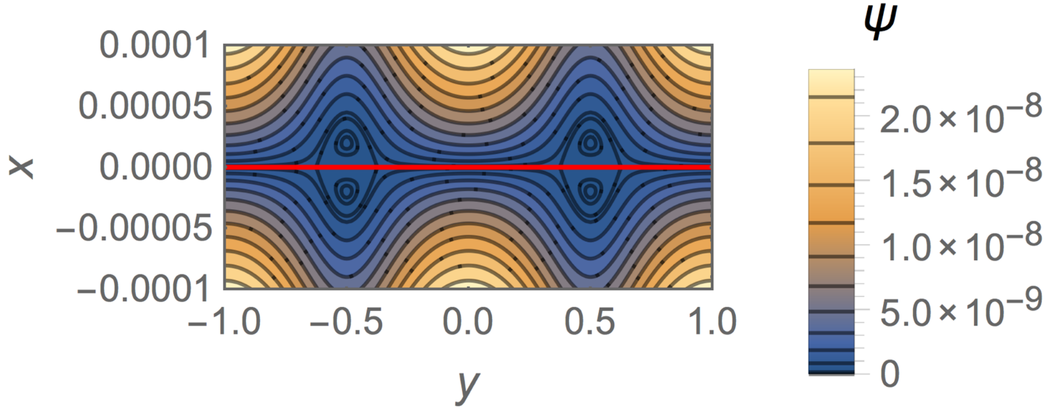
$$\rho = \rho_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right) ⇨ \; p = p_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right)$$

(N.B. *incompressible* in limit $v/C_\mathrm{s} \to 0$) and

$$\nabla\times\mathbf{v} = \alpha_{0i} \exp\left(-\frac{v^2}{2\tau_i}\right)\mathbf{v}$$

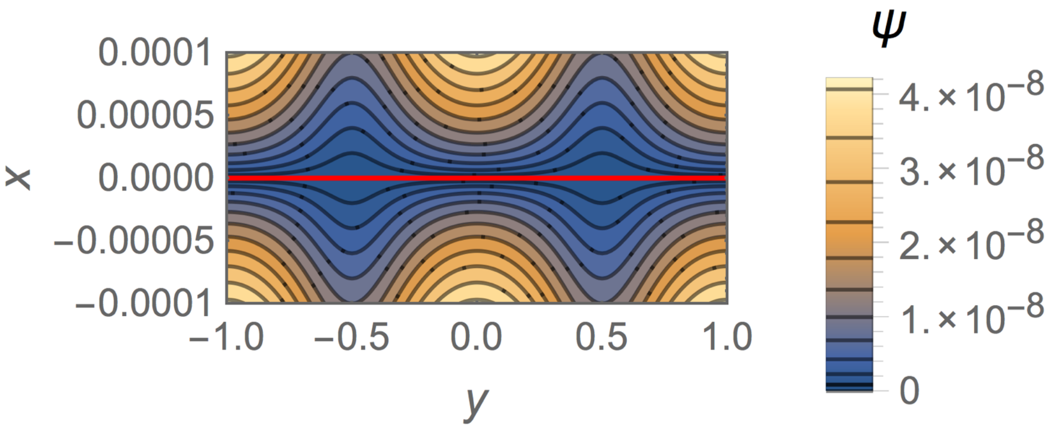
Almost isomorphous to **B** equation: should be implementable in SPEC. Derivable variationally?

Slow limit? Switch-on slab boundary ripple:*



Ripple amplitude:
α = 0.003
Current density exhibits sign reversal

2-region MRxMHD Hahm-Kulsrud model: mirror-image ripple top and bottom excites modulated current sheet at *x* = 0



