Cantori, chaotic coordinates and temperature gradients in chaotic magnetic fields

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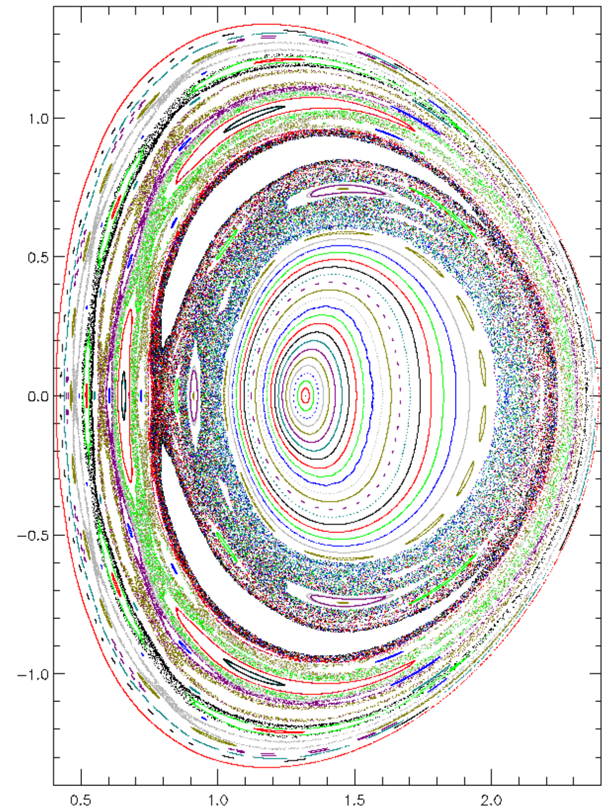
1 **Motivation:** Error fields, 3D effects, etc. create "fieldline chaos" in magnetically confined plasmas, which deterioates confinement.

2 **Method:** Compare invariant and *almost* invariant structures of fieldline flow to isotherms, where T satisfies
$\kappa_\parallel \nabla_\parallel^2 T + \kappa_\perp \nabla_\perp^2 T = 0$, with $\kappa_\parallel / \kappa_\perp = 10^{10}$.

3 **We found:**

1) isotherms coincide with cantori;

2) $T = T(s)$ is a surface function
in "chaotic coordinates" based on "ghost surfaces".

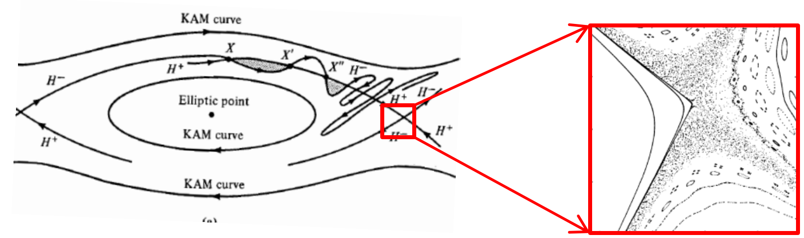


eg. M3D simulation of CDX-U

With increasing non-axisymmetry, the flux surfaces become increasingly “broken”

- Invariant flux surfaces are destroyed near “resonances”, $\omega = n / m$, n, m are integers
  construction of action-angle coordinates for perturbed system fails because of “small-denominators”

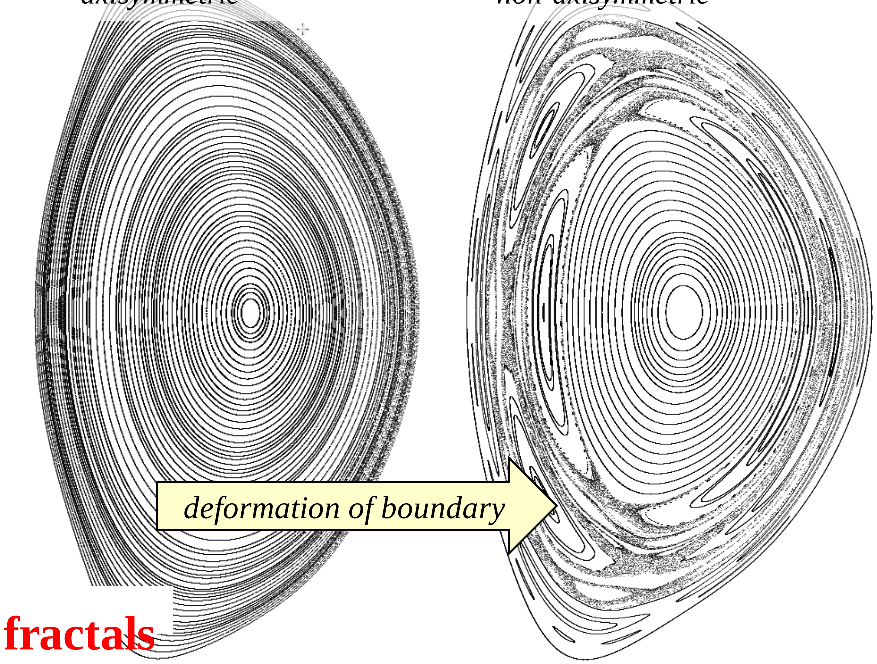
- Magnetic islands (resonance zones) form
  chaotic, “irregular” field lines emerge,
  that wander **seemingly randomly** over a volume



separatrix splitting, unstable manifold, *“chaotic tangle”*

- Confinement deteriorates,
  the pressure is flat inside islands and chaos

Poincaré Plot of DIIID *axisymmetric* | *Poincaré Plot of DIIID* *non-axisymmetric*

