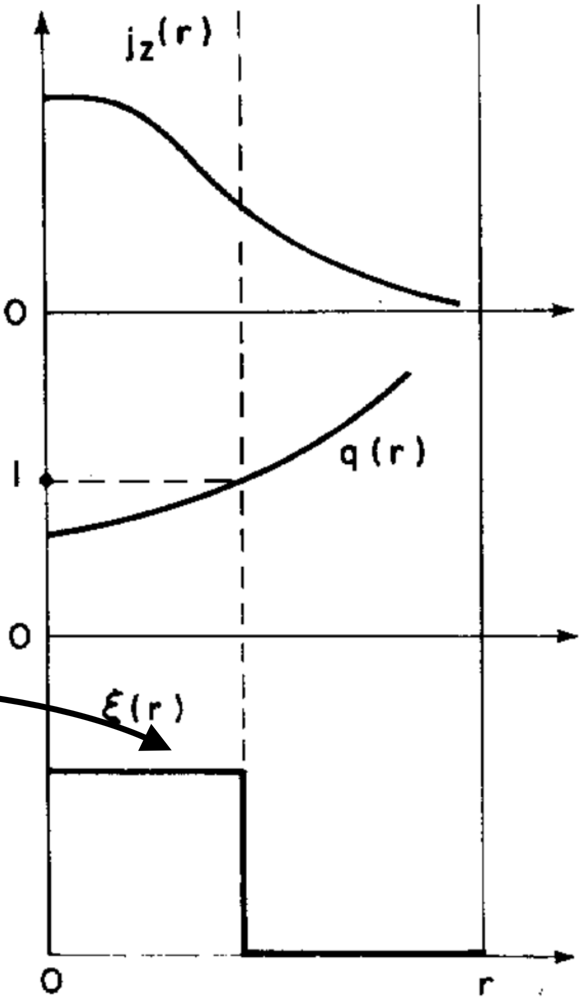
IDEAL MHD FAILS AT RATIONAL SURFACES

Breakdown of perturbation theory:

Following Rosenbluth, Dagazian & Rutherford, [Phys. Fluids **16**, 1894 (1973)]

" .. we digress to discuss briefly the standard perturbation theory approach to such nonlinear problems, .. which is not applicable here due to the singular nature of the lowest order step function solution for $\boldsymbol{\xi}$"



" In the absence of such singularity we could formally expand .."

$$\boldsymbol{\xi} \;=\; \epsilon\,\boldsymbol{\xi}_1 + \epsilon^2\,\boldsymbol{\xi}_2 + \epsilon^3\,\boldsymbol{\xi}_3 + \ldots$$

$$\delta\mathbf{B}[\boldsymbol{\xi}] \equiv \nabla\times(\boldsymbol{\xi}\times\mathbf{B}),$$
$$\delta p[\boldsymbol{\xi}] \equiv (\gamma-1)\boldsymbol{\xi}\cdot\nabla p - \gamma\nabla\cdot(p\boldsymbol{\xi})$$

Equilibrium and perturbed equations

$$\begin{array}{lcl}
\mathbf{F}[\mathbf{x}] & \equiv & \nabla\, p_0 - \mathbf{j}_0 \times \mathbf{B}_0 = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_1] & \equiv & \nabla\delta p[\boldsymbol{\xi}_1] - \delta\mathbf{j}[\boldsymbol{\xi}_1]\times\mathbf{B}_0 - \mathbf{j}_0\times\delta\mathbf{B}[\boldsymbol{\xi}_1] = 0 \\
\mathcal{L}_0[\boldsymbol{\xi}_2] & = & \nabla\delta p[\boldsymbol{\xi}_2] - \delta\mathbf{j}[\boldsymbol{\xi}_2]\times\mathbf{B}_0 - \mathbf{j}_0\times\delta\mathbf{B}[\boldsymbol{\xi}_2] = \delta\mathbf{j}[\boldsymbol{\xi}_1]\times\delta\mathbf{B}[\boldsymbol{\xi}_1] \\
\mathcal{L}_0[\boldsymbol{\xi}_3] & = & \ldots = \ldots
\end{array}$$

"However, since $\mathcal{L}_0$ is a singular operator .. this equation cannot, in general, be solved, .."

"leads, of course, to successively worse divergences in this perturbation theory approach which therefore breaks down .."

"we must abandon the perturbation theory approach.."

The singularity also affects Newton iterative solvers: $\mathbf{x}_{i+1} \equiv \mathbf{x}_i - \nabla\mathbf{F}^{-1}\cdot\mathbf{F}[\mathbf{x}_i]$

IDEAL MHD FAILS AT RATIONAL SURFACES

$\nabla p = \mathbf{j} \times \mathbf{B}$ yields $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$. $\qquad$ $\mathbf{j}$ is current-*density*, $\quad$ current $= \int_{\mathcal{S}} \mathbf{j} \cdot ds$.

Write $\mathbf{j} = \sigma \mathbf{B} + \mathbf{j}_\perp$, $\;\nabla \cdot \mathbf{j} = 0$ yields $\boxed{\mathbf{B} \cdot \nabla \sigma = -\nabla \cdot \mathbf{j}_\perp}$ $\qquad$ (1)

Nested flux surfaces allows (ψ, θ, ζ) s.t.

$$\begin{aligned} \mathbf{B} &= \nabla\psi \times \nabla\theta + \iota\!\!\!- \;\nabla\zeta \times \nabla\psi \\ \sqrt{g}\,\mathbf{B} \cdot \nabla &= \partial_\zeta + \iota\!\!\!- \;\partial_\theta \\ \sqrt{g}\,\mathbf{B} \cdot \nabla\zeta &= 1 \end{aligned}$$

Fourier, $\sigma \equiv \sum_{m,n} \sigma_{m,n}(\psi) e^{i(m\theta - n\zeta)}$, Eqn(1) becomes $\boxed{(\iota\!\!\!-\, m - n)\sigma_{m,n} = i(\sqrt{g}\nabla \cdot \mathbf{j}_\perp)_{m,n}}$ $\quad$ (2)

Resonant, parallel current-density : $\sigma_{m,n} = \underbrace{\dfrac{g_{m,n}(x)\, p'(x)}{x}}_{\text{Pfirsch-Schlüter}} + \Delta_{m,n} \underbrace{\delta_{m,n}(x)}_{\delta\text{-function}}$, $\quad$ where $x \equiv \iota\!\!\!- - n/m$.

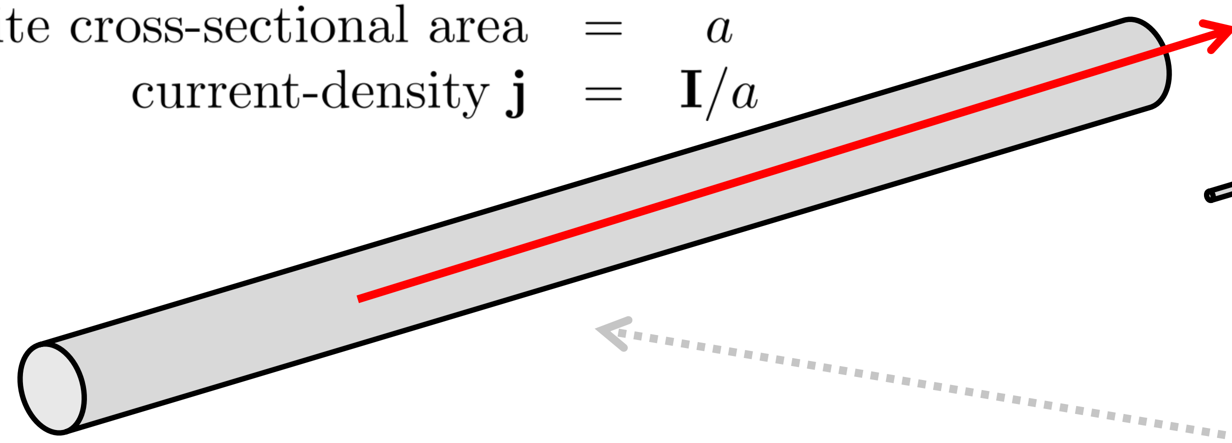
The δ-function current-density is integrable, e.g.

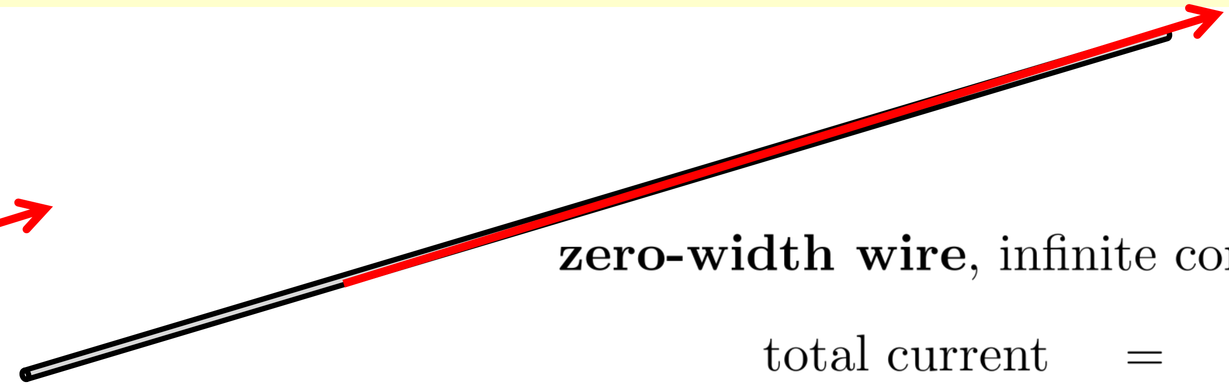
$$\int_{-\infty}^{+\infty} f(x)\delta(x)dx = f(0), \quad \int_{-\infty}^{\bar{x}} \delta(x)dx = H(\bar{x}) = \textit{Heaviside step function}, \; xH' = 0,$$

and is an acceptable mathematical idealization of localized currents.

thin wire, finite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{finite cross-sectional area} &= a \\ \text{current-density } \mathbf{j} &= \mathbf{I}/a \end{aligned}$$





zero-width wire, infinite conductivity,

$$\begin{aligned} \text{total current} &= \mathbf{I} \\ \text{zero cross-sectional area} &\to 0 \\ \text{current-density } \mathbf{j} &\to \mathbf{I}\,\delta(x) \\ \mathbf{B} &= \int_{\mathcal{V}} \frac{\mathbf{j}\times\mathbf{r}}{r^3} dv \\ &= \int_{\mathcal{V}} \frac{\mathbf{I}\,\delta(\mathbf{x})\times\mathbf{r}}{r^3} dv \end{aligned}$$

$L \gg a$

Approximating a localized current-density by a δ-function current density

1. is acceptable for a **macroscopic** physical model that assumes **infinite conductivity**, and
2. is mathematically-tractable (one just needs to accommodate discontinuous solutions).

$$\begin{aligned} \text{Net current through cross-section} \quad \int_{\mathcal{S}} \mathbf{j}\cdot d\mathbf{s} &= \int d\psi \int d\theta \, \sqrt{g}\, \mathbf{j}\cdot\nabla\zeta \\ &= \int_{-\epsilon}^{+\epsilon} dx \int_0^{2\pi} d\theta \;\; \Delta_{m,n}\; \delta_{m,n}(x)\; e^{i(m\theta-n\zeta)} \sqrt{g}\,\mathbf{B}\cdot\nabla\zeta \\ \sqrt{g}\,\mathbf{B}\cdot\nabla\zeta = 1 \qquad &= 0 \\ i.e. \quad &\text{no discontinuity in rotational-transform} \end{aligned}$$

The pressure-driven 1/x current density gives infinite parallel currents through certain surfaces.

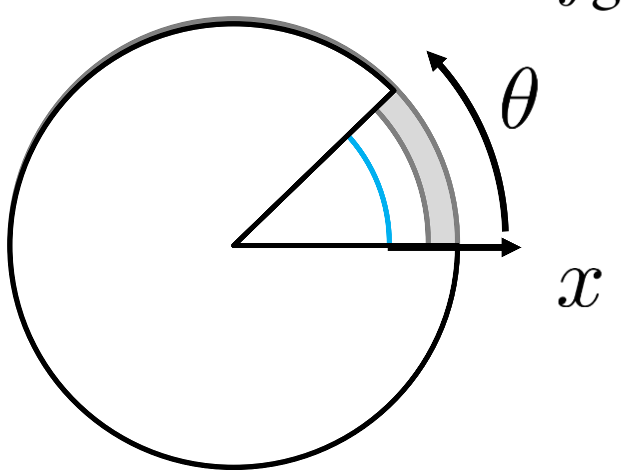
Parallel current-density $$\mathbf{j}_\parallel = \sum_{m,n}\left[\frac{g_{m,n}\,p'}{x} + \Delta_{m,n}\delta_{m,n}(x)\right] e^{(im\theta - in\zeta)}\mathbf{B}.$$

Parallel current through cross-section $$\int_{\mathcal{S}} \mathbf{j}_\parallel \cdot d\mathbf{s} = \int d\psi \int d\theta \;\sqrt{g}\,\mathbf{j}_\parallel \cdot \nabla\zeta$$

$$= \int_\epsilon^\delta dx \int_0^{\pi/m} d\theta \;\frac{g_{m,n}\,p'}{x} e^{i(m\theta-n\zeta)}\sqrt{g}\,\mathbf{B}\cdot\nabla\zeta$$

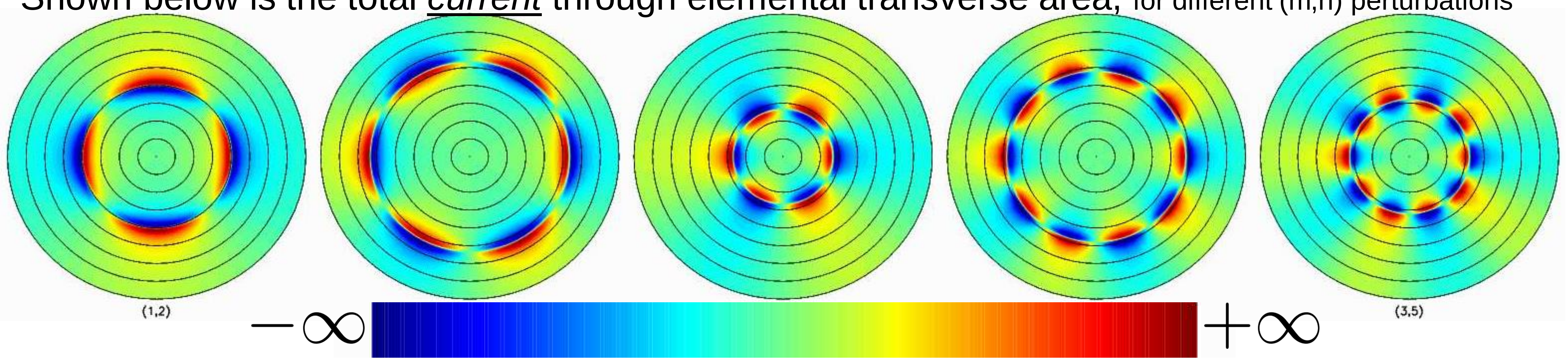
$$= g_{m,n,0}\,p'_0 \,\frac{2}{m}\int_\epsilon^\delta dx \,\frac{1}{x}$$

$$= g_{m,n,0}\,p'_0 \,\frac{2}{m}\left(\ln\delta - \ln\epsilon\right) \to \infty \text{ as } \epsilon \to 0.$$

