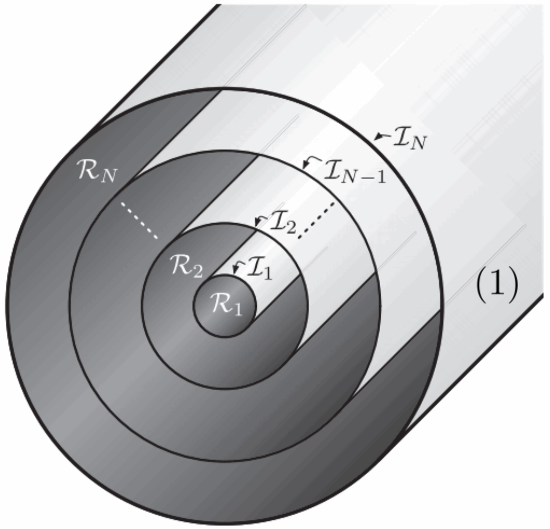
Multi-Region Relaxed MHD: background

1. Motivated by a theorem of [O. Bruno & P. Laurence, Comm. Pure. Appl. Math. **19**, 717 (1996)]
   "We establish an existence result for 3D MHD equations for $p \neq const.$ in tori without symmetry. More precisely, our theorems insure the existence of sharp boundary solutions"

2. MRxMHD Energy Functional

$$\mathcal{F} \equiv \sum_{i=1}^{N} \left[\int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \left(\int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv - H_i \right) \right], \quad (1)$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \text{ on } \partial \mathcal{R}_i$$



3. Implemented numerically in the Stepped Pressure Equilibrium Code (SPEC)
   [S.R.Hudson, R.L.Dewar *et al.*, Phys. Plasmas **19**, 112502 (2012)]

4. SPEC to model self-organized helical states in RFP
   [G.R.Dennis, S.R.Hudson *et al.*, Phys. Rev. Lett. **111**, 055003 (2013)]

5. MRxMHD & SPEC shown to recover ideal-MHD in limit
   [G.R.Dennis, S.R.Hudson *et al.*, Phys. Plasmas **20**, 032509 (2013)]

6. MRxMHD extended to include flow
   [G.R. Dennis, S.R.Hudson *et al.* Phys. Plasmas **21**, 042501 (2014)]

7. MRxMHD extended to include flow, pressure anisotrophy
   [G.R. Dennis, S.R.Hudson *et al.*, Phys. Plasmas **21**, 072512 (2014)]

8. SPEC used to compute singular current densities in ideal-MHD
   [J. Loizu, S. Hudson *et al.*, Phys. Plasmas **22**, 022501 (2015)]

Multi-Region Relaxed MHD: recent theoretical developments
R.L. Dewar, M. Lingam

1. "Variational formulation of relaxed and multi-region relaxed magnetohydrodynamics"
   [R.L. Dewar, Z. Yoshida *et al.*, J. Plasma Phys. **81**, 515810604 (2015)]

2. "Multi-region relaxed Hall magnetohydrodynamics with flow"
   [Manasvi Lingam, Hamdi M. Abdelhamid & Stuart R. Hudson, Phys. Plasmas **23**, 082103 (2016)]

3 "Penetration of a resonant magnetic perturbation in an adiabatically rippled plasma slab"
   [Robert L. Dewar, Stuart R. Hudson *et al.*, Phys. Plasmas, submitted (2016)]

SPEC: recent applications/developments
J.Loizu, S.R. Hudson, S. Lazerson

1. "Existence of three-dimensional ideal-MHD equilibria with current sheets"
   [J. Loizu, S.R. Hudson *et al.* Phys. Plasmas **22**, 090704 (2015)]

2. "Pressure-driven amplification and penetration of resonant magnetic perturbations"
   [J. Loizu, S.R. Hudson *et al.* Phys. Plasmas **23**, 055703 (2016)]

3. "Verification of the ideal magnetohydrodynamic response at rational surfaces in the VMEC code"
   [S. Lazerson, J. Loizu *et al.* Phys. Plasmas **23**, 012507 (2016)]

4. "Verification of the SPEC code in stellarator geometries"
   [J. Loizu, S.R. Hudson & C. Nührenberg, submitted, Phys. Plasmas, (2016)]

We have resolved a long-standing issue regarding perturbed, "ideal" equilibria.