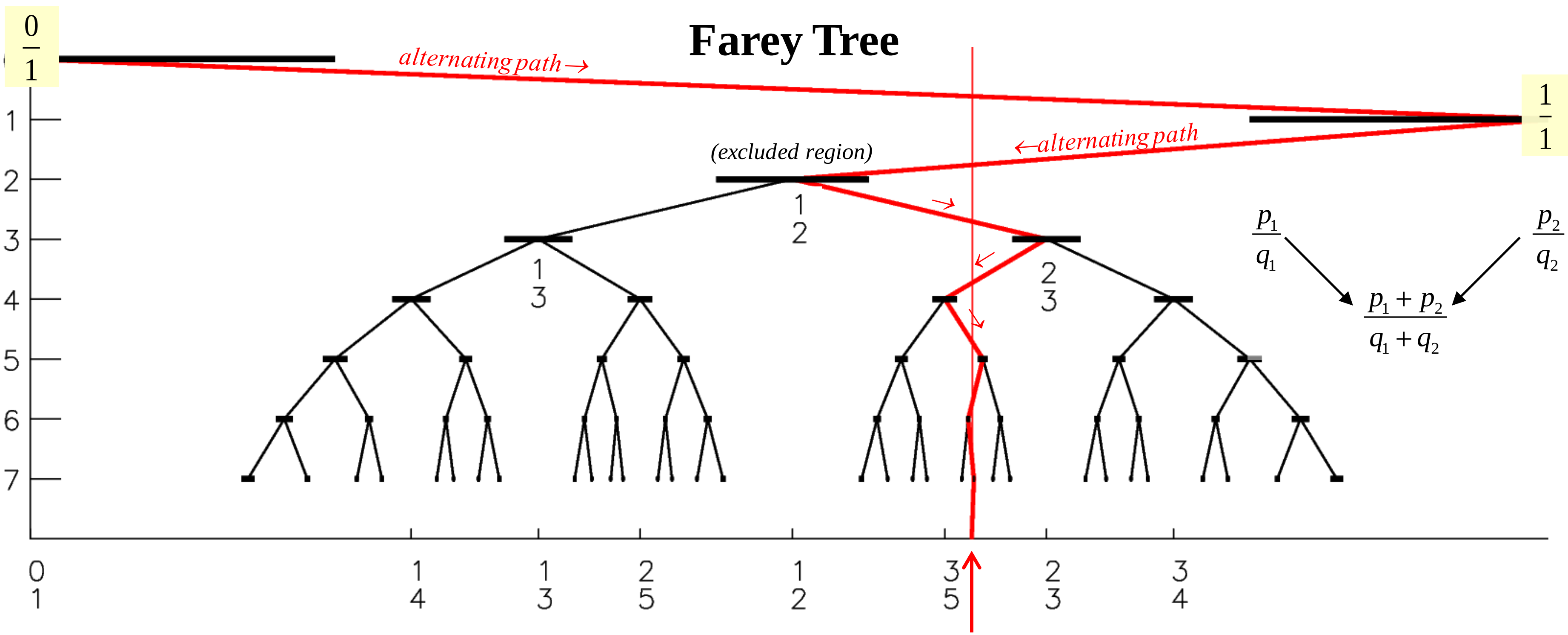


- The calculation of three-dimensional partially-chaotic equilibria must

1) Be consistent with theoretical plasma physics
2) Be consistent with experimental results
3) **Be consistent with Hamiltonian chaos theory**
4) **employ numerical methods that accommodate fractals**

WHERE TO START? START WITH CHAOS

The fractal structure of chaos is related to the structure of numbers



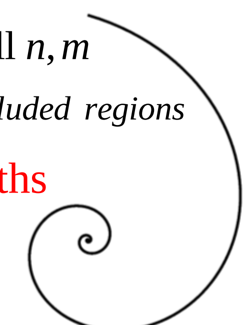
islands & chaos emerge at every rational → about each rational n/m, introduce excluded region, width r/m^k

KAM Theorem → flux surface can survive if $|\omega - n/m| > r/m^k$, for all n, m

(Kolmogorov, Arnold, Moser) *we say that ω is "strongly-irrational" if ω avoids all excluded regions*

Greene's residue criterion → the most robust flux surfaces are associated with alternating paths

→ Fibonacci ratios $\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \frac{13}{21}, \frac{21}{34}, \ldots$