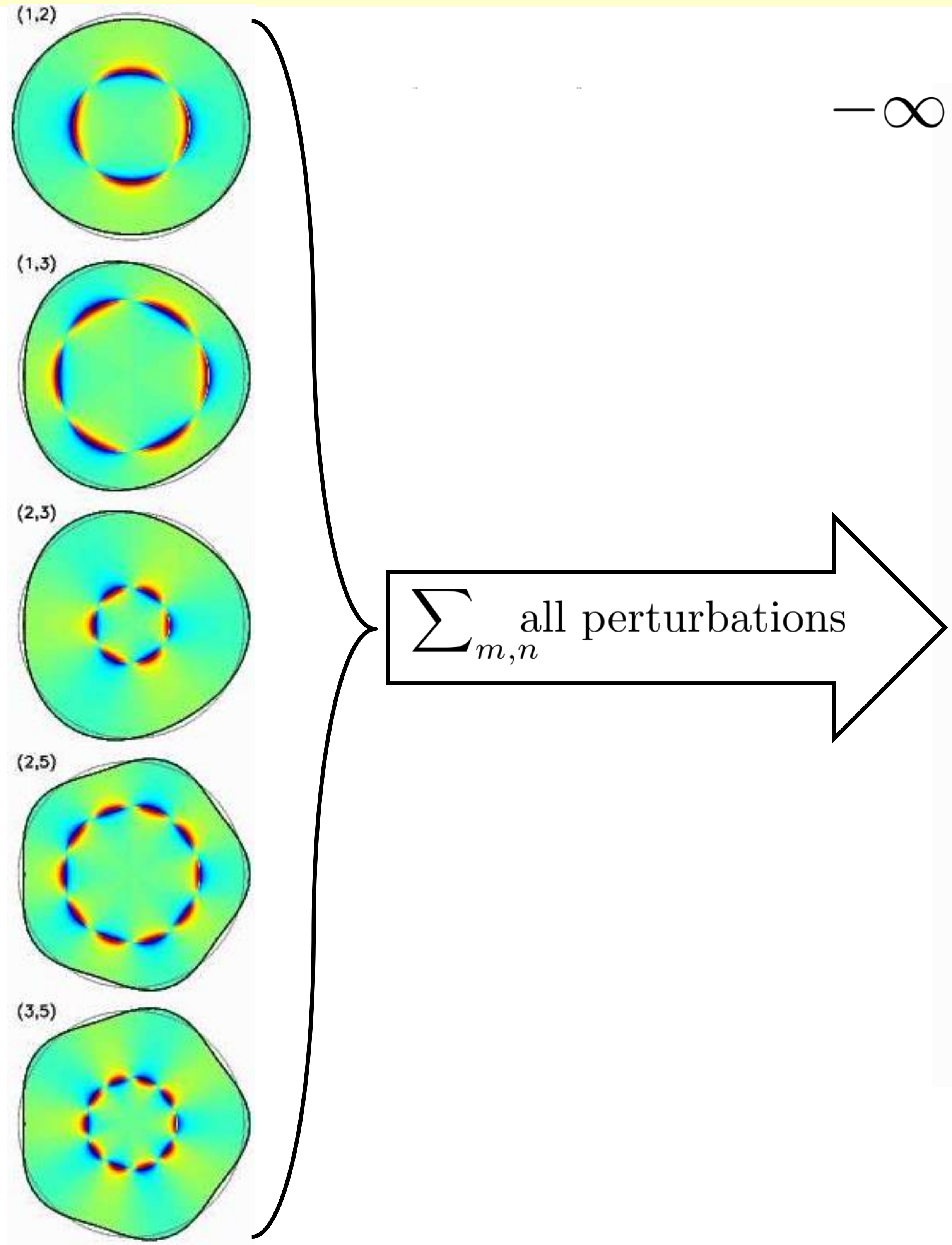


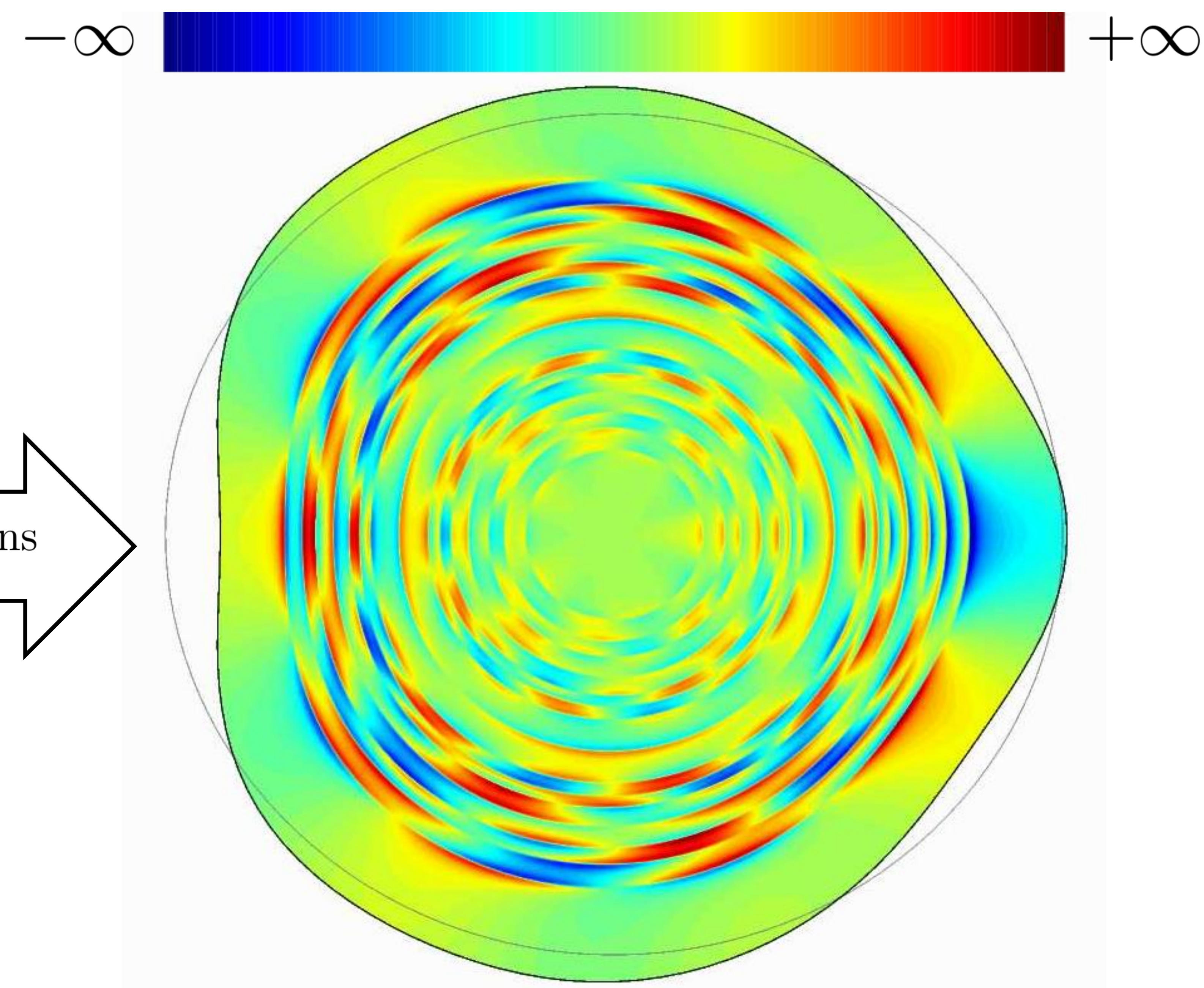
The problem is *NOT* a lack of numerical resolution.
Is a dense collection of alternating infinite currents physical?

Shown below is the total <u>current</u> through elemental transverse area, for different (m,n) perturbations



In arbitrary, three-dimensional geometry, "solutions" to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with smooth profiles and nested surfaces are nonsense.





Dense collection of alternating infinite currents is not an acceptable solution.

If there are rational surfaces, then we must choose:

1. flatten pressure near rationals, smooth pressure; ✗
2. flatten pressure near rationals, fractal pressure; ✗
3. flatten pressure near rationals, discontinuous pressure; ✓
4. restrict attention to "healed" configurations [Weitzner, PoP **21**, 022515, 2014], [Zakharov, JPP **81**, 515810609, 2015]

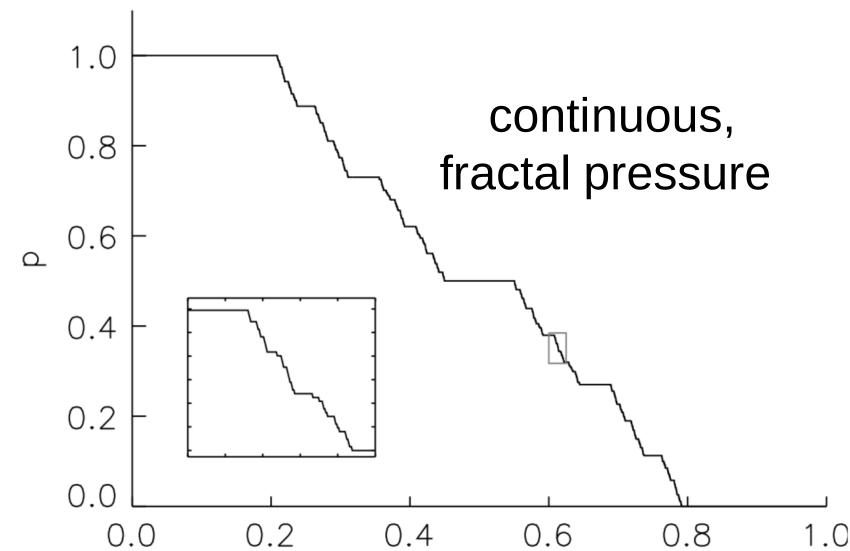
1. Locally-flattened, smooth pressure:

if (*i.*) $p'(x) = 0$ if $|x - n/m| < \epsilon_{m,n}$, $\forall(n,m)$,

and (*ii.*) $p'(x)$ is continuous, then $p'(x) = 0$, $\forall x$. **No pressure!**

2. "Diophantine" pressure profile: e.g. from KAM theory

$$p'(x) = \begin{cases} 1, & \text{if } |x - n/m| > r/m^k, \quad \forall(n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |x - n/m| < r/m^k, \quad \exists(n,m), \end{cases}$$



$p'(x)$ is discontinuous on an uncountable infinity of points,

Not computationally tractable.

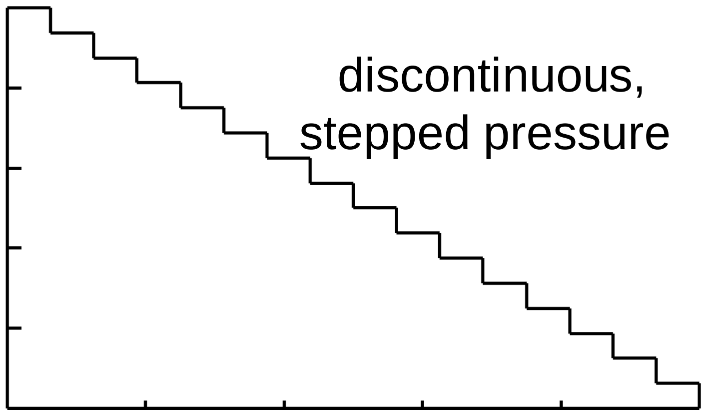
e.g. cannot constrain topology of non-integrable **B** to match fractal pressure

"The function p is continuous but its derivative is pathological." Grad, Phys. Fluids 10, 137 (1967)]

3. "Stepped" pressure profile: ✓

Existence of Three-Dimensional Toroidal MHD Equilibria with Nonconstant Pressure
[Bruno & Laurence, Commun. Pure Appl. Math. **49**, 717 (1996)]

" . . our theorems insure the existence of sharp boundary solutions for tori
whose departure from axisymmetry is sufficiently small;
they allow for solutions to be constructed with an arbitrary number of pressure jumps."



Culmination of long history of "waterbag" or "sharp-boundary" equilibria
[Potter, "Waterbag methods in magnetohydrodynamics", Methods in Computational Physics, **16**, 43 (1976)]
[Berk et al., Phys. Fluids, **29**, 3281 (1986)]
[Kaiser & Salat Phys. Plasmas **1**, 281 (1994)]

Relaxed MHD $\leftarrow$ Multi-Region relaxed MHD $\rightarrow$ Ideal MHD

[Taylor, Phys. Rev. Lett. **33**, 1139 (1974)] [Dewar, Hole, Hudson, et al., circa 2006] [Kruskal & Kulsrud, Phys. Fluids **1**, 265 (1958)]

$N_V = 1$ Relaxed MHD

$$\mathcal{F} \equiv \underbrace{\int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv}_{energy} - \frac{\mu}{2} \underbrace{\int_{\mathcal{R}} \mathbf{A} \cdot \mathbf{B} \, dv}_{helicity},$$

$\delta \mathbf{B} \equiv \nabla \times \delta \mathbf{A}$ is arbitrary in $\mathcal{R}$
$(\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ on $\partial \mathcal{R})$
$+$ constrained flux

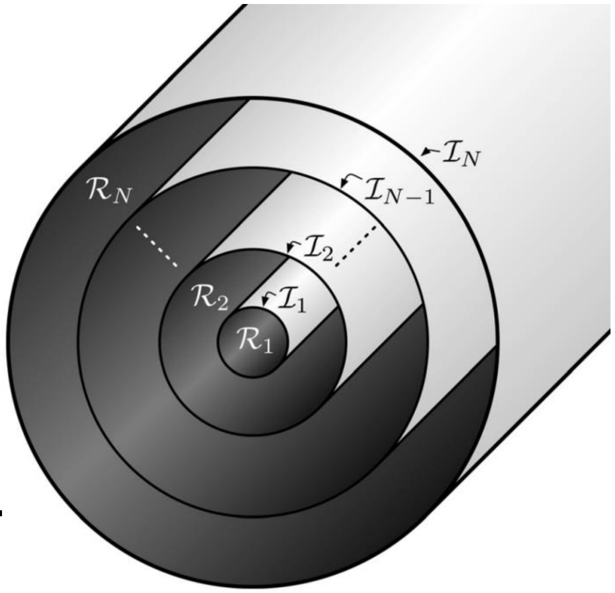
$$\delta \mathcal{F} = 0, \quad p = p_0, \quad \nabla \times \mathbf{B} = \mu \mathbf{B} \ \text{in } \mathcal{R};$$

$N_V < \infty$ MRx MHD

$$\mathcal{F} \equiv \sum_{i=1}^{N_V} \left\{ \int_{\mathcal{R}_i} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv - \frac{\mu_i}{2} \int_{\mathcal{R}_i} \mathbf{A} \cdot \mathbf{B} \, dv \right\},$$

$\delta \mathbf{B}_i \equiv \nabla \times \delta \mathbf{A}_i$ is arbitrary in $\mathcal{R}_i$
$\delta \mathbf{B}_i = \nabla \times (\boldsymbol{\xi} \times \mathbf{B}_i)$ on $\partial \mathcal{R}_i$
$+$ constrained fluxes in $\mathcal{R}_i$

$$\delta \mathcal{F} = 0, \quad p = p_i, \quad \nabla \times \mathbf{B} = \mu_i \mathbf{B} \ \text{in } \mathcal{R}_i; \quad \left[\left[p + \frac{B^2}{2} \right]\right] = 0 \text{ across } \partial \mathcal{R}_i;$$



$\boxed{\textbf{S}\text{tepped } \textbf{P}\text{ressure } \textbf{E}\text{quilibrium } \textbf{C}\text{ode}}$

[Hudson, Dewar et al., Phys. Plasmas **19**, 112502 (2012)]

$N_V = \infty$ Ideal MHD

$$\mathcal{F} \equiv \int_{\mathcal{R}} \left(\frac{p}{\gamma - 1} + \frac{B^2}{2} \right) dv,$$

$\delta \mathbf{B} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ in $\mathcal{R}$
(fluxes & helicity conserved)

$$\delta \mathcal{F} = 0, \quad p = p(\psi), \quad \nabla p = \mathbf{j} \times \mathbf{B} \ \text{in } \mathcal{R}.$$

Compute the 1/x and δ-function current densities in perturbed geometry
Self-consistent solutions require **infinite shear**

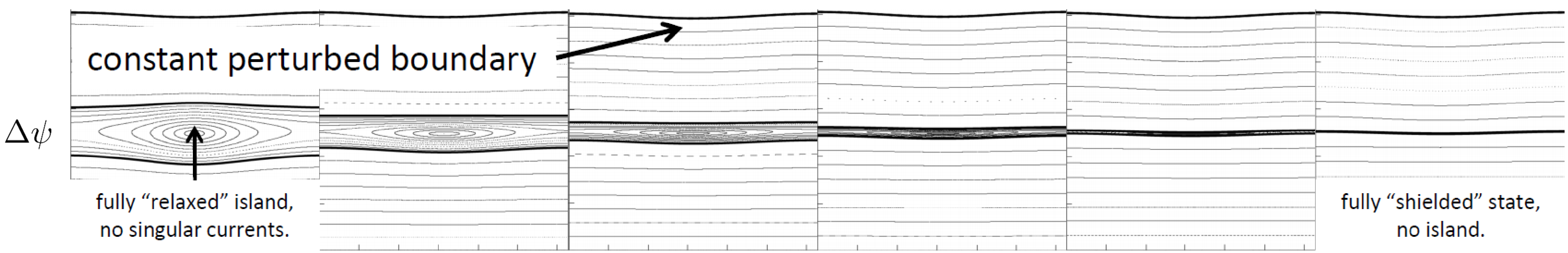
Cartesian, slab geometry with an $(m,n) = (1,0)$ resonantly-perturbed boundary

i. $N_V = 3$ MRxMHD calculation, no pressure, $\iota(\psi)$ given discretely,

ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$, shear $\equiv \Delta\iota/\Delta\psi = x^{\alpha-\beta}$, $\boxed{\beta > \alpha}$.

iii. island forced to vanish,

iv. resonant $\delta_{m,n}$-function current-density appears as tangential discontinuity in $\mathbf{B}$.