J. Loizu, S.R. Hudson *et al.*

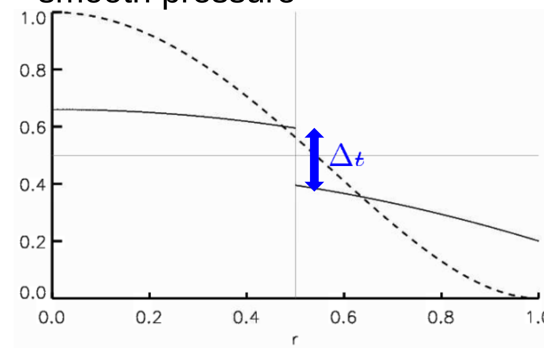
Rational surfaces result in non-integrable current singularities . . .
Resolution: a "sheet current" arises that produces a *discontinuous* rotational-transform

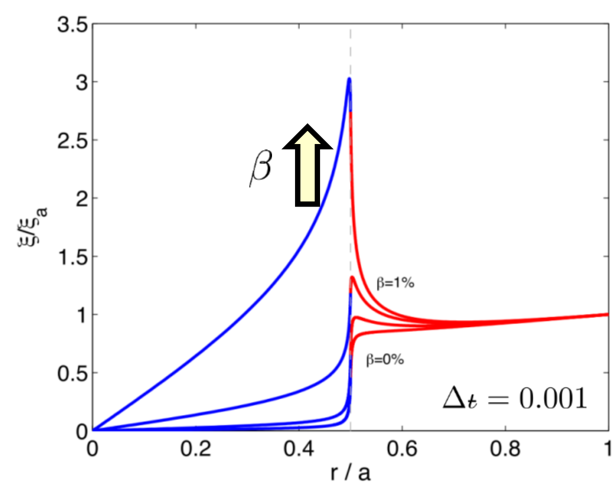
1. Solutions to $\nabla p = \mathbf{j} \times \mathbf{B}$ in 3D geometry, with non-overlapping nested flux surfaces, must have "sheet currents" that produce discontinuities in the rotational-transform.
2. Exact verification calculations (in cylindrical geometry) of SPEC were performed.

New Physics Insight :

1. there is penetration, and magnification of the perturbation *inside* the resonant surface by pressure.