THEN, ADD PLASMA PHYSICS

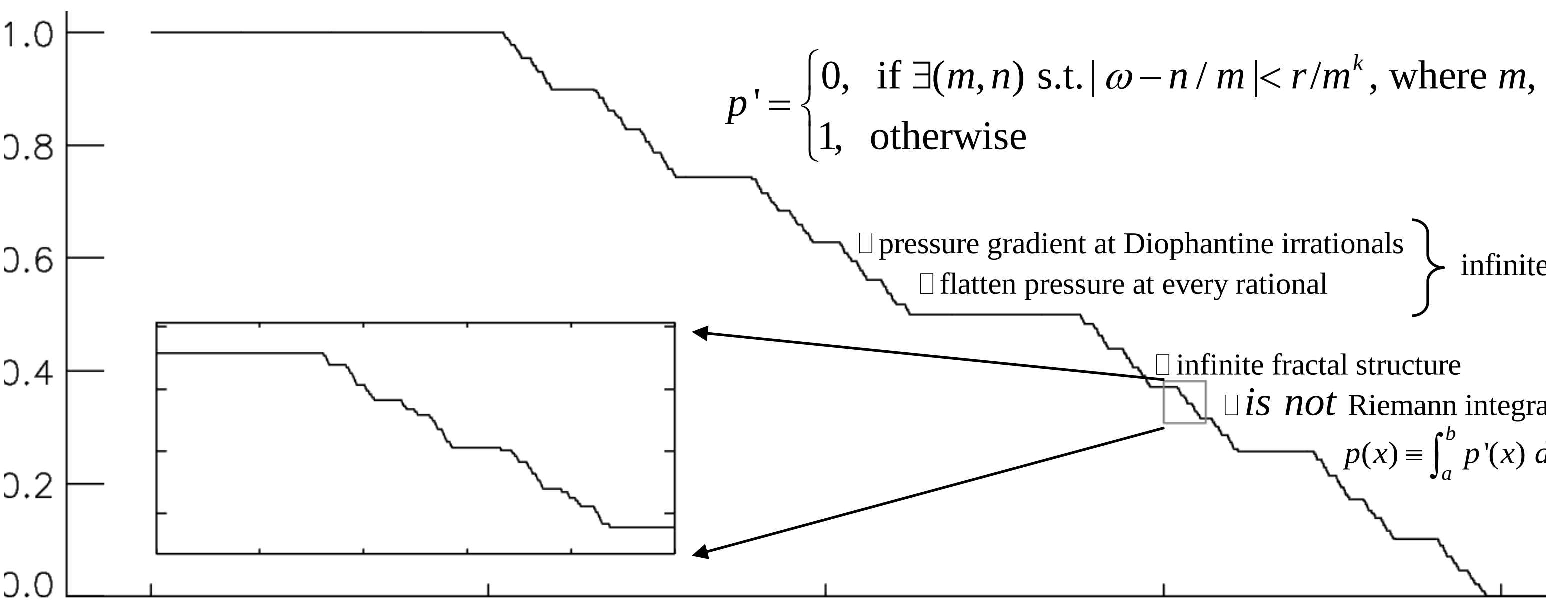
Force balance means the pressure is a *“fractal staircase”*

- $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$, implies that $\boldsymbol{B}.\nabla p=0$ i.e. pressure is constant along a field line

- Pressure is flat across the rationals (assuming no “pressure” source inside the islands)
  → *islands and chaos at every rational* → *chaotic field lines wander about over a volume*

- Pressure gradients supported on the “most-irrational” irrationals
  → *surviving “KAM” flux surfaces confine particles and pressure*

Diophantine Pressure Profile is it pathological?

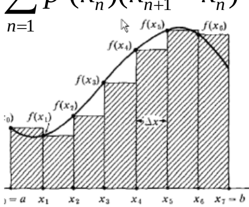


$$p' = \begin{cases} 0, & \text{if } \exists (m,n) \text{ s.t.} \,|\omega - n/m| < r/m^k, \text{ where } m,\ n \text{ are integer} \\ 1, & \text{otherwise} \end{cases}$$

pressure gradient at Diophantine irrationals
flatten pressure at every rational
} infinitely discontinuous

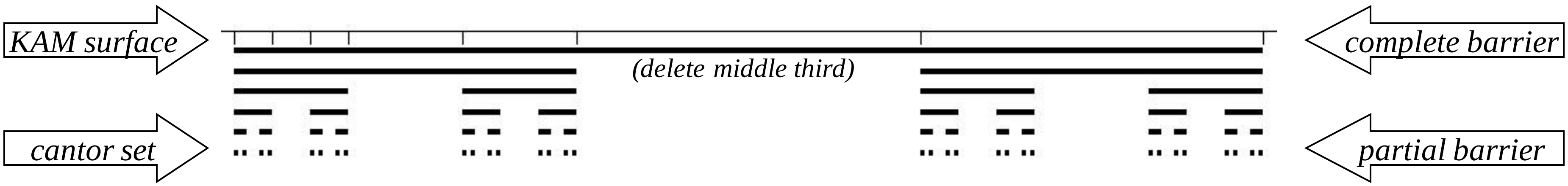
infinite fractal structure
is not Riemann integrable

$$p(x) \equiv \int_a^b p'(x)\, dx \neq \lim_{N\to\infty} \sum_{n=1}^{N} p'(x_n)(x_{n+1} - x_n)$$