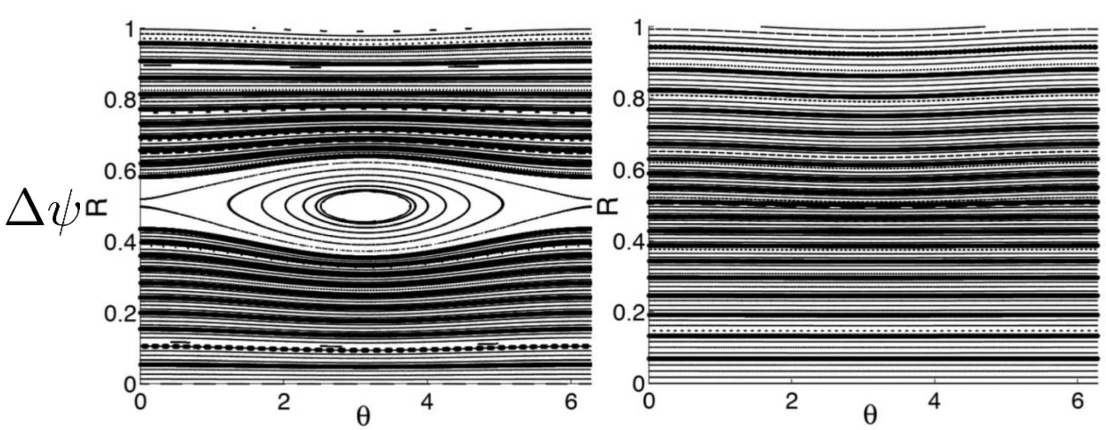


i. $N_V =$ large MRxMHD calculation, stepped pressure ≈ smooth pressure,

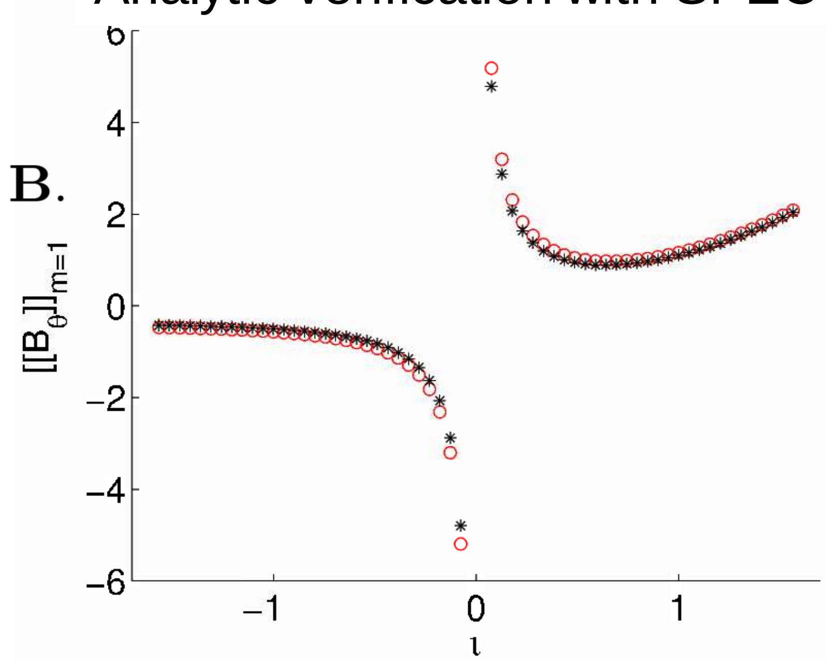
ii. take limit $\Delta\psi \equiv x^\beta$, $\iota_i = -x^\alpha/2$, $\iota_{i+1} = +x^\alpha/2$,

iii. island forced to vanish,

iv. resonant p'/x current-density appears as tangential discontinuity in $\mathbf{B}$.



Analytic verification with SPEC



[Loizu, Hudson et al., Phys. Plasmas **22**, 022501 (2015)]

Infinite gradient ≈ discontinuity.

Introduce new solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with discontinuous transform

1. Cylindrical geometry with an $(m, n) = (2, 1)$ resonantly-perturbed boundary

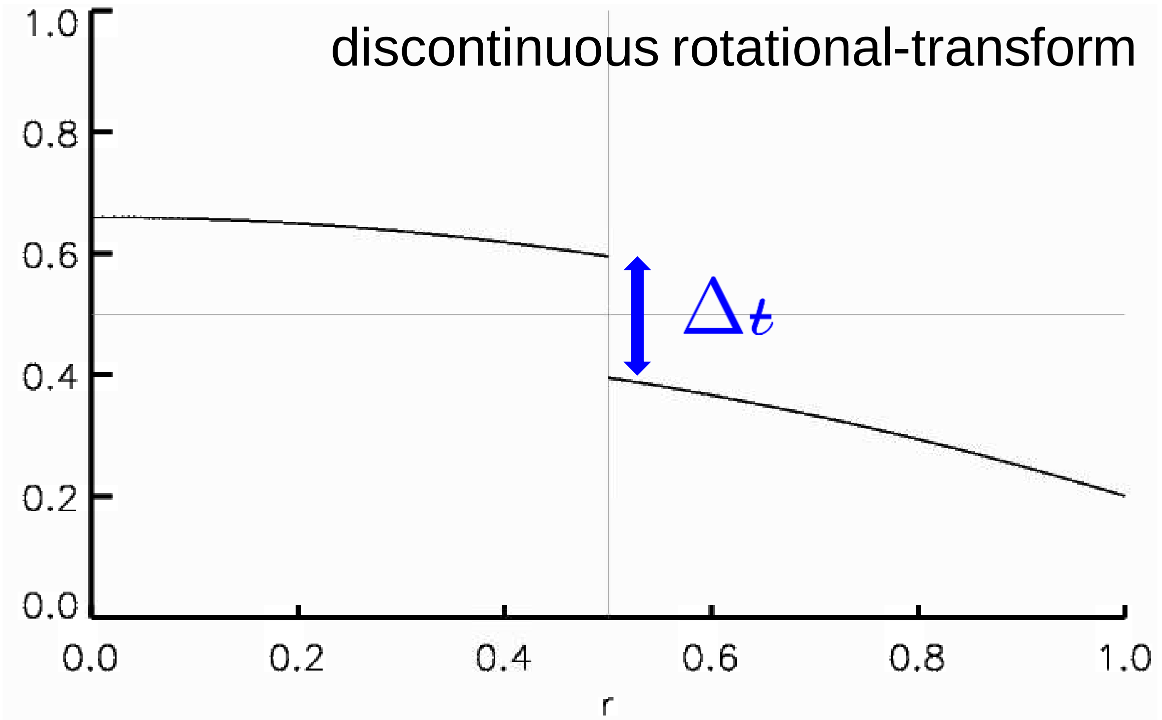
   i. $\boxed{p = 0,}$ $\iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota,$

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1 + \iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \boxed{} - \delta\mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta\mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
   for $\Delta\iota \to 0$, recover step-function solution



Perturbation penetrates into the core

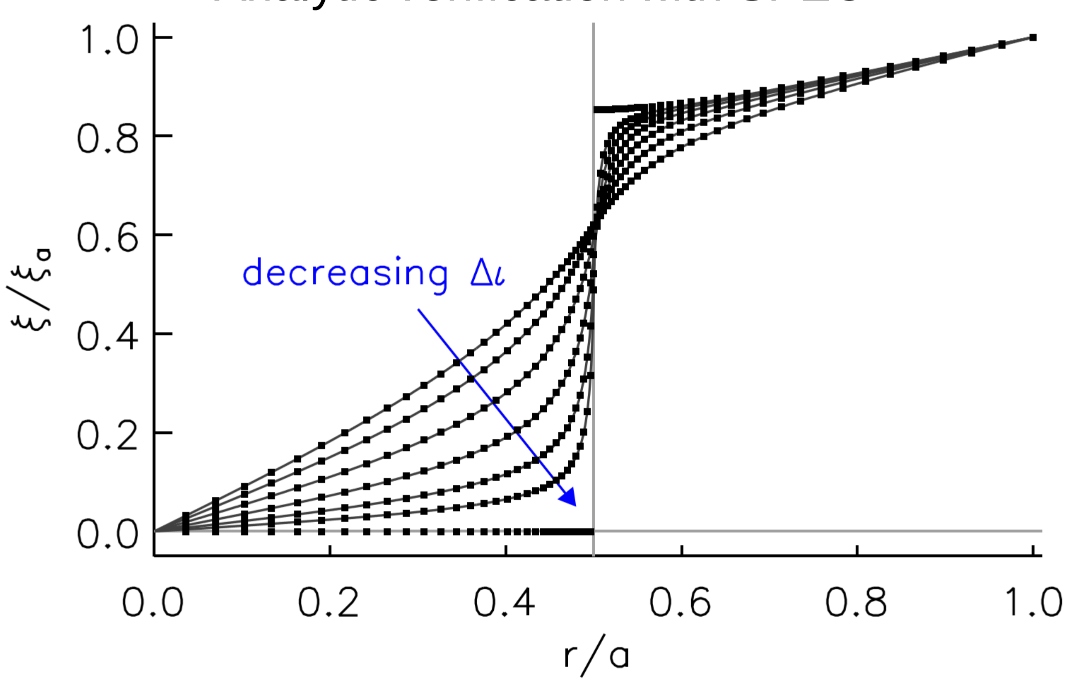
2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$

Analytic verification with SPEC

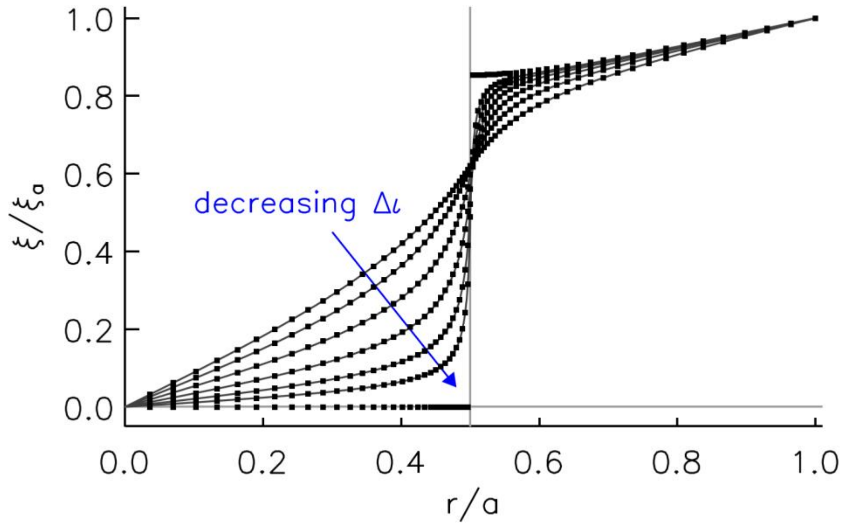


[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

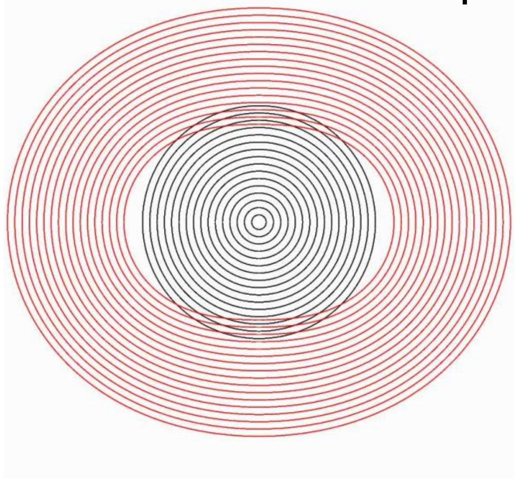
Necessary condition for non-overlapping of perturbed surfaces
Existence of non-linear solutions

1. **Condition for non-overlapping perturbed surfaces**

$$\left|\frac{\partial \xi}{\partial r}\right|_{max} < 1$$



Discontinuously-perturbed flux surfaces overlap!

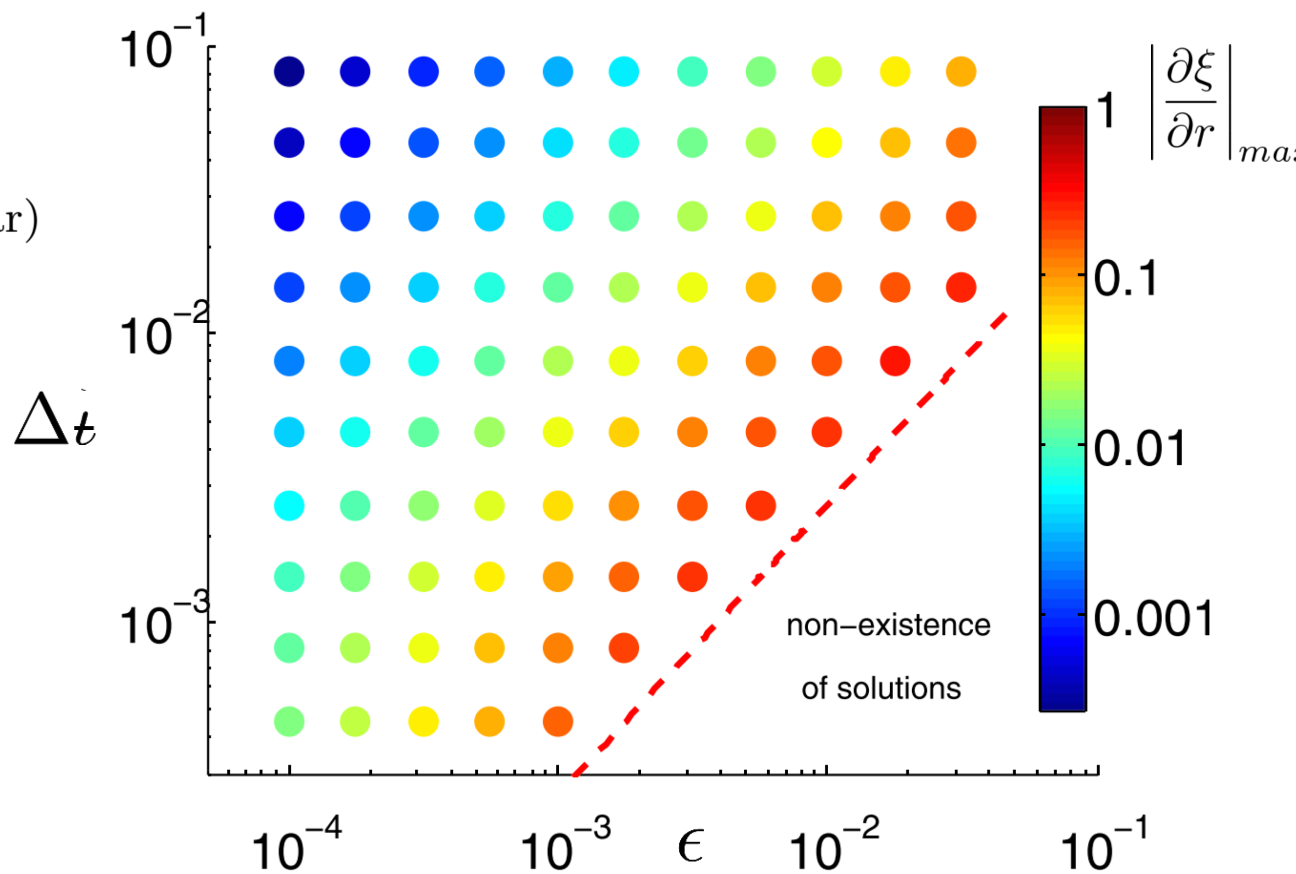


2. **An asymptotic analysis near the rational surface**
gives the *sine-qua-non* condition *(an indispensable condition, element, or factor; something essential)*

$$\Delta\iota > \Delta\iota_{min}, \quad \text{where } \Delta\iota_{min} \equiv 2\iota_s' \xi_s$$

(analysis for cylindrical, zero-β; general result probably similar)

3. **If this condition is violated, non-linear solutions do not exist.**
   i. Shown is ξ', as computed using non-linear SPEC calculations, as a function of $(\epsilon, \Delta\iota)$
   ii. SPEC fails in ideal-limit, i.e. $N_V \rightarrow \infty$, when $\Delta\iota < \Delta\iota_{min}$



[Loizu, Hudson et al., Phys. Plasmas **22**, 090704 (2015)]

Infinite gradient ≈ discontinuity.