discontinuous rotational-transform
smooth pressure

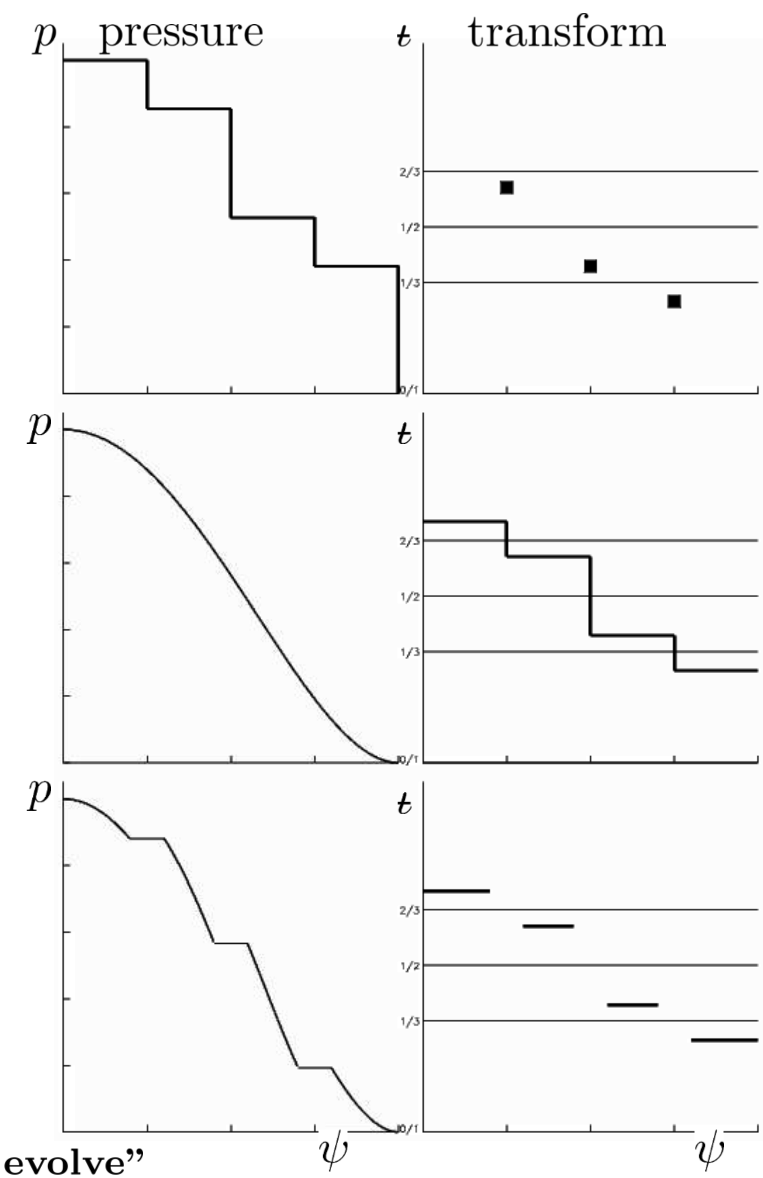




[1] J. Loizu, S.R. Hudson *et al.*, Phys. Plasmas **22**, 090704 (2015)
[2] J. Loizu, S.R. Hudson *et al.*, Phys. Plasmas **23**, 055703 (2016)
[3] J. Loizu, 2015 International Sherwood Fusion Theory Conference, ***Invited Talk***
[4] J. Loizu, 57th Annual Meeting of the APS Division of Plasma Physics, ***Invited Talk***
[5] S.R. Hudson, 2016 International Sherwood Fusion Theory Conference, ***Invited Talk***
[6] J. Loizu, 2016 Joint Varenna - Lausanne International Workshop, ***Invited Talk***
[7] S.R. Hudson, 26th IAEA Fusion Energy Conference, ***Invited Talk***

The classes of general, tractable 3D MHD equilibria are:

1. **Stepped-pressure equilibria,**
   i. Bruno & Laurence states
   ii. extrema of MRxMHD energy functional
   iii. transform constrained discretely
   iv. pressure discontinuity at ι = irrational
   v. allows for islands, magnetic fieldline chaos
2. **Stepped-transform equilibria,**
   i. introduced by Loizu, Hudson *et al.*
   ii. extrema of ideal MHD energy functional
   iii. transform (almost) everywhere irrational
   iv. arbitrary, smooth pressure
   v. continuously-nested flux surfaces
3. **Or, a combination of the above.**
   i. **each can be computed using SPEC**
   ii. suggests VMEC, NSTAB, should be modified to allow for discontinuous transform



Q. **How does a state with continuous transform "ideally evolve" into a 3D state with discontinuous transform?**
implications for ideal stability if no accessible 3D state exists?

SPEC: ongoing development/applications
S.R. Hudson

1. RECENT code improvements:

   i. finite-elements replaced by Chebshev polynomials

   e.g. $$\mathbf{A} \equiv \sum_{l,m,n}^{L,M,N} [\alpha_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\theta + \beta_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\zeta]$$

   ii. linearized equations

   iii. Cartesian, cylindrical, toroidal geometry

   iv. detailed online documentation,
   `http://w3.pppl.gov/~shudson/Spec/spec.html`

   v. easy-to-use, easy-to-edit, graphical user interface

2. ONGOING physics applications

   i. W7-X vacuum verification calculations, OP1.1 [completed]