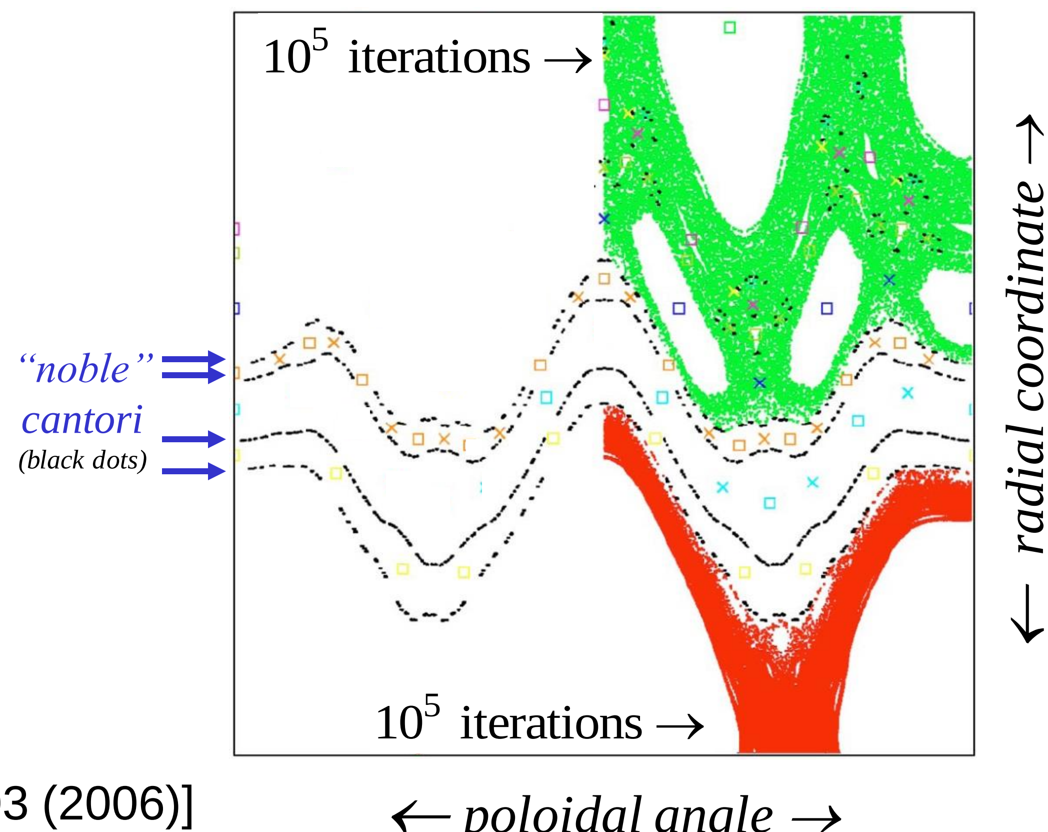


Fieldline transport is restricted by irrational field-lines

The irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict fieldline transport even after the onset of chaos.



Poincaré plot (model field → next slide)



KAM surfaces **stop**
radial field-line transport.

Cantori ≡ "broken KAM surfaces"
do not stop, but do slow down
radial field-line transport

[S.R. Hudson, Phys. Rev. E **74**, 056203 (2006)]

Cantori are approximated by high-order periodic orbits;

High-order (minimizing) periodic orbits are located using variational methods.

1. Magnetic fieldlines are stationary curves, $\mathcal{C}$, of the action, $S[\mathcal{C}] \equiv \int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$, where $\mathbf{A} = \psi \nabla \theta - \chi(\psi, \theta, \zeta) \nabla \zeta$, and $\chi = \frac{1}{2}\psi^2 + \sum k_{m,n}(\psi) \cos(m\theta - n\zeta)$.

2. Setting $\delta S = 0$ gives $\dot{\theta} = \dfrac{B^\theta}{B^\zeta}$ and $\dot{\psi} = \dfrac{B^\psi}{B^\zeta}$.

3. A piecewise-linear, $\theta(\zeta) \equiv \theta_i + \dfrac{(\theta_{i+1} - \theta_i)}{(\zeta_{i+1} - \zeta_i)}(\zeta - \zeta_i)$, trial curve

   allows analytic evaluation of the action integral, $S = S(\theta_0, \theta_1, \theta_2, \dots)$.

4. To find (p, q) periodic curves, use Newton's method to find $\dfrac{\partial S}{\partial \theta_i} = 0$, with constraint $\zeta_N = \zeta_0 + 2\pi q$ and $\theta_N = \theta_0 + 2\pi p$

   robust = not sensitive to Lyapunov error

5. Two types of periodic orbit:

   0 : stable, action minimax,

   X : unstable, action-minimizing, $\rightarrow$ cantori as $p/q \rightarrow$ irrational.

Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.