Introduce new solutions to $\nabla p = \boldsymbol{j} \times \boldsymbol{B}$ with discontinuous transform & pressure

1. Cylindrical geometry with an $(m,n) = (2,1)$ resonantly-perturbed boundary

   i. $\boxed{p = p_0(1 - 2r^2 + r^4)}$, $\iota(r) = \iota_0 - \iota_1 r^2 \pm \Delta\iota$,

   ii. compute cylindrically symmetric equilibrium
   $$\frac{dp}{dr} + \frac{1}{2}\frac{d}{dr}\left[B_z(1+\iota^2 r^2)\right] + r\iota^2 B_z^2 = 0$$

   iii. compute linearly perturbed equilibrium:
   $$\mathcal{L}_0[\boldsymbol{\xi}] \equiv \boxed{\nabla \delta p} - \delta \mathbf{j}[\boldsymbol{\xi}] \times \mathbf{B}_0 - \mathbf{j}_0 \times \delta \mathbf{B}[\boldsymbol{\xi}] = 0$$
   for $\Delta\iota > 0$, $\mathcal{L}_0$ is non-singular,

   iv. solved analytically
   $$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0$$

   v. for $\Delta\iota > \Delta\iota_{min}, \partial\xi/\partial r < 1$, non-overlapping perturbed surfaces
   for $\Delta\iota > 0$, $\boldsymbol{\xi}$ is continuous and smooth,
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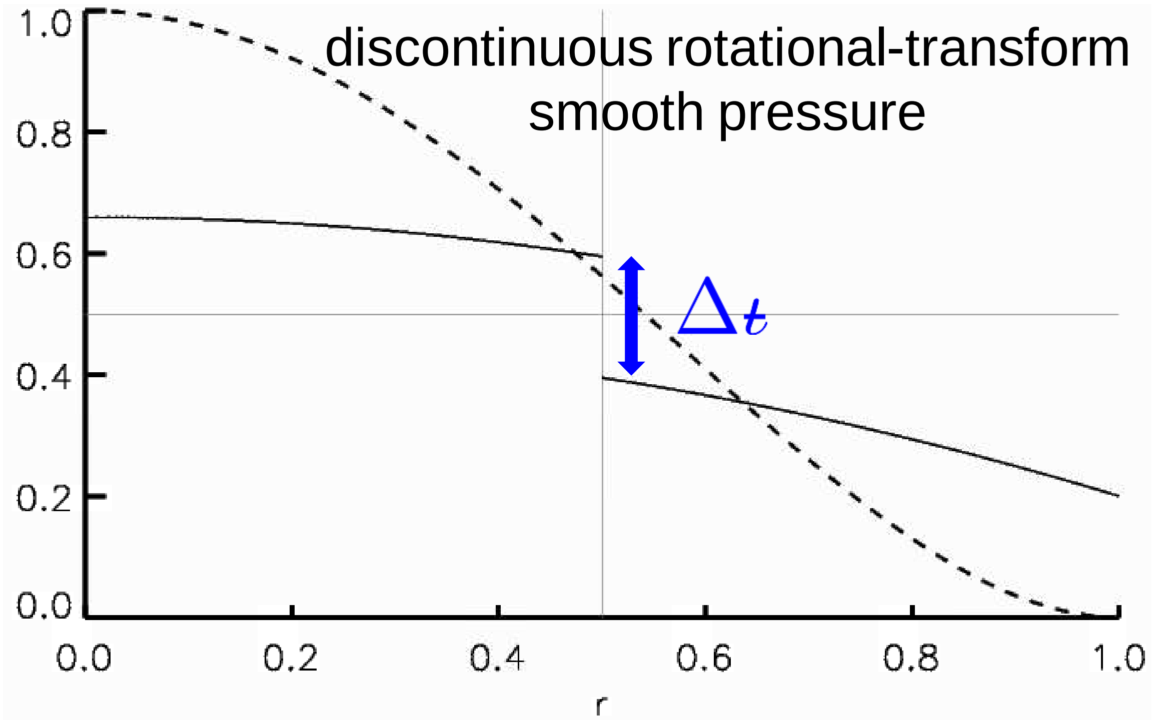


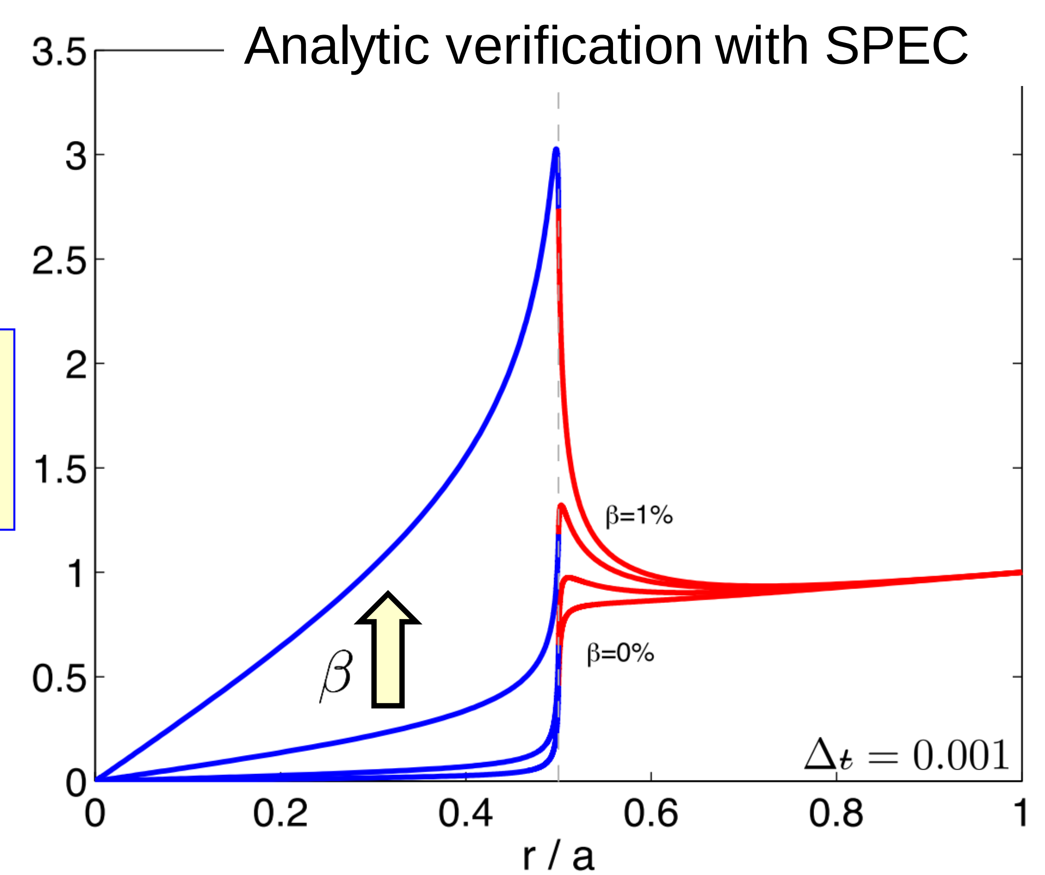
Perturbation amplified by pressure near and inside "resonant" surface

2. Comparison with SPEC

   i. construct large N_V MRxMHD calculation,

   ii. "linearized" SPEC calculation: $||\boldsymbol{\xi}_{exact} - \boldsymbol{\xi}_{linear}|| \sim N_V^{-1}$

   iii. nonlinear SPEC calculation: $||\boldsymbol{\xi}_{linear} - \boldsymbol{\xi}_{nonlinear}|| \sim \epsilon^2$



[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

SPEC	allows	discontinuous	profiles: exact	agreement
VMEC	assumes	smooth	profiles: approximate	agreement

1. VMEC assumes smooth profiles

   and smooth profiles imply discontinuous displacement

2. but, VMEC enforces nested flux surfaces

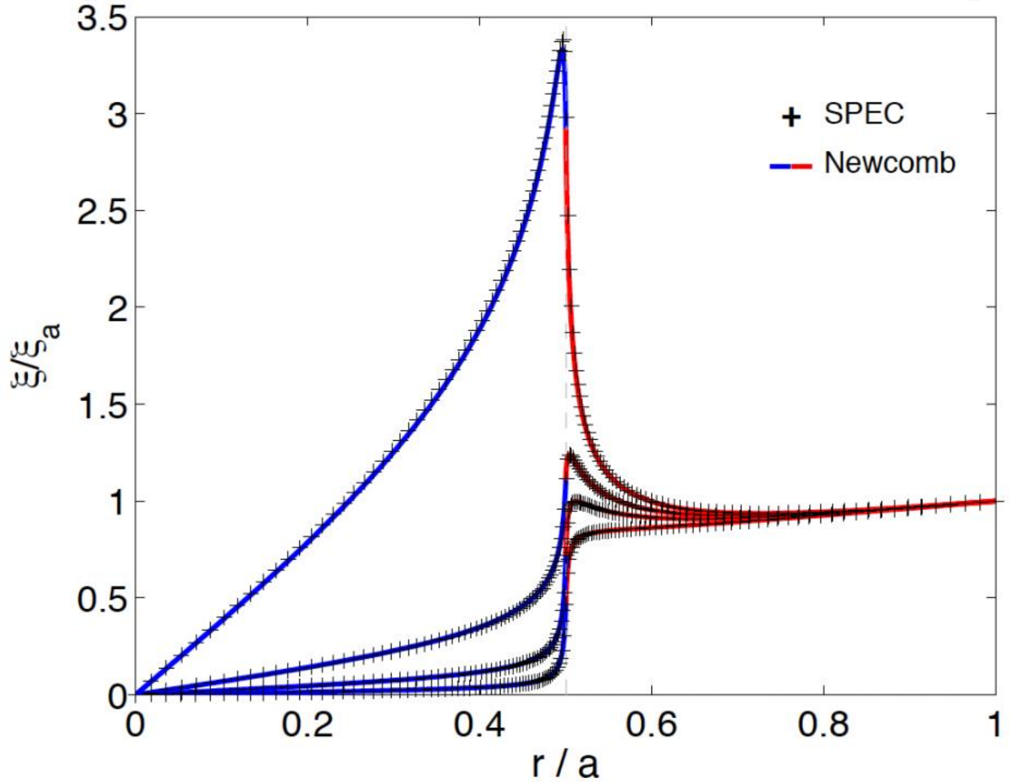
   nested flux surfaces in 3D imply $\dfrac{\partial \xi}{dr} < 1$ displacement from 2D

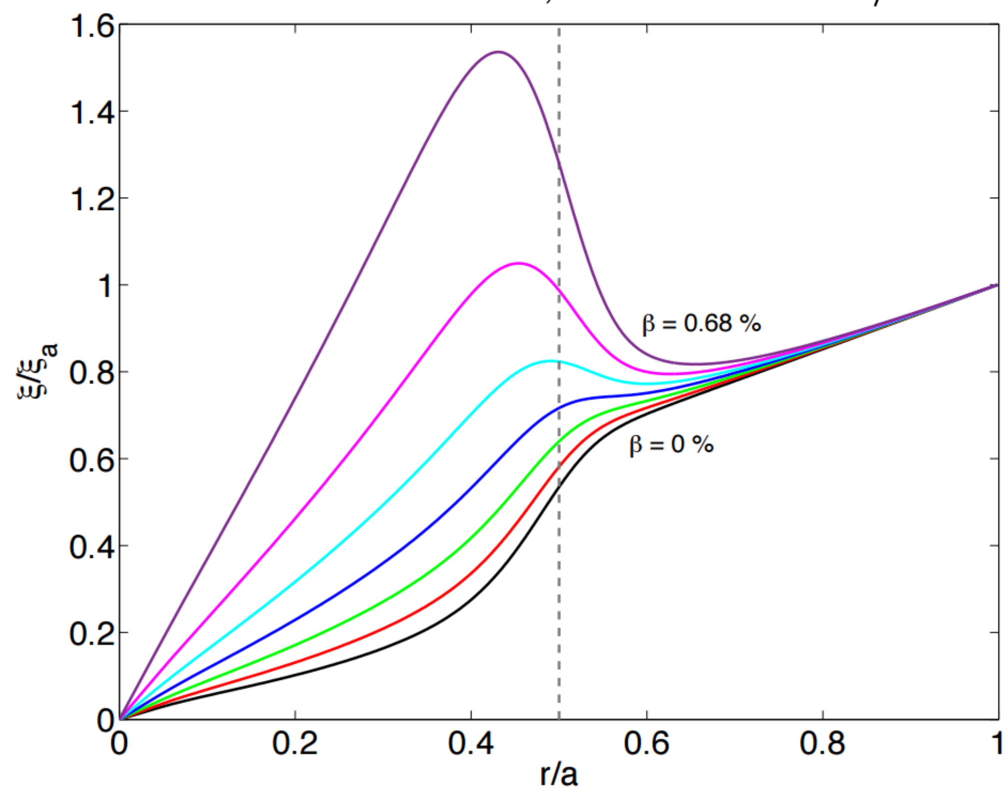
   and this is consistent only with discontinuous transform with $\Delta\iota > \Delta\iota_{min}$

3. Empirical study (i.e. radial convergence) shows that

   VMEC qualitatively reproduces self-consistent, perturbed solution

   interpretation: finite radial resolution implies an "effective" $\Delta\iota \sim \iota' h$, where $h \equiv 1/N$?





[Loizu, Hudson, …, Lazerson, ..., Phys. Plasmas **23**, 055703 (2016)]

Convergence studies using VMEC

[Lazerson, Loizu et al., Phys. Plasmas **23**, 012507 (2016)]

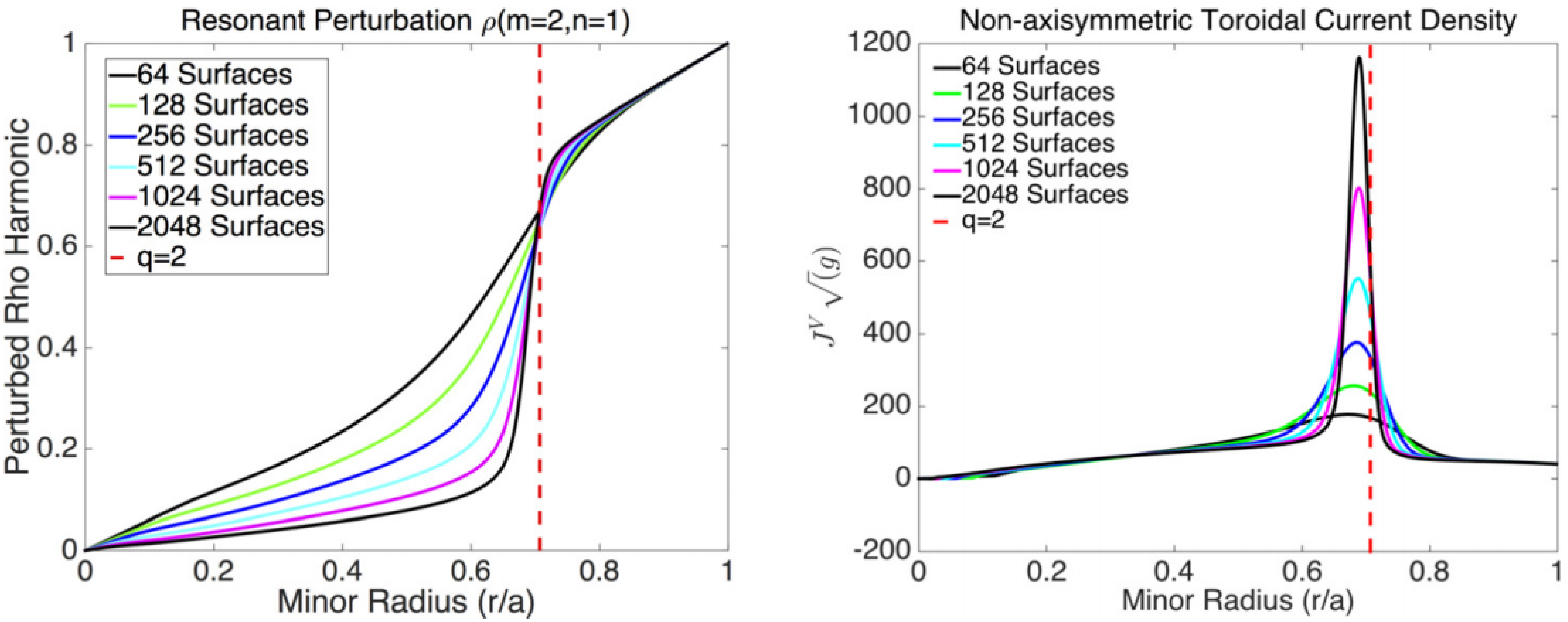


FIG. 2. Profile of the perturbed ρ harmonic (left) and the $m=2\ n=1$ component of the toroidal current density (right) showing dependence on radial resolution at fixed shear. Boundary perturbation 1×10^{-4} of minor radius. The $q=2$ surface is located at $s=0.5$ $(r/a \sim 0.7)$ in this plot. Note that the toroidal current density includes a Jacobian factor.