Larger ripple amplitude:
α = 0.005
No sign reversal so half-islands disappear

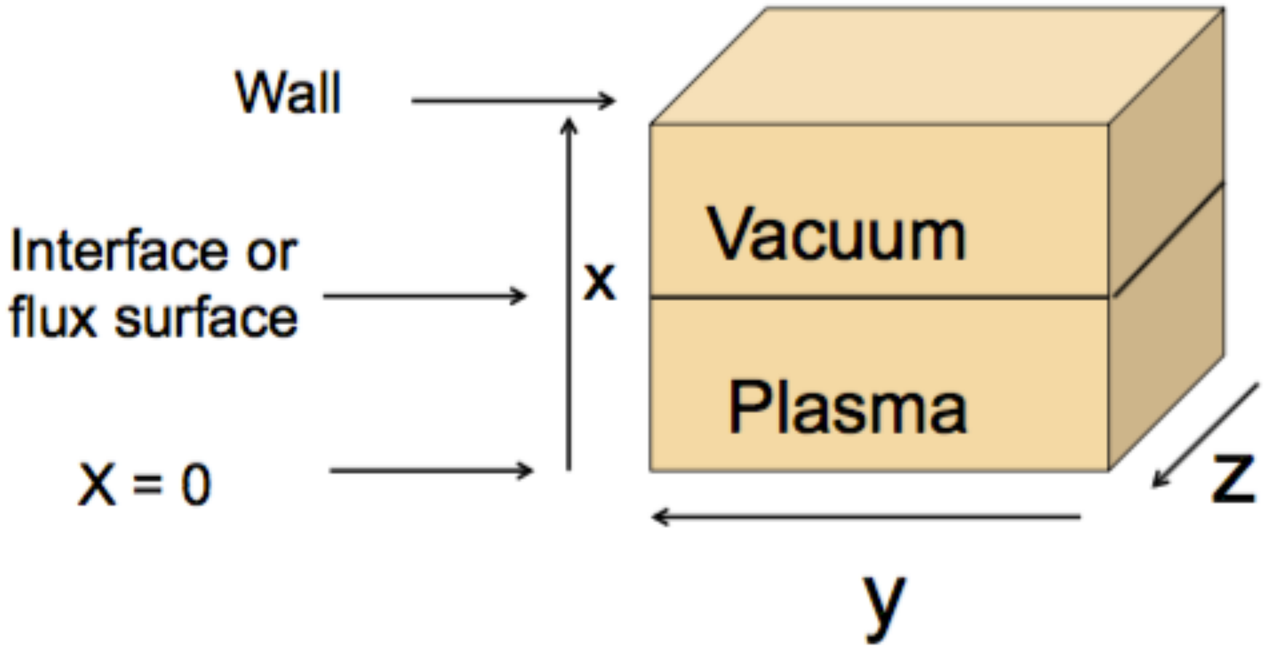
*From APS DPP 2014 poster

Full *t*-dependence: linear modes in slab

B_1 a superposit^n of "Beltrami waves" in plasma ($\mu > 0$)

and

vacuum ($\mu = 0$)



MRxMHD: sound waves in plasma (ρ_0 = const > 0, $\tau > 0$)

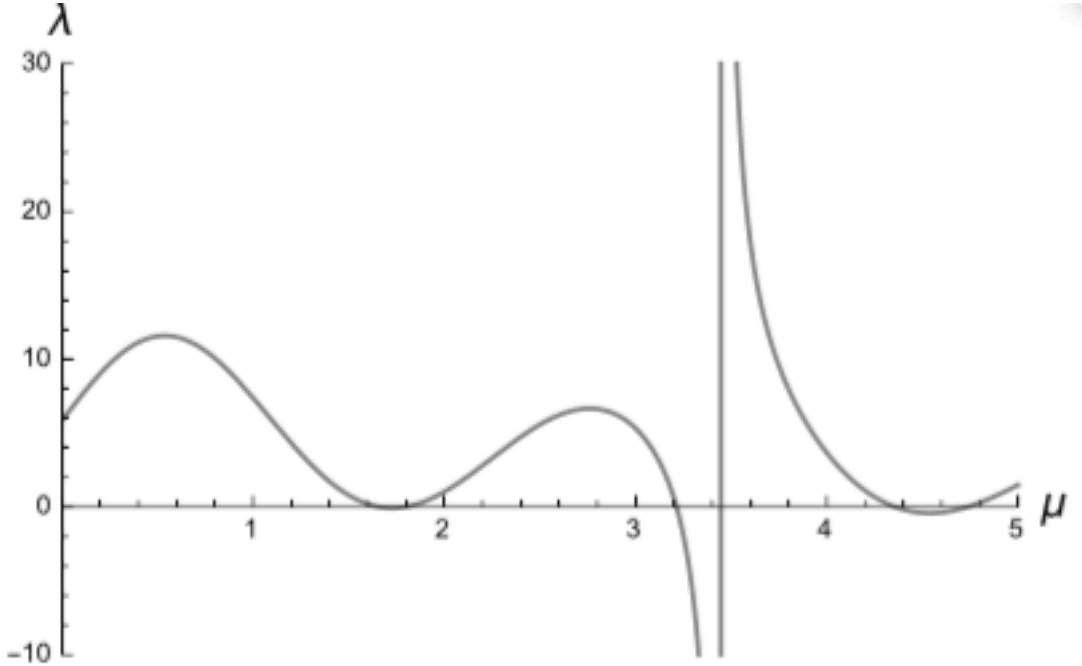
Results to be reported at ICPP, Kaoshiung, Taiwan 2016

MRxMHS+: $\lambda = \omega^2$ with $\rho_0 = \delta(x\text{-}a)$ ⇨ no sound waves

Hole et al, Nucl Fusion **47**, 746 (2007), etc

Alexis Tuen's MSc thesis 2016

Loading all mass on interfaces causes problems:



- Growth rate goes to ∞ when Newcomb node goes through interface
- Growth rate zero if wall or **k.B**=0 is at interface

Conclusion

- Action-based MRxMHD shows great promise
  - Very simple
  - Includes reconnection and flow in natural way
- We need to check physical reasonability of predictions in simple models
- Need both to extend SPEC and build a new time-evolution code