[C. Golé, J. Differ. Equations **97**, 140 (1992), R.S. MacKay and M.R. Muldoon, Phys. Lett. A **178**, 245 (1993)]

1. At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\dfrac{\partial}{\partial\theta_j}\dfrac{\partial}{\partial\theta_i}S$, with negative eigenvalue indicates the direction in which the action integral decreases.

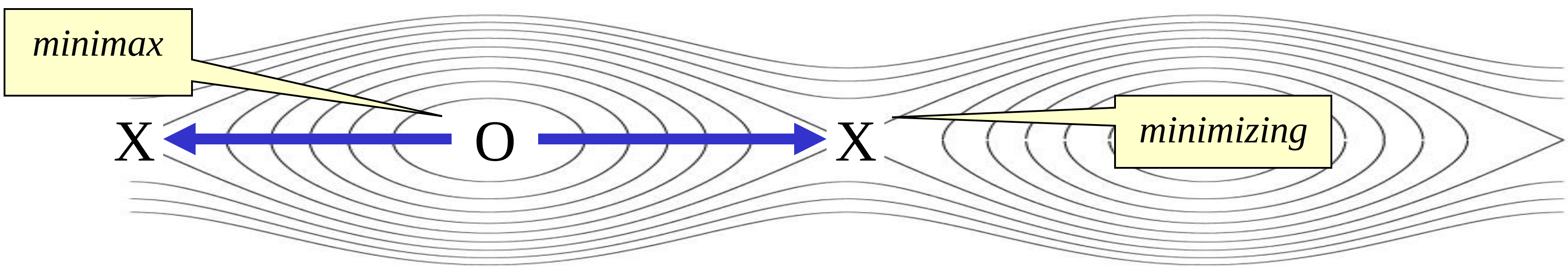
2. Pushing a trial curve from the minimax p/q orbit down the action gradient flow to the minimizing (unstable) p/q orbit defines "ghost surfaces".

$$\text{Action Gradient Flow: } \frac{\partial\theta_i}{\partial\alpha} = -\frac{\partial S}{\partial\theta_i}$$

problem: action-gradient flow is small near integrable limit

where α is arbitrary integration parameter (new angle).

3. Ghost surfaces may be thought of as rational coordinate surfaces that pass through magnetic islands (resonance zones), i.e. "replacement" flux surfaces.



Ghost-surfaces are identical to quadratic-flux-minimizing surfaces (if using appropriate angles).

1. Quadratic-flux-minimizing surfaces minimize

$$\varphi \equiv \frac{1}{2}\int_{\mathcal{S}} \left(\frac{\partial S}{\partial \theta}\right)^2 d\theta d\zeta = \frac{1}{2}\int_{\mathcal{S}} (\sqrt{g}\mathbf{B}\cdot\mathbf{n})^2 \; d\theta d\zeta.$$

2. A constrained variational principle for rational pseudo-orbits exists:

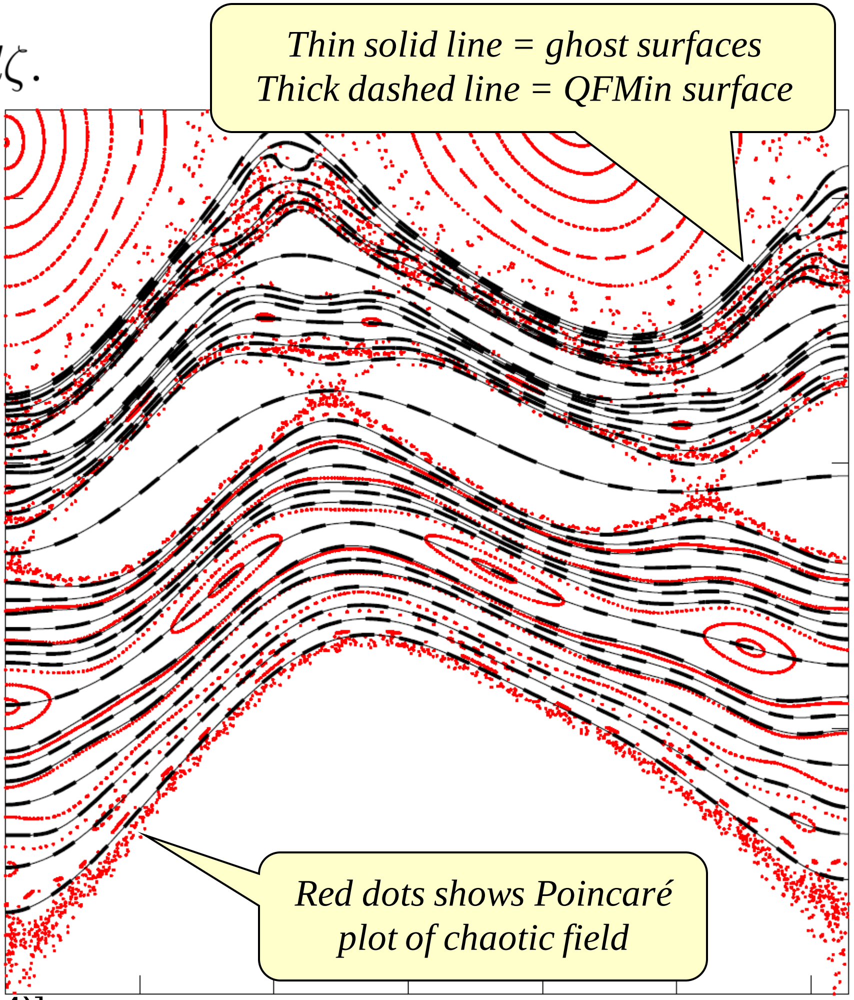
$$S[\mathcal{C}] = \int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{l} - \nu \left(\int \theta \, d\zeta - a\right).$$