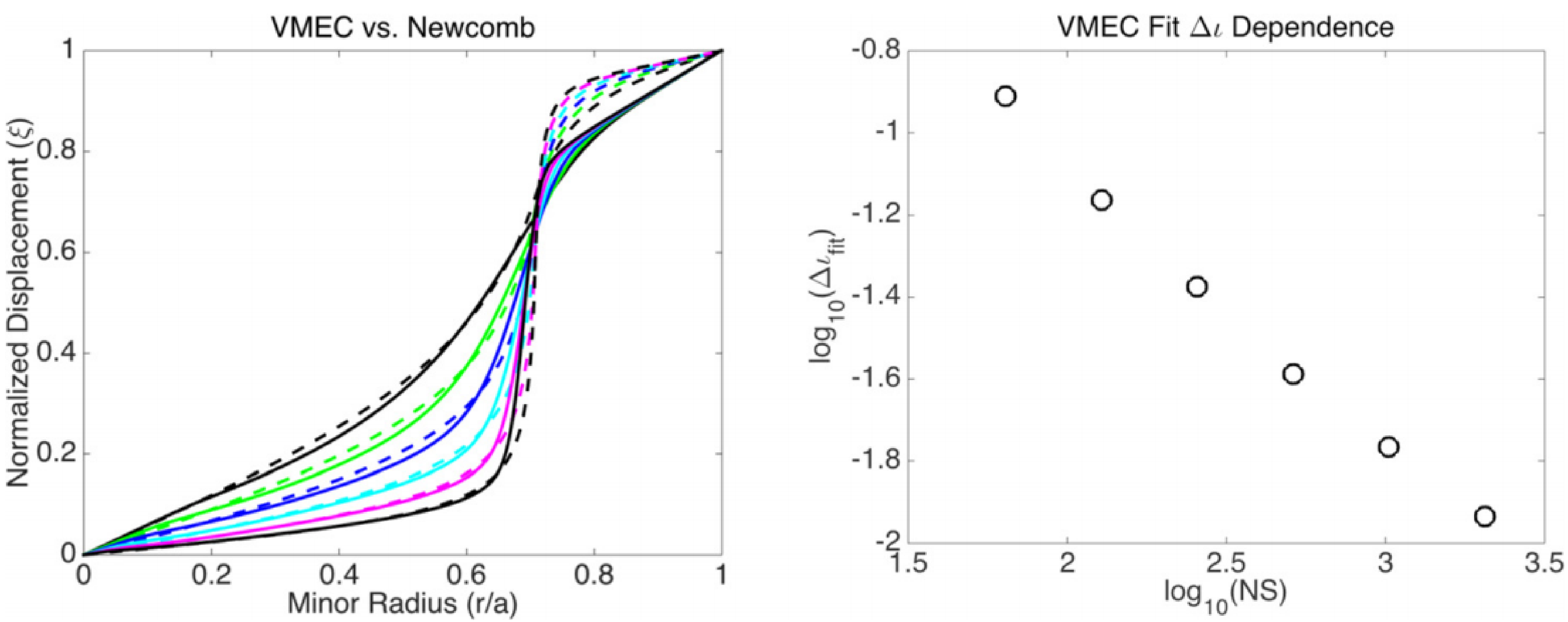
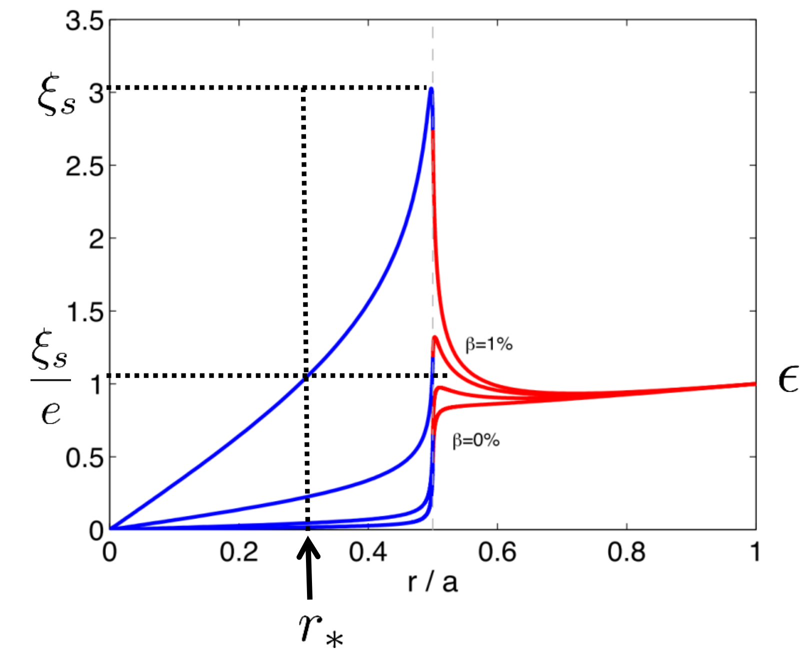


FIG. 5. Comparison of VMEC response (solid) to Loizu's solution to Newcomb's equation (dotted) (left) and the effective $\Delta\iota$ necessary to fit each curve (right). The colors are the same as those in Figure 2, and NS refers to the number of radial grid points.

Amplification and penetration as stability boundary is approached

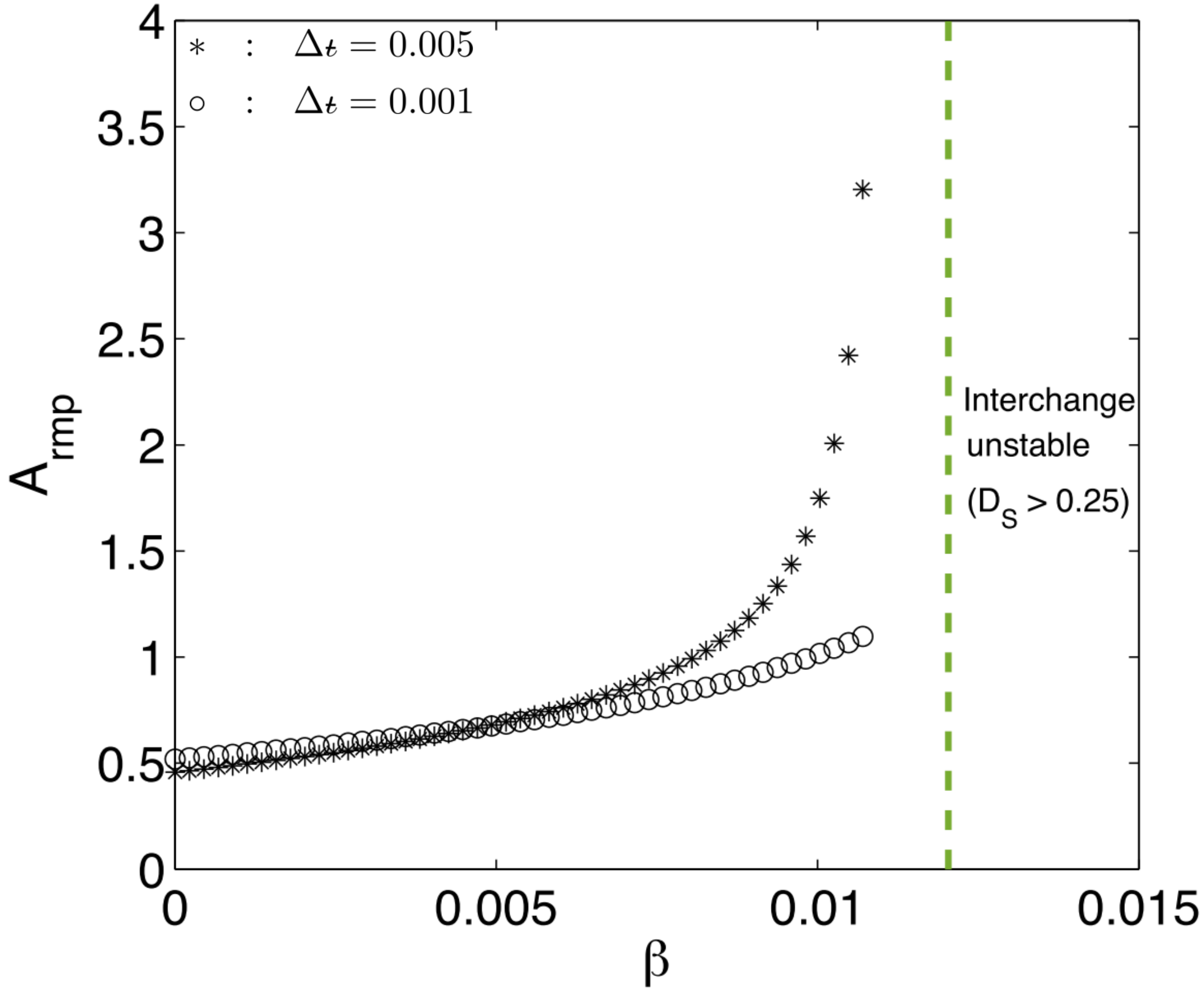
1. Can define a measure of

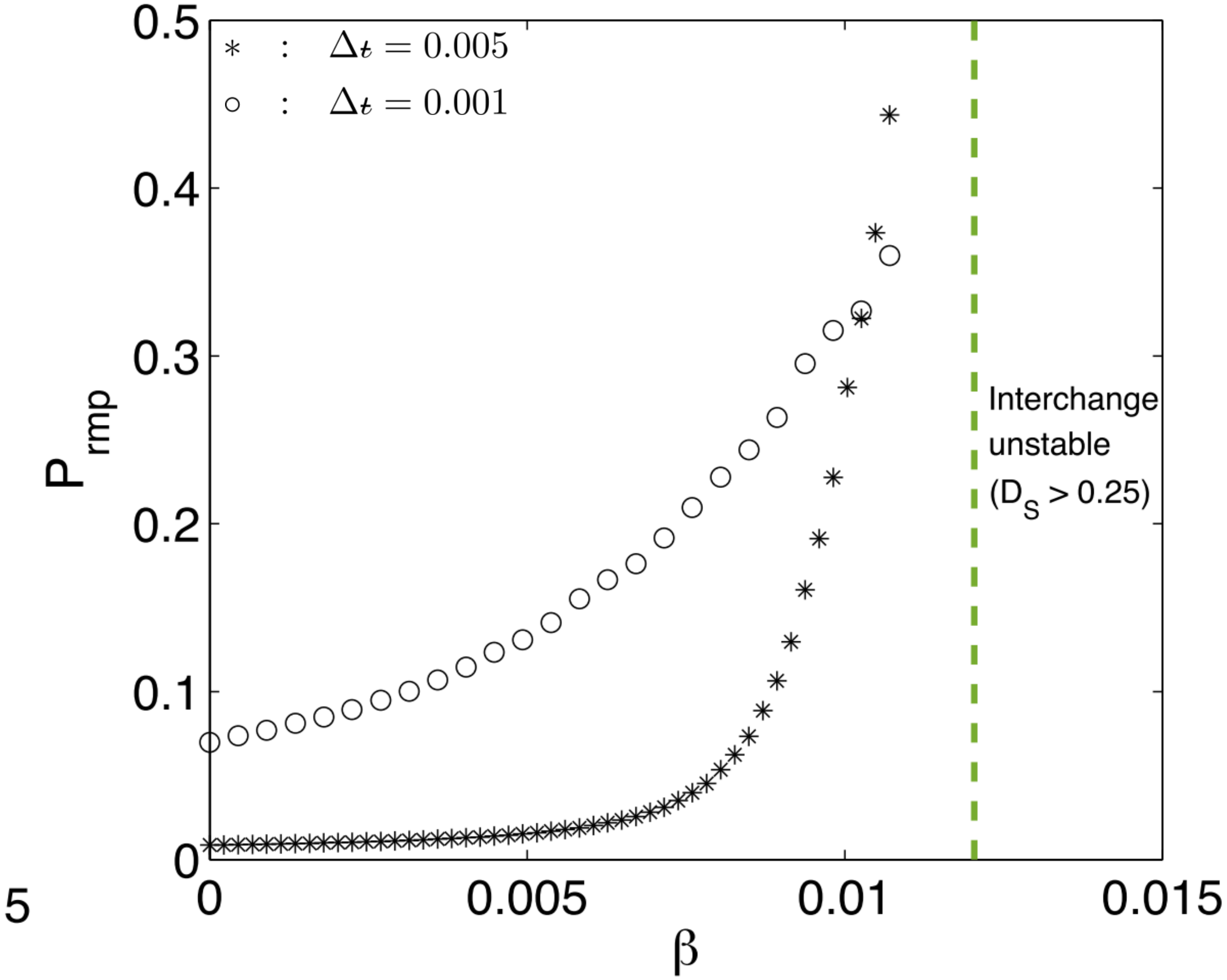
$$\text{"Amplification"} \quad A_{rmp} = \xi_s/\epsilon, \quad \text{where } \epsilon \equiv \text{boundary deformation}$$
$$\text{"Penetration"} \quad P_{rmp} = 1 - r_*/r_s, \quad \text{where } \xi(r_*) \equiv \xi_s/e$$



2. A necessary condition for interchange stability in a screw pinch is given by the Suydam criterion, $\boxed{D_S \equiv -\left(\frac{2p'\iota^2}{rB_z^2\iota'^2}\right)_s < \frac{1}{4}.}$

3. Amplification and penetration of RMP **fantastically increased** as stability limit approached.





[Loizu, Hudson et al., Phys. Plasmas **23**, 055703 (2016)]

Now, including pressure and an island . . .
Amplification and penetration of the RMP is still present.

1. Now, include a "relaxed" region,

   i. $\Delta\psi_t \equiv$ toroidal flux in relaxed region.

   ii. $\Delta\iota\!\!\!-\equiv$ jump in transform across relaxed region.

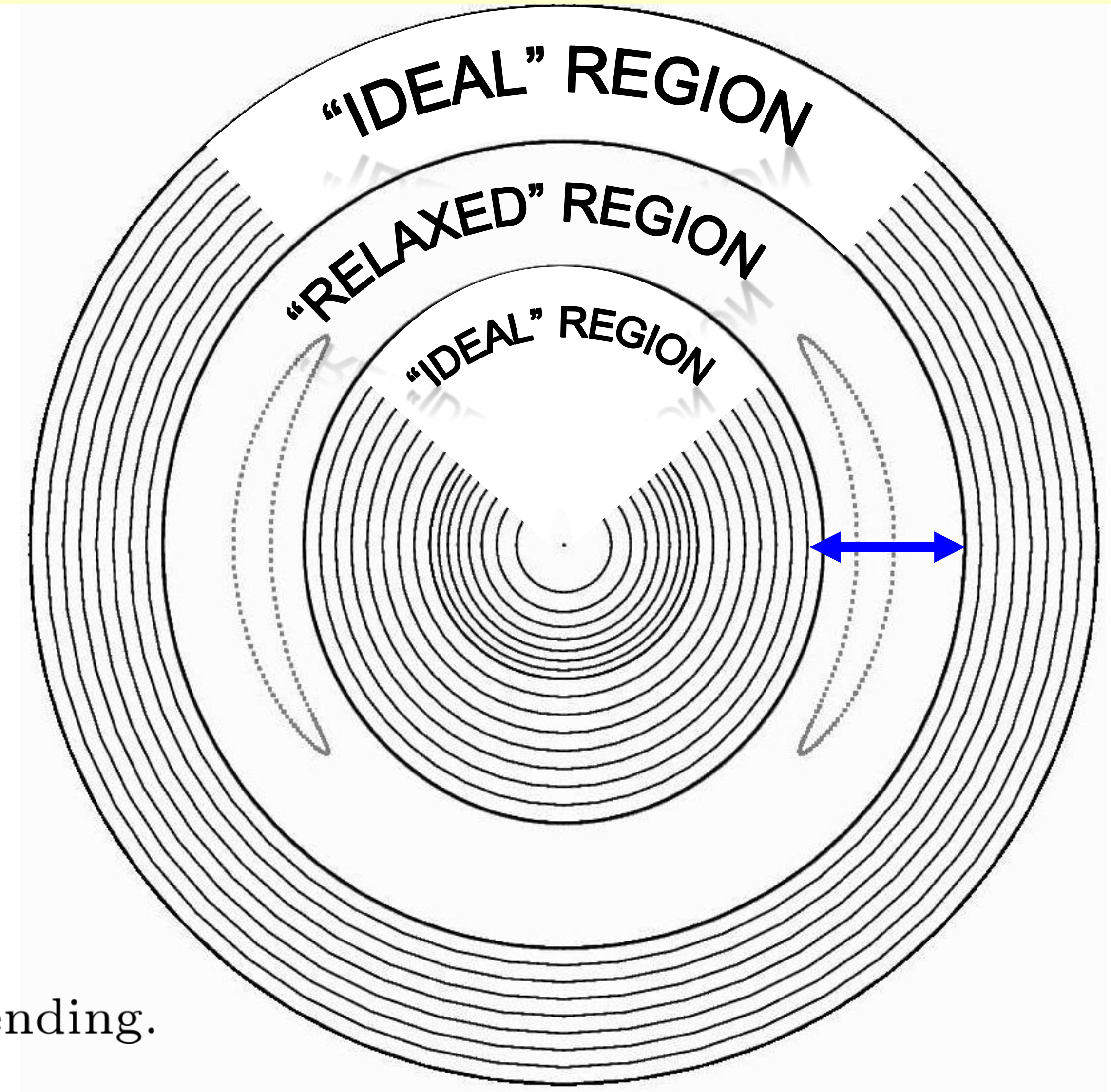
   so that an island is allowed to form.