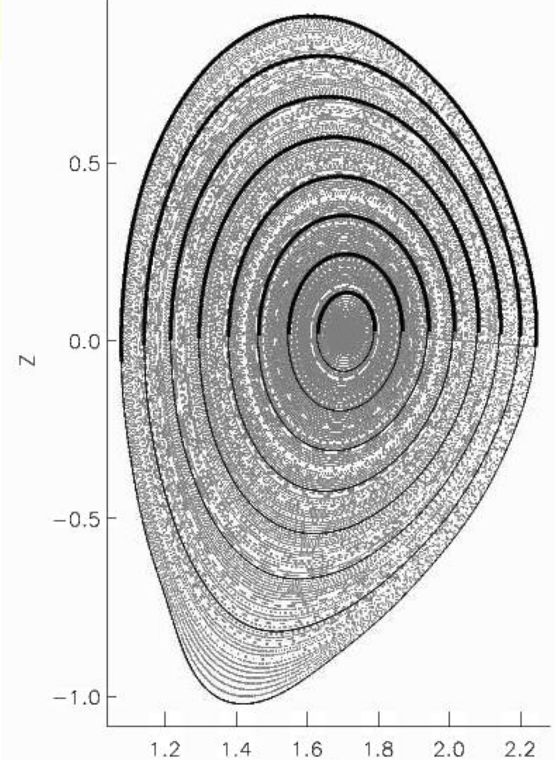
   ii. non-stellarator symmetric, e.g. DIIID, [completed]

   iii. free-boundary, [completed]

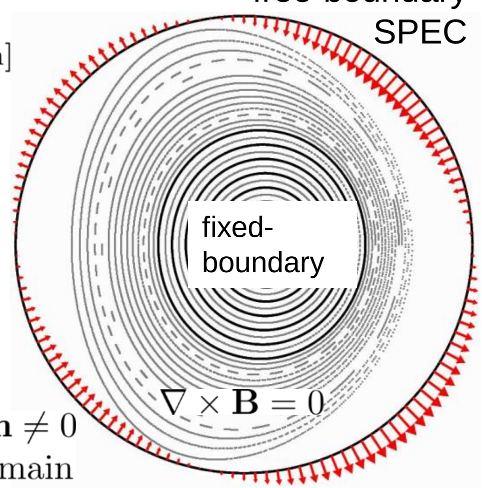
   iv. including flow, anisotrophy, . . [under construction]

   v. MRxMHD linear stability, [under construction]

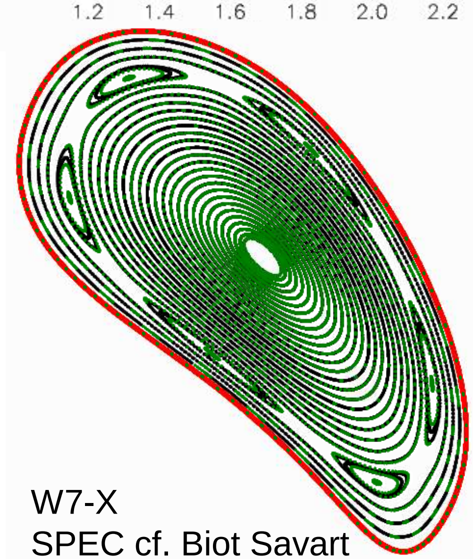
DIIID: SPEC cf. VMEC



free-boundary
SPEC



$\mathbf{B} \cdot \mathbf{n} \neq 0$
on ∂Domain



W7-X
SPEC cf. Biot Savart

A new approach to stellarator coil design
Caoxiang Zhu & S.R. Hudson

1. Previous methods (NESCOIL, COILOPT) vary angular location of coils on a "winding surface", but perhaps this is over-constrained.

2. We are investigating a new approach motivated by "The Fundamental Theorem of Curves" : every regular curve in three-dimensional space, with non-zero curvature, has its shape (and size) completely determined by its curvature and torsion,

$$\kappa_n(s) \equiv \sum_m \kappa_{n,m} \exp(ims), \quad \tau_n(s) \equiv \sum_m \tau_{n,m} \exp(ims).$$

   Let $\delta \mathbf{x} \equiv \{\kappa_{n,m}, \tau_{n,m}\}$ be degrees of freedom of N discrete coils.