3. Freedom in the choice of angles, $\sqrt{g}(\theta,\zeta)$, exploited so that ghost-surfaces = quadratic-flux minimizing surfaces.

4. This approach provides

   (i) intuitive understanding,

   (ii) faster algorithm.



[R.L Dewar, S.R Hudson & P. Price, Phys. Lett. A **194**, 49 (1994)]
[S.R. Hudson & R.L. Dewar, Phys. Lett. A, **373**, 4409 (2009)]
[R.L.Dewar, S.R.Hudson & A.M.Gibson, J. Plasma Fusion Res. SERIES **9**, 487 (2010)]

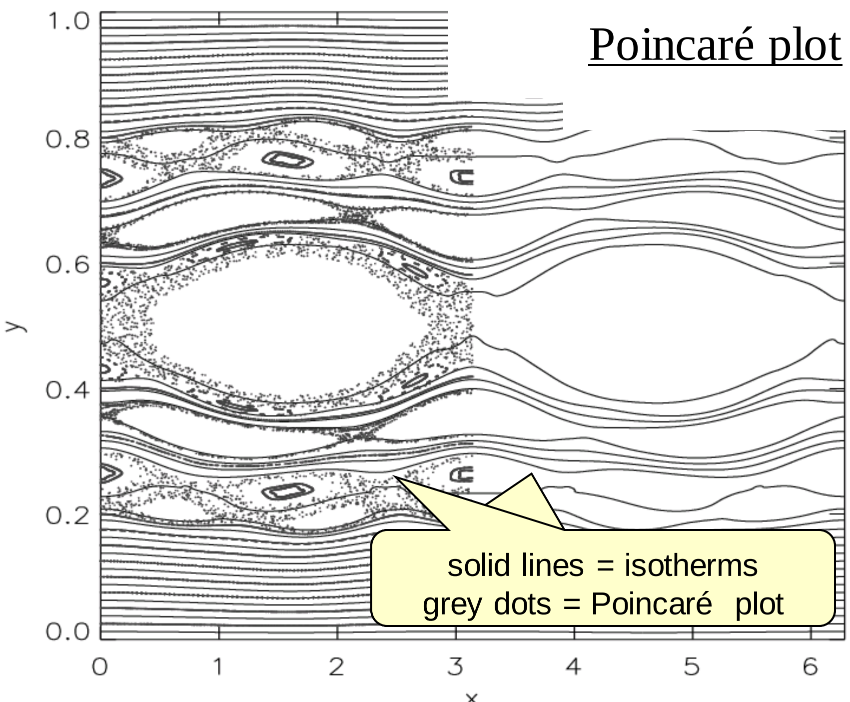
Numerically solving anisotropic heat transport exploits field-alligned coordinates *(α,β,η)*

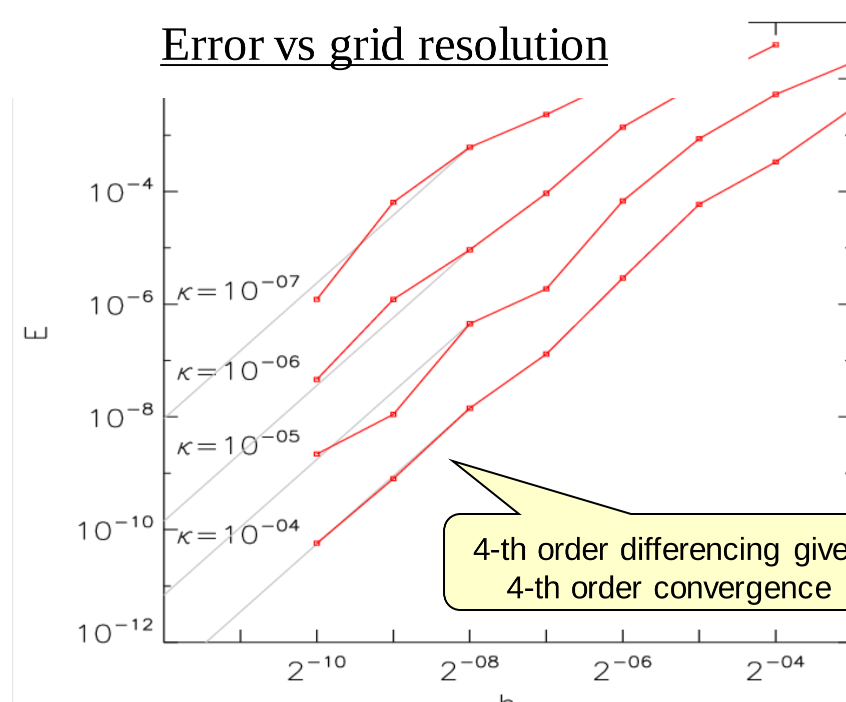
1. Heat flux $\nabla \cdot \mathbf{q} = 0$, where $\mathbf{q} = \mathbf{b} \cdot \nabla T \, \kappa_{\|} \, \mathbf{b} + \kappa_{\perp} \nabla_{\perp} T$, strongly anisotropic.