2. SPEC calculations indicate that

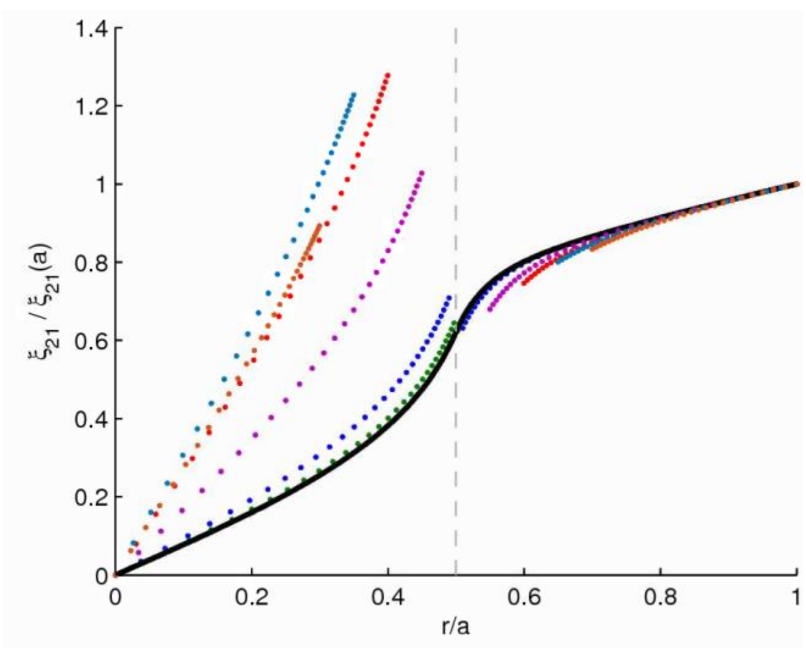
   i. The perturbation still penetrates.

   ii. The perturbation is still amplified by pressure.

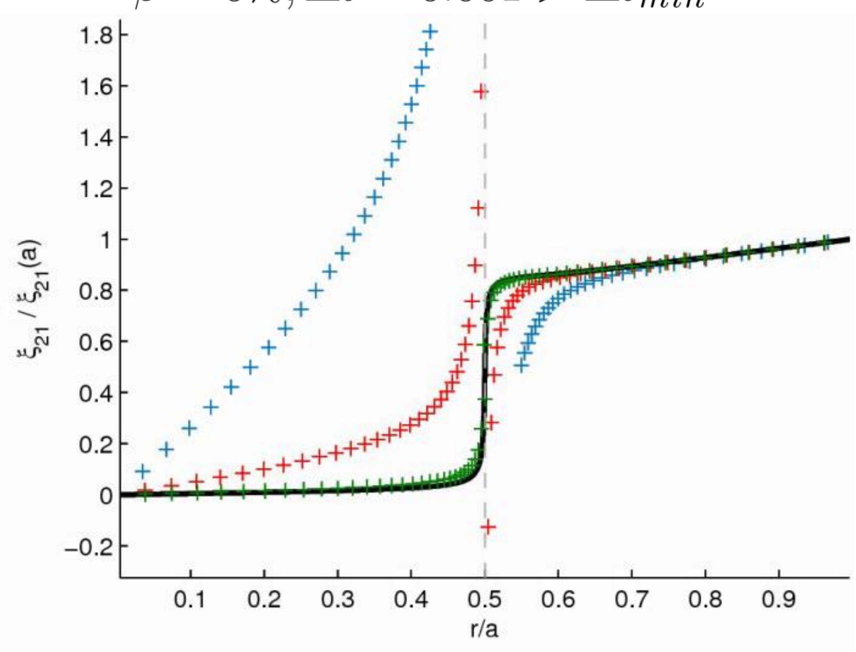
3. Precise comparison of SPEC cf. tearing mode theory pending.

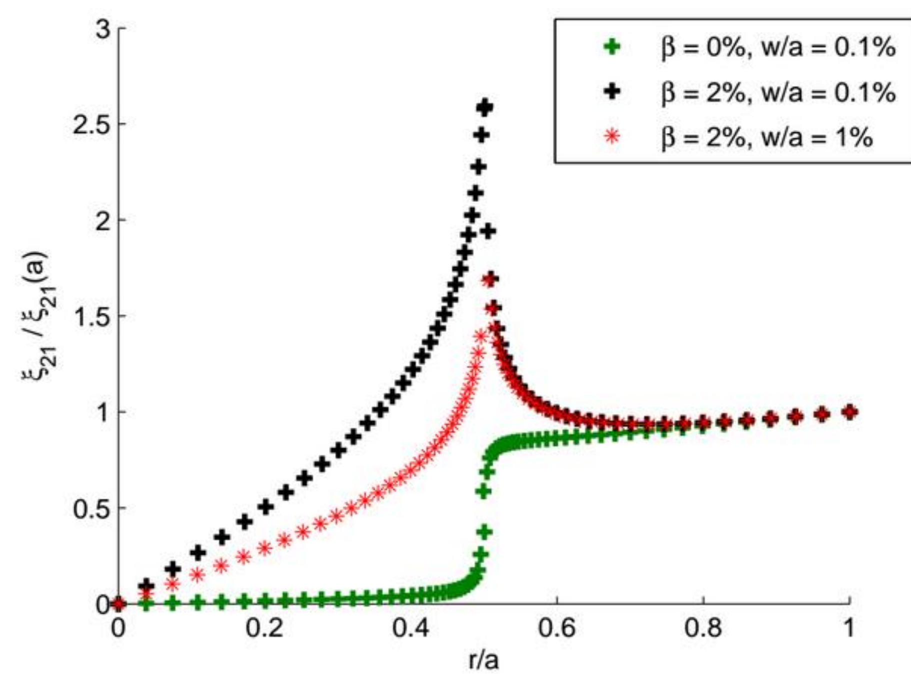


$\beta = 0\%, \Delta\iota\!\!\!- = 0.050 > \Delta\iota\!\!\!-_{min}$



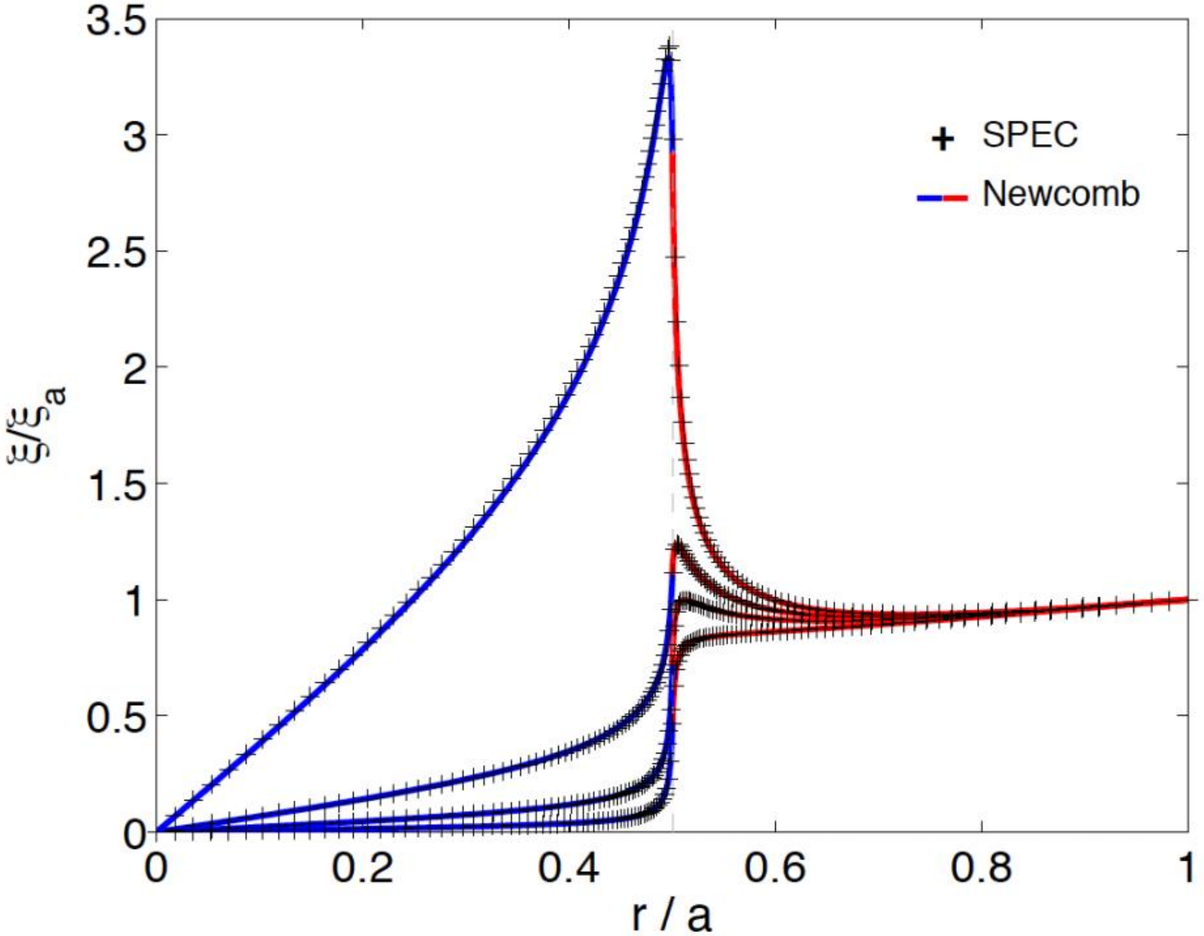
$\beta = 0\%, \Delta\iota\!\!\!- = 0.001 > \Delta\iota\!\!\!-_{min}$



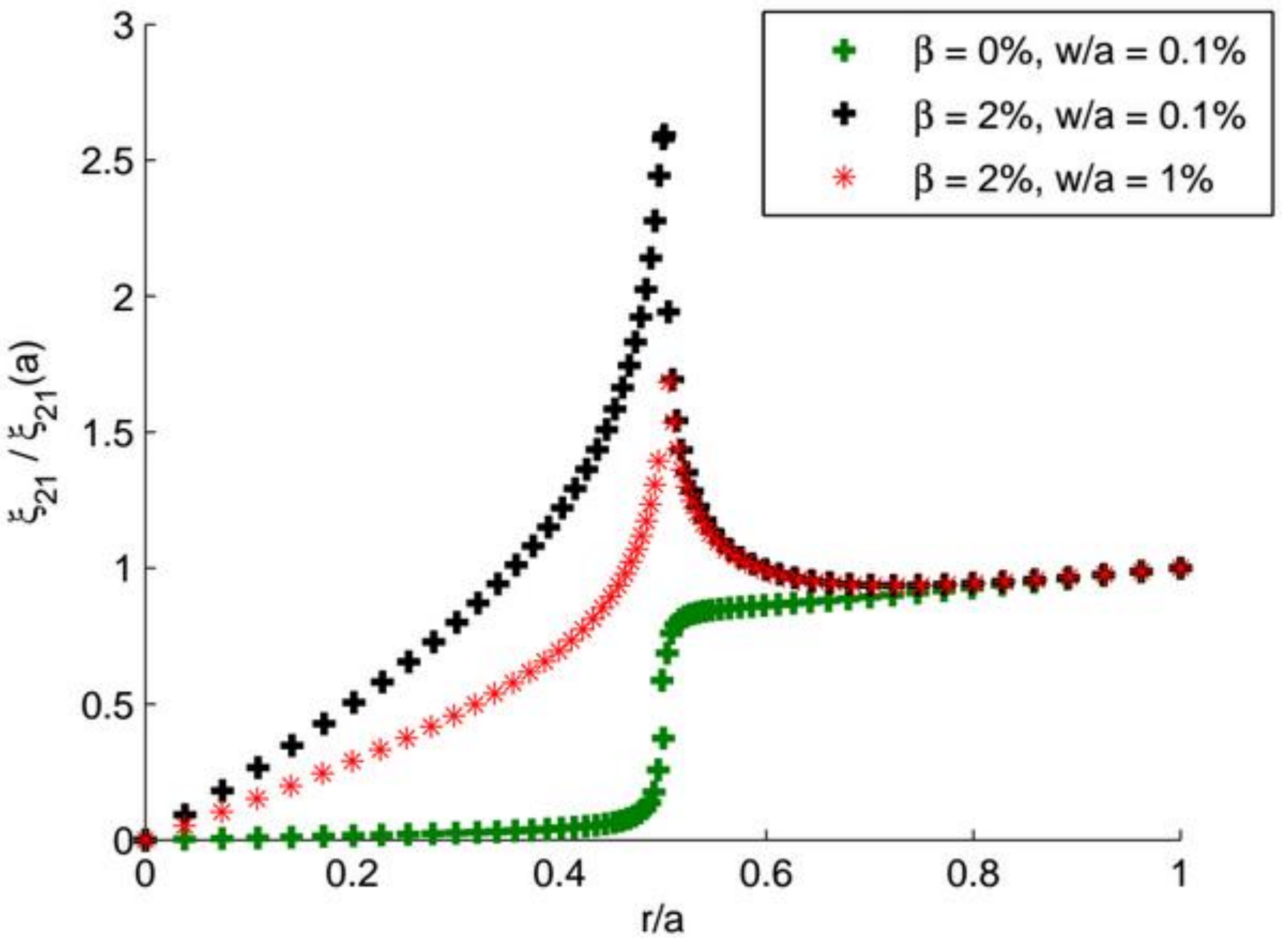


Discontinuous transform solution cf. "Tearing" solution

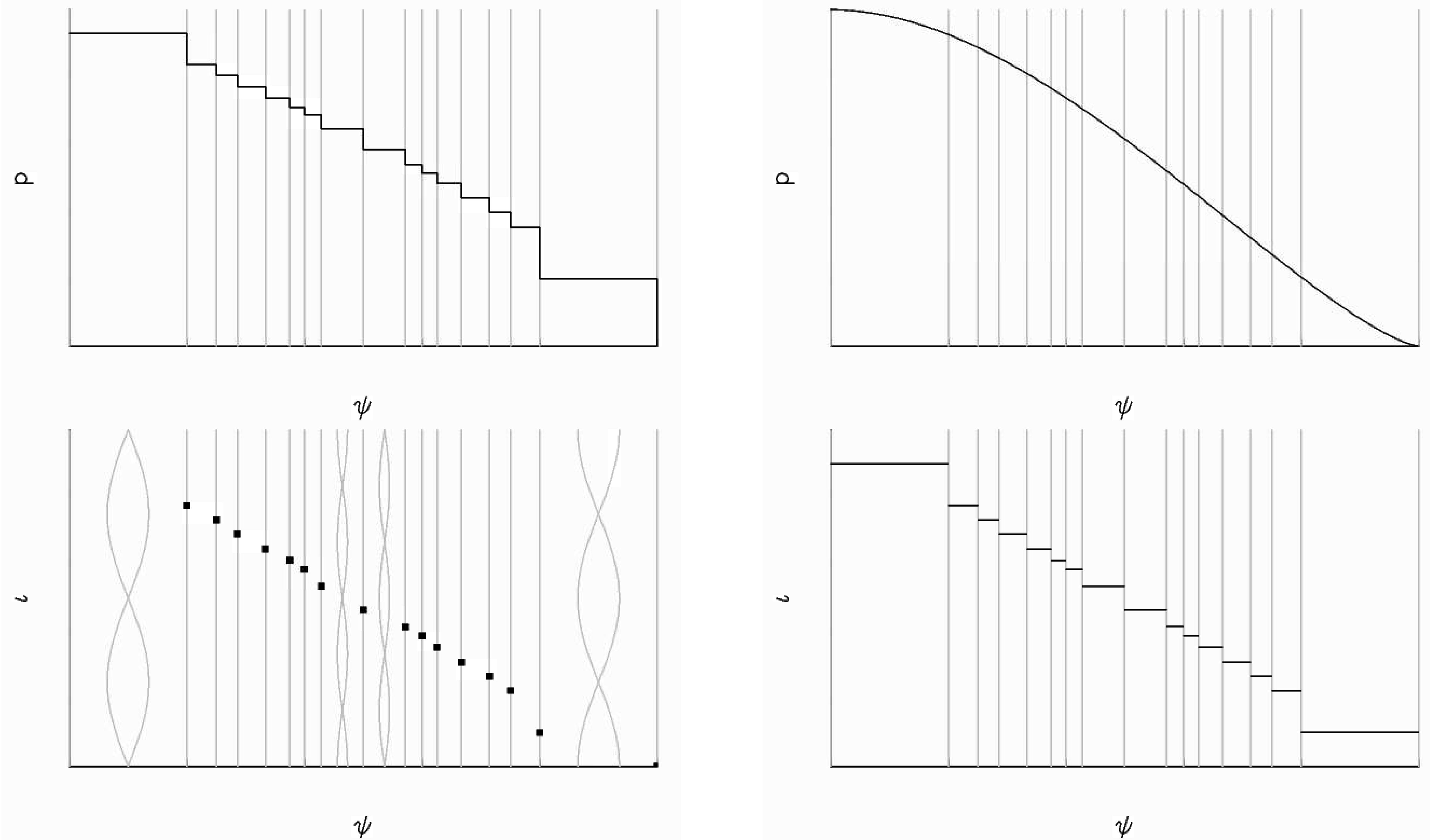
Discontinuous transform with no island (ideal)



Continuous transform with island (tearing)



Two classes of solutions for discontinuous 3D MHD equilibria:
STEPPED PRESSURE and STEPPED TRANSFORM



[Bruno & Laurence, Commun. Pure Appl. Math **49**, 717 (1996)]
[Loizu, Hudson *et al*., Phys. Plasmas **22**, 090704 (2015)]

Multi-Region relaxed-ideal MHD Energy Functional alternating ideal, relaxed, ideal, relaxed MHD regions.