   Alternatively, use Cartesian representation, e.g. $\mathbf{x} = \{x_{n,m}, y_{n,m}, z_{n,m}\}$.

3. Coil geometry is varied to minimize

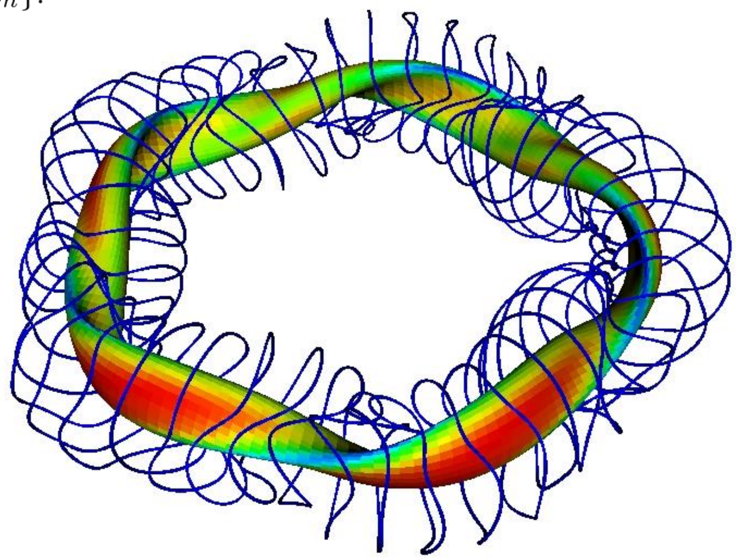
$$\mathcal{F}[\mathbf{x}] \equiv \int_{\mathcal{S}} (\mathbf{B} \cdot \mathbf{n})^2 \, d\mathbf{s} + \omega_L \sum_n (\text{length})^2 + \text{other constraints}$$

4. Differential flow can find minimizing coil geometry

$$\frac{\partial \mathbf{x}}{\partial t} = -\nabla \mathcal{F}[\mathbf{x}].$$

5. A parallelized, Newton method quickly finds extrema

$$\delta \mathbf{x} = -\nabla^2 \mathbf{F}^{-1} \cdot \nabla \mathbf{F}[\mathbf{x}].$$



W7-X coils produced by Knotopt (from initial circular coils)

Investigating the fractal structure of “Diophantine” equilibria

Brian Kraus, S.R. Hudson

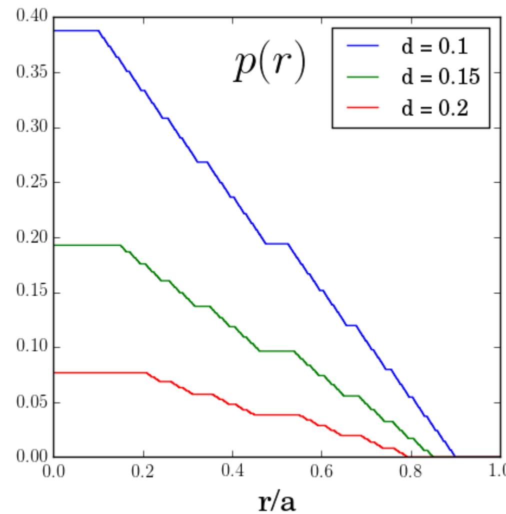
1. To avoid pressure-driven, non-physical parallel currents near rational surfaces, require

$$p'(\psi) = 0 \text{ where } |\iota\!\!\!-(\psi) - n/m| < \epsilon, \quad \forall n, m.$$

2. KAM theorem: “an irrational flux surface will exist (for small perturbations) if the rotational-transform is sufficiently irrational, i.e. if $\iota\!\!\!-$ satisfies a Diophantine condition”.

3. Consider “Diophantine” pressure profile

$$p'(\psi) = \begin{cases} 1, & \text{if } |\iota\!\!\!-(\psi) - n/m| > d/m^2, \quad \forall n, m, \\ 0, & \text{otherwise.} \end{cases}$$



4. Consider cylindrical geometry
(i.e. no chaos, but consider fractal pressure).

5. What are the fractal properties of p, B_z, J_θ, etc. ?