2. Parallel relaxation employs field-alligned coordinates, $\mathbf{B} = \nabla\alpha \times \nabla\beta$, so parallel derivative is accurate, $\nabla_{\|}^2 T = \dfrac{\partial^2 T}{\partial \eta^2} = B^{\zeta} \dfrac{\partial}{\partial \zeta}\left(\dfrac{B^{\zeta}}{B^2}\dfrac{\partial T}{\partial \zeta}\right)$.

3. Perpendicular relaxation simply $\nabla_{\perp}^2 T = \dfrac{\partial^2 T}{\partial \alpha^2} + \dfrac{\partial^2 T}{\partial \beta^2}$.

4. Sparse linear system solved iteratively on numerical grid, resolution $= 2^{12} \times 2^{12}$.

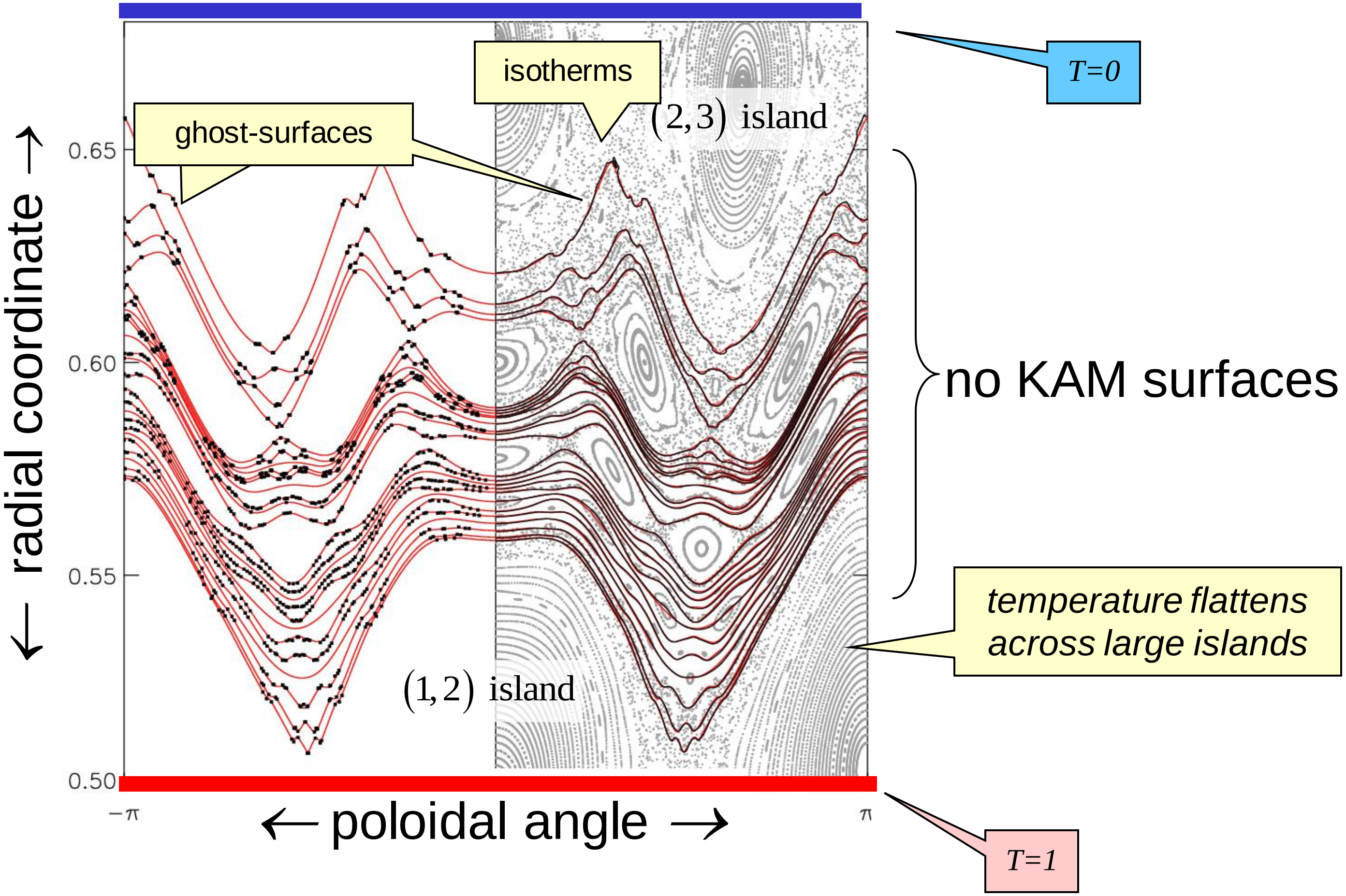




Isotherms coincide with cantori and ghost-surfaces!