[Kruskal & Kulsrud (1958)] [Taylor (1974)]

$$W = \underbrace{\int_{\mathcal{R}_1}\left(\frac{p}{\gamma-1}+\frac{B^2}{2}\right)dv}_{\substack{\delta\mathbf{B}=\nabla\times(\boldsymbol{\xi}\times\mathbf{B})\\ \delta p=(\gamma-1)\boldsymbol{\xi}\cdot\nabla p-\gamma\nabla\cdot(p\boldsymbol{\xi})}} + \underbrace{\int_{\mathcal{R}_2}\left(\frac{p}{\gamma-1}+\frac{B^2}{2}\right)dv}_{\substack{\delta\mathbf{B}\;=\;arbitrary\\ \delta p\;=\;arbitrary}} + \underbrace{\int_{\mathcal{R}_3}\left(\frac{p}{\gamma-1}+\frac{B^2}{2}\right)dv}_{\substack{\delta B\;=\;ideal\\ \delta p\;=\;ideal}} + \dots$$

$$+ \quad \mu\left[\int_{\mathcal{R}_2}\mathbf{A}\cdot\mathbf{B}\,dv - H_2\right]$$

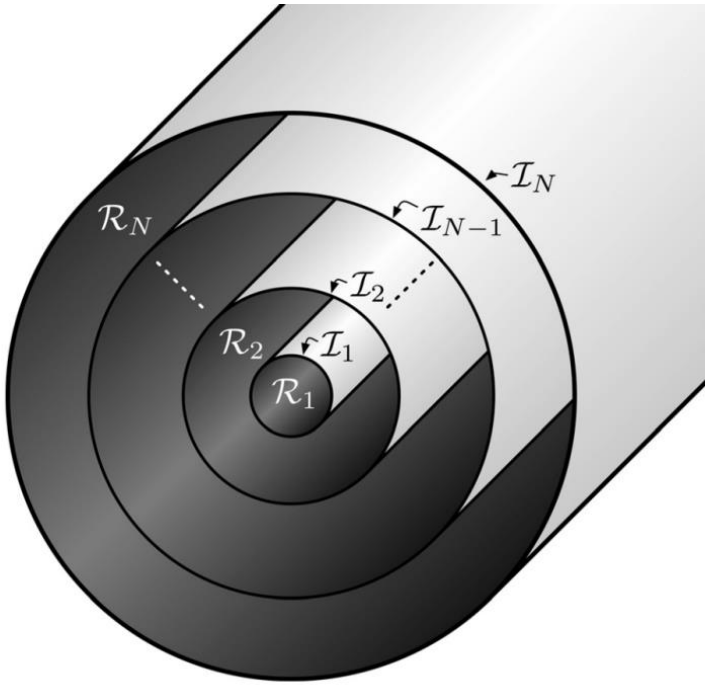
"IDEAL" "TAYLOR RELAXED" "IDEAL"

$$\delta W = \int_{\mathcal{R}_1}\boldsymbol{\xi}\cdot(\nabla p-\mathbf{j}\times\mathbf{B})\,dv + \int_{\mathcal{R}_2}\boldsymbol{\xi}\cdot(\nabla\times\mathbf{B}-\mu\mathbf{B})\,dv + \int_{\mathcal{R}_3}\boldsymbol{\xi}\cdot(\nabla p-\mathbf{j}\times\mathbf{B})\,dv + $$

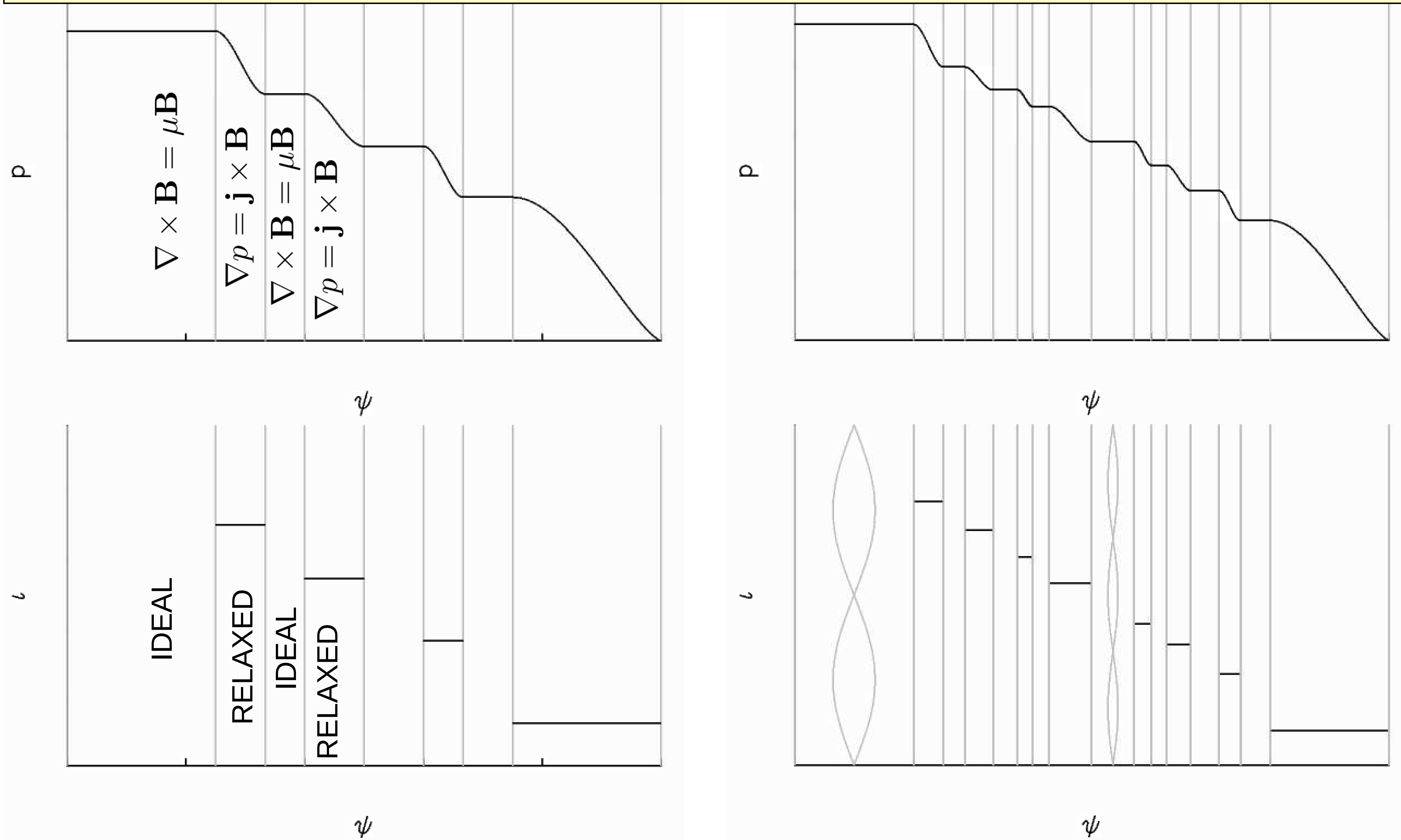
nested flux surfaces allows for islands

$p' \neq 0$ $p' = 0$

$$\iota = \underbrace{\frac{p_1+\gamma\,p_2}{q_1+\gamma\,q_2}}_{\text{“noble”}} \qquad \mu = \frac{\mathbf{j}\cdot\mathbf{B}}{B^2} = const.$$



General solution for smooth 3D MHD equilibria: Multi-Region, relaXed-Ideal, MHD



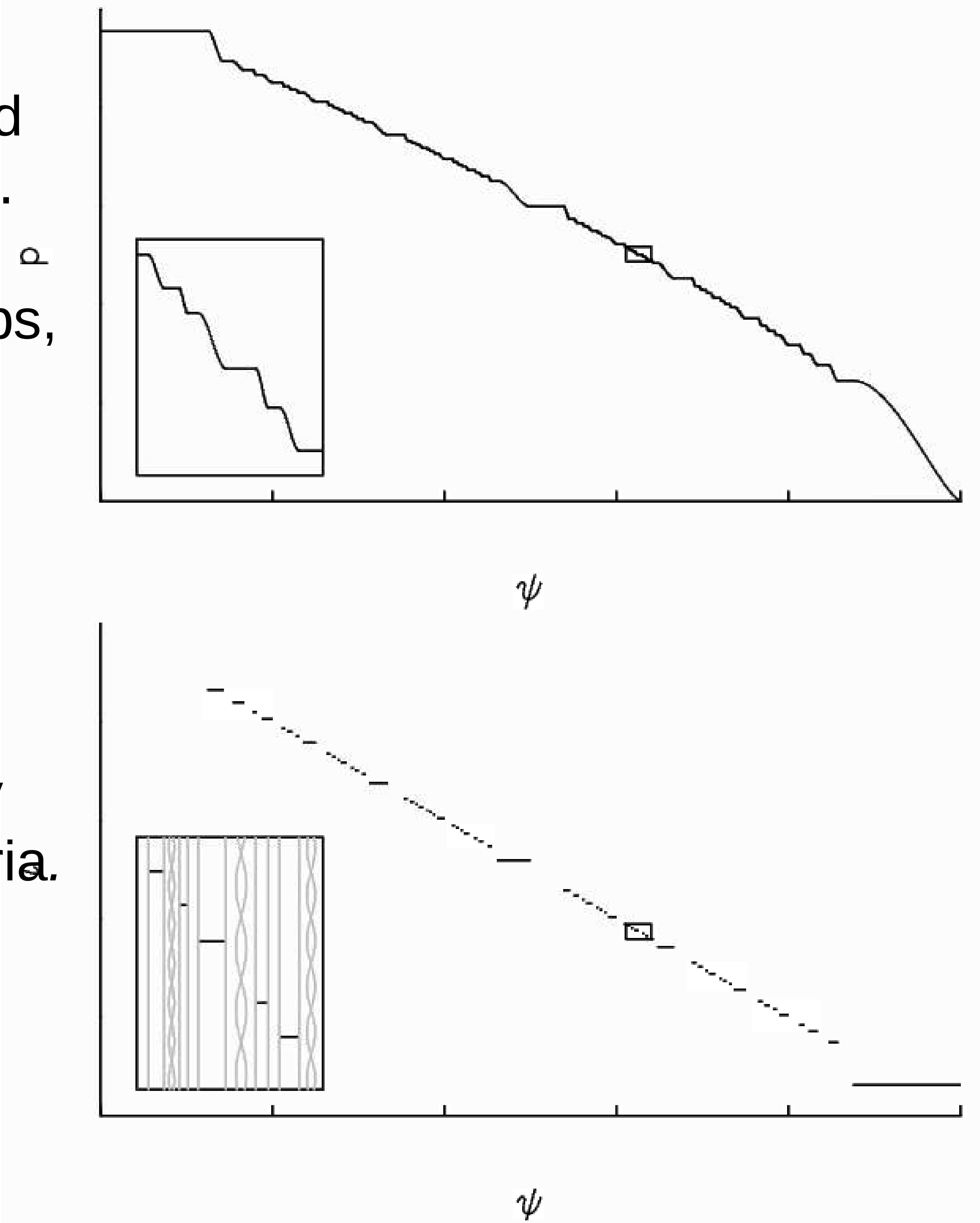
[Hudson & Kraus, J. Plasma Phys., submitted (2017)]

Can reliably, systematically approach *fractal* equilibria.

1. Fractals can only be treated numerically by taking limits.

2. With a finite number of steps, extrema of the MRxiMHD energy functional have smooth pressure gradients *and* magnetic islands and chaos.

3. Can reliably, systematically approximate fractal equilibria.



Given continuous, non-integrable $\boldsymbol{B}$, $\boldsymbol{B}.\nabla p = 0$ implies p is fractal. Given fractal p, what is continuous, non-integrable $\boldsymbol{B}$?

- **Defn.** An equilibrium code computes the magnetic field consistent with a given p and e.g. given $\iota\!\!$-.
- **Theorem.** The topology of $\mathbf{B}$ is partially dictated by p.

  $\hookrightarrow$ Where $p' \neq 0$, $\mathbf{B} \cdot \nabla p = 0$ implies $\mathbf{B}$ must have flux surfaces.

  $\hookrightarrow$ Where $p' = 0$, $\mathbf{B}$ can have islands, chaos and/or flux surfaces.

TRANSPORT: given $\mathbf{B}$, solve for p.

1. Given general, non-integrable magnetic field, $\mathbf{B} = \nabla \times [\psi \nabla \theta - \chi(\psi, \theta, \zeta) \nabla \zeta]$

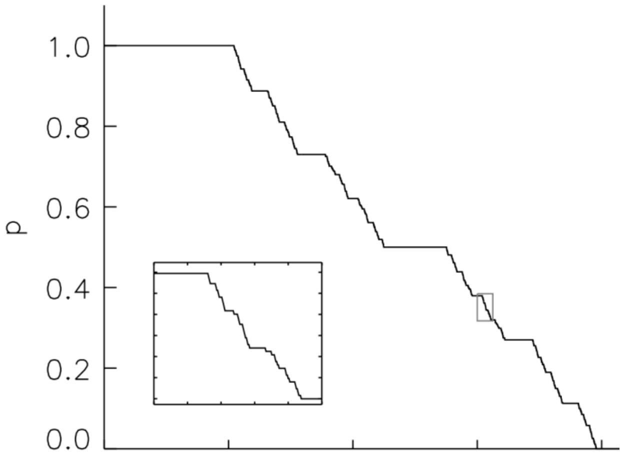
   i. fieldline Hamiltonian: $\chi(\psi, \theta, \zeta) = \chi_0(\psi) + \sum_{m,n} \chi_{m,n}(\psi) e^{i(m\theta - n\zeta)}$

2. KAM theorem: for suff. small perturbation, "sufficiently irrational" flux surfaces survive

   i. if $\iota\!\!$- satisfies a "Diophantine" condition, $|\iota\!\!\text{-} - n/m| > r/m^k, \forall (n,m)$, **excluded interval about every rational**

   ii. need e.g. Greene's residue criterion to determine if flux-surface$_{\iota\!\!\text{-}}$ exists; lot's of work;

3. With $\mathbf{B} \cdot \nabla p = 0$, i.e. infinite parallel transport, pressure profile must be fractal:



$$p'(\iota\!\!\text{-}) = \begin{cases} 1, & \text{if } |\iota\!\!\text{-} - n/m| > r/m^k, \quad \forall (n,m), \text{ e.g. } r = 0.2,\ k = 2, \\ 0, & \text{if } |\iota\!\!\text{-} - n/m| < r/m^k, \quad \exists (n,m), \end{cases}$$

$p'(x)$ is discontinuous on an uncountable infinity of points; impossible to discretize accurately;

EQUILIBRIUM: given p, solve for $\mathbf{B}$.

Q. *Given* a fractal p', how can the topology of $\mathbf{B}$ be constrained to enforce $\mathbf{B} \cdot \nabla p = 0$?

   i. e.g. if $p(\psi)$ is continuous and smooth, nowhere zero, then $\mathbf{B}$ *must* be integrable, i.e. $\chi_{m,n}(\psi) = 0$

   ii. if $p'(\psi)$ is fractal, then what are $\chi_{m,n}(\psi) = ?$

Ongoing development of SPEC

1. Code improvements:

   i. finite-elements replaced by Chebshev polynomials

   e.g. $\mathbf{A} \equiv \sum_{l,m,n}^{L,M,N} [\alpha_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\theta + \beta_{l,m,n} T_l(s) \cos(m\theta - n\zeta) \nabla\zeta]$

   ii. linearized equations

   iii. Cartesian, cylindrical, toroidal geometry

   iv. detailed online documentation,
   `http://w3.pppl.gov/~shudson/Spec/spec.html`

   v. easy-to-use, easy-to-edit, graphical user interface

2. Physics applications

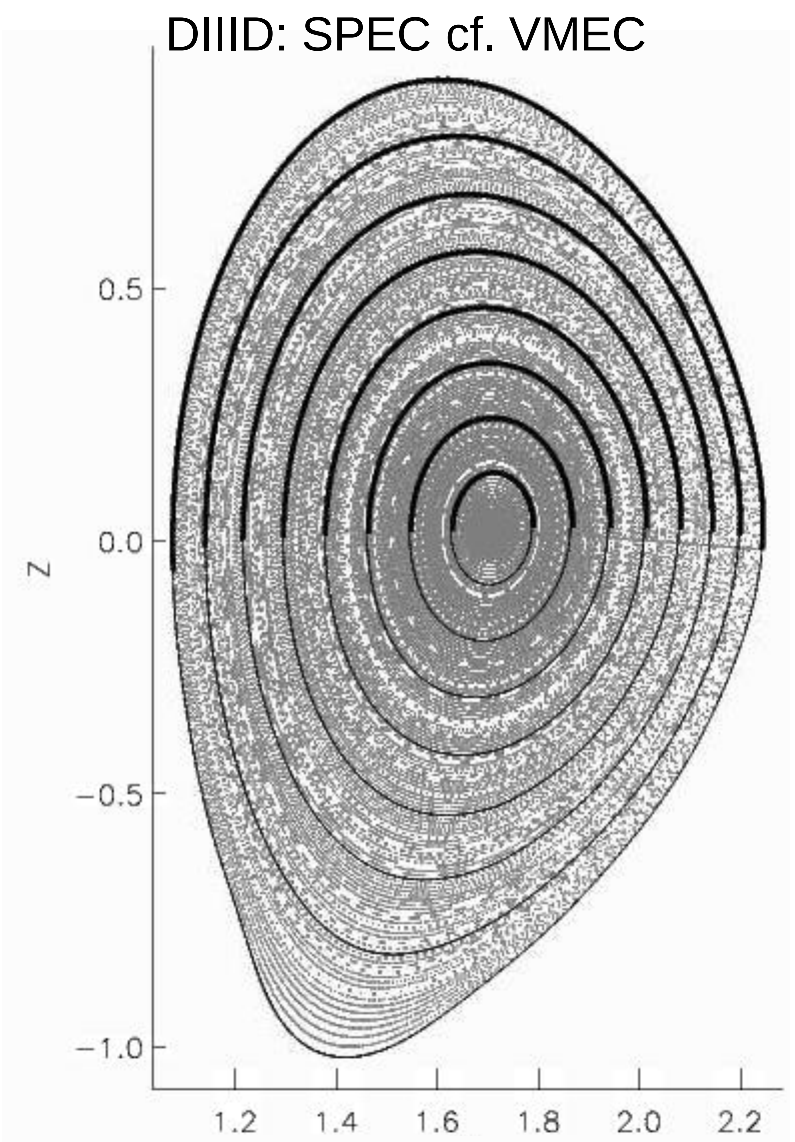
   i. W7-X vacuum verification calculations, OP1.1 [completed]

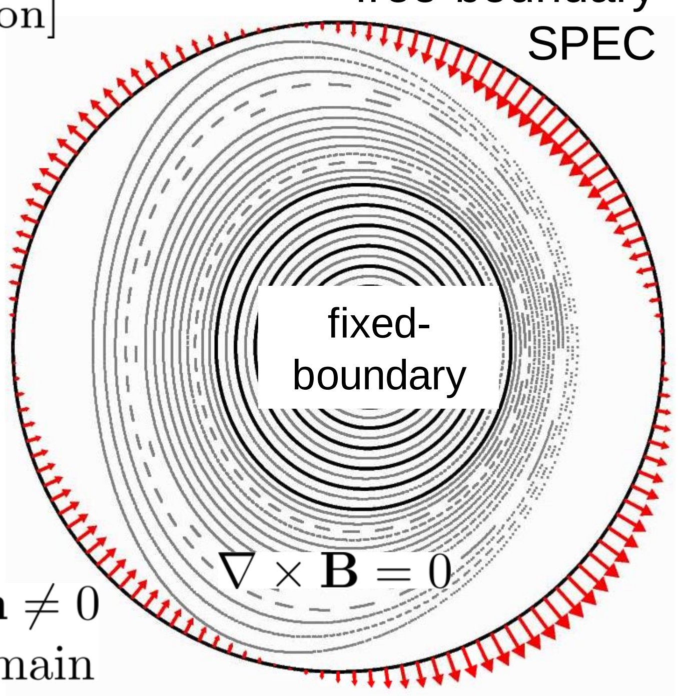
   ii. non-stellarator symmetric, e.g. DIIID, [completed]

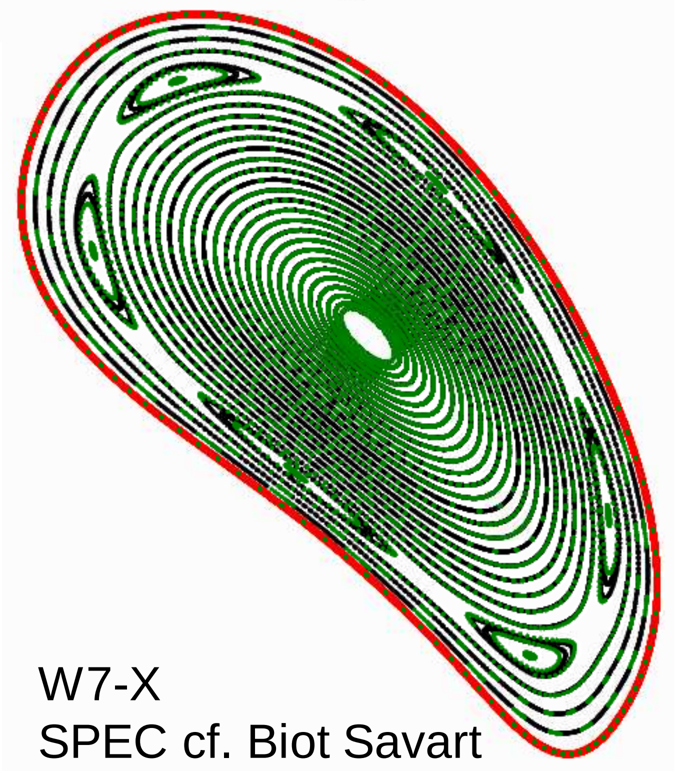
   iii. free-boundary, [completed]

   iv. including flow, [under construction]

   v. MRxMHD linear stability, [under construction]







Published SPEC convergence / verification calculations

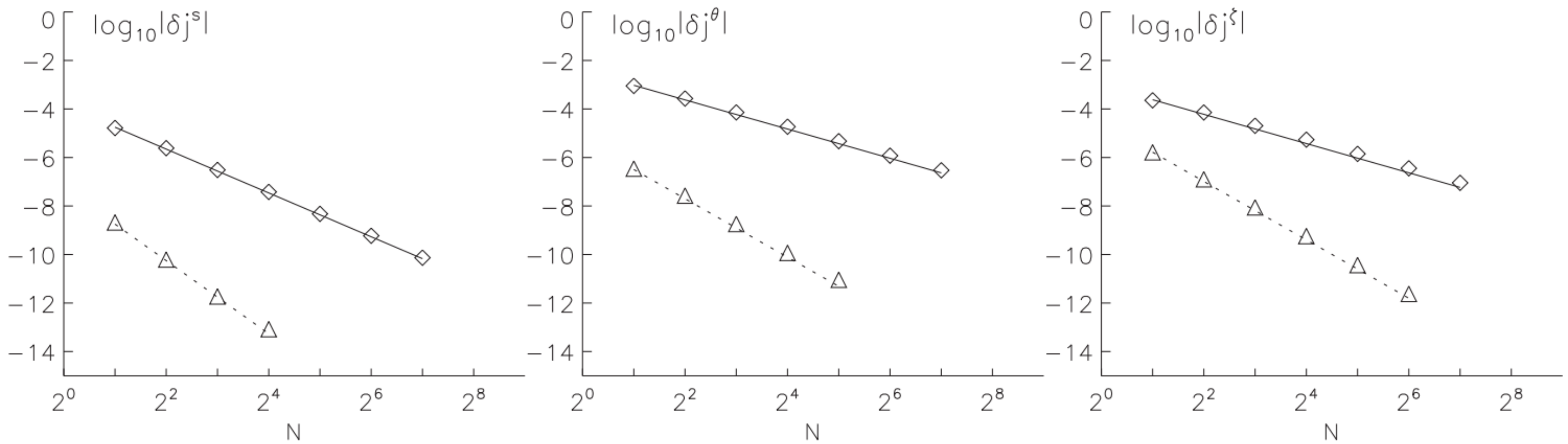


FIG. 2. Scaling of components of error, $\delta\mathbf{j} \equiv \mathbf{j} - \mu\mathbf{B}$, with respect to radial resolution. The diamonds are for the $n = 3$ (cubic) basis functions, the triangles are for the $n = 5$ (quintic) basis functions. The solid lines have gradient -3, -2, and -2, and the dotted lines have gradient -5, -4, and -4.

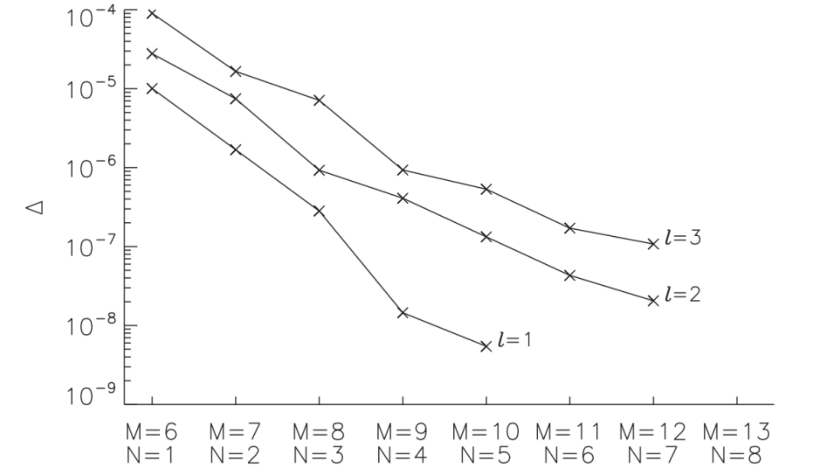


FIG. 6. Difference between finite M, N approximation to interface geometry, and a high-resolution reference approximation (with $M = 13$ and $N = 8$), plotted against Fourier resolution.

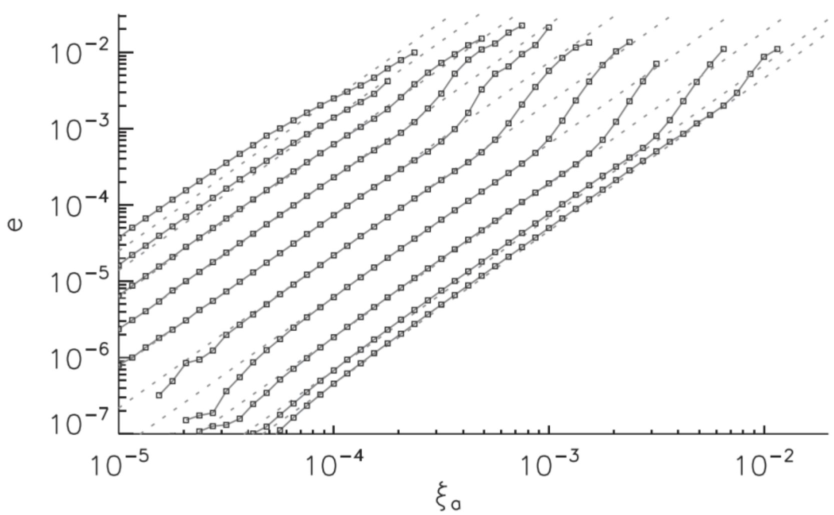


FIG. 2. Convergence of the error between linear and nonlinear SPEC equilibria as ξ_a is decreased, and for different values of $\Delta\!\!\!\iota$, ranging from 10^{-4} (upper curve) to 10^{-1} (lower curve).

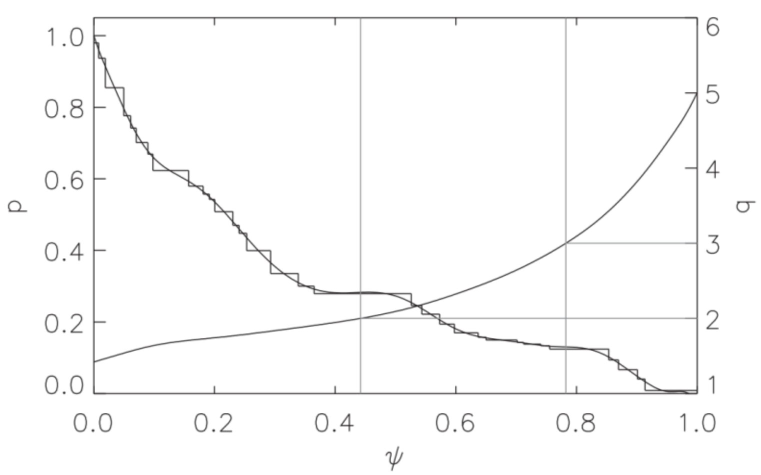


FIG. 7. Pressure profile (smooth) from a DIIID reconstruction using STELLOPT and stepped-pressure approximation. Also, shown is the inverse rotational transform $\equiv$ safety factor.

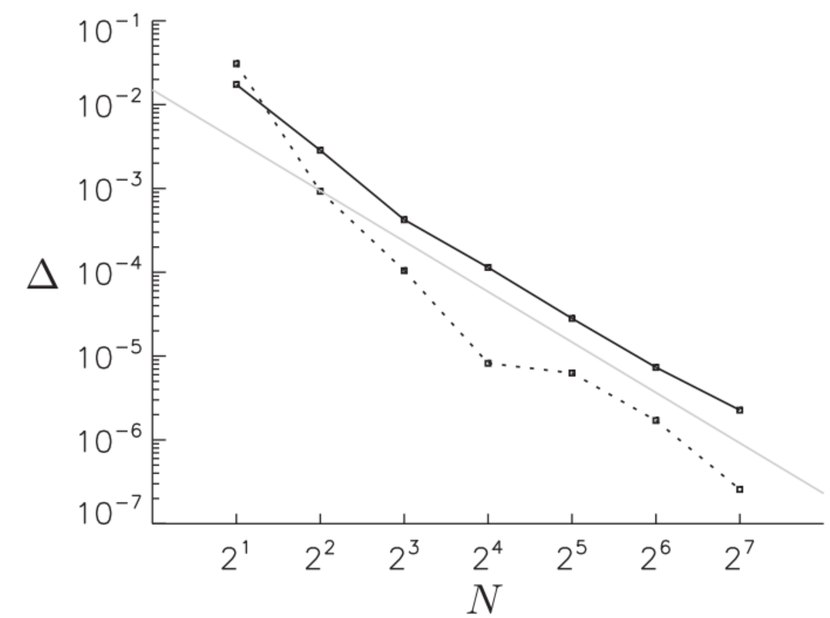
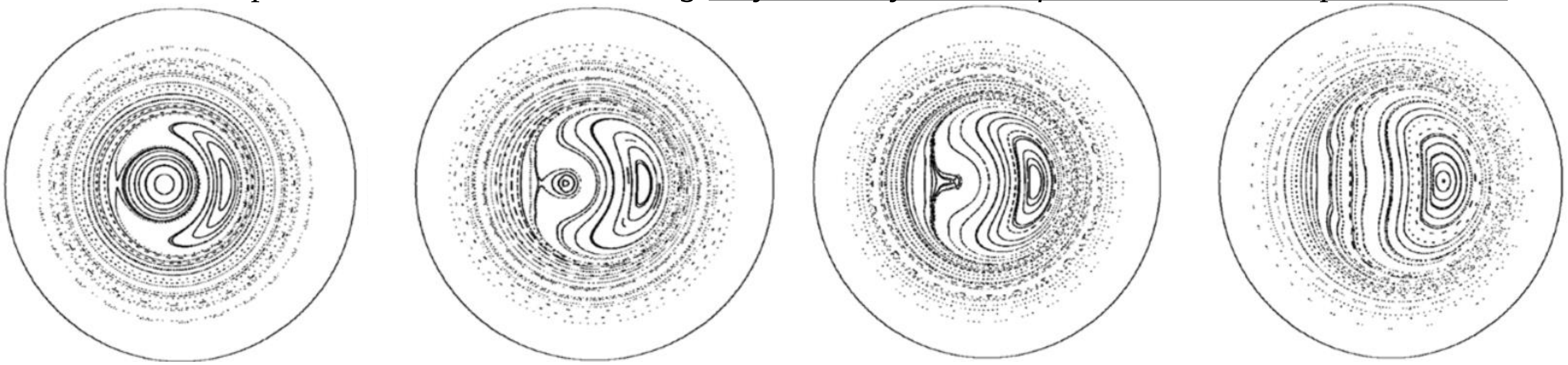