Ghost-surface for high-order periodic orbits “fill in the gaps” in the (irrational) cantori.

Ghost-surfaces and isotherms are almost indistinguishable; suggests *T=T(s)*.



[S.R. Hudson & J. Breslau, Phys. Rev. Lett. **100**, 095001 (2008)]

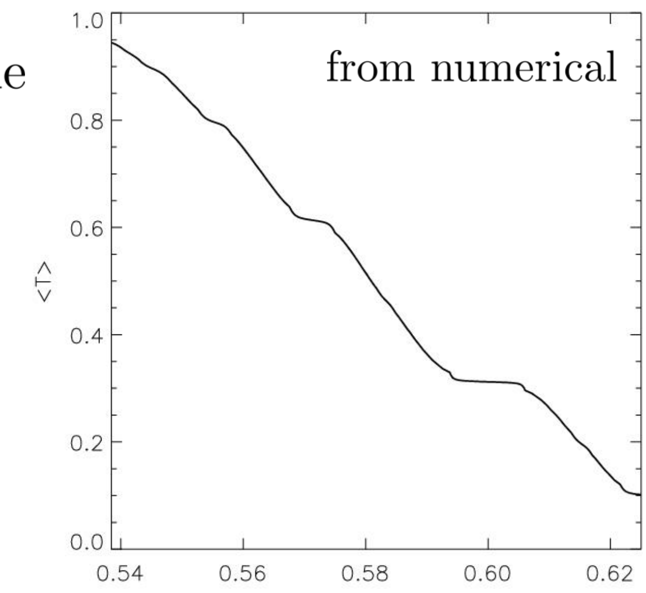
Chaotic-coordinates simplifies temperature profile to a smoothed fractal (devils) staircase.

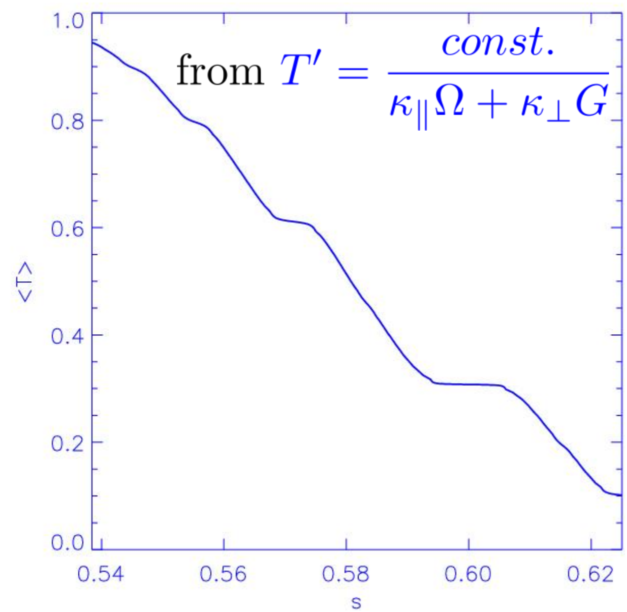
Ghost-surfaces can be used as radial coordinate surfaces.

1. From $0 = \dfrac{\partial}{\partial s}\displaystyle\int_V \nabla\cdot\mathbf{q}\,dv = \dfrac{\partial}{\partial s}\displaystyle\int_{\partial\mathcal{V}} \mathbf{q}\cdot\mathbf{n}\,d\sigma$, assume $T = T(s)$ to derive $T' = \dfrac{const.}{\kappa_\parallel\Omega + \kappa_\perp G}$
   for quadratic-flux $\Omega = \displaystyle\int g^{ss}\frac{B_n^2}{B^2}\,d\sigma$, and metric $G = \int g^{ss}d\sigma$, $g^{ss} = \nabla s\cdot\nabla s$.
2. In the "ideal limit", $\kappa_\perp \to 0$, $T'(s) \to \infty$ on irrational KAM surfaces (where $\Omega = 0$).
3. Non-zero $\kappa_\perp$ ensures $T(s)$ is smooth; $T'(s)$ peaks on minimal Ω surfaces = noble cantori.

Temperature Profile

$$\frac{\kappa_\parallel}{\kappa_\perp} = 10^{10}$$





[S.R. Hudson, Phys. Plasmas **16**, 010701 (2009)]

Summary

1. In chaotic magnetic fields, anisotropic heat transport is restricted by irrational fieldlines = cantori.

2. Ghost surfaces are identical (depending on angle) to quadratic-flux-minimizing surfaces; a simple numerical construction is introduced.

3. Interpolating between rational (p/q) ghost surfaces allows “chaotic magnetic coordinates”, or “action-angle coordinates for non-integrable Hamiltonian systems”.

4. In chaotic coordinates, the temperature profile takes a simple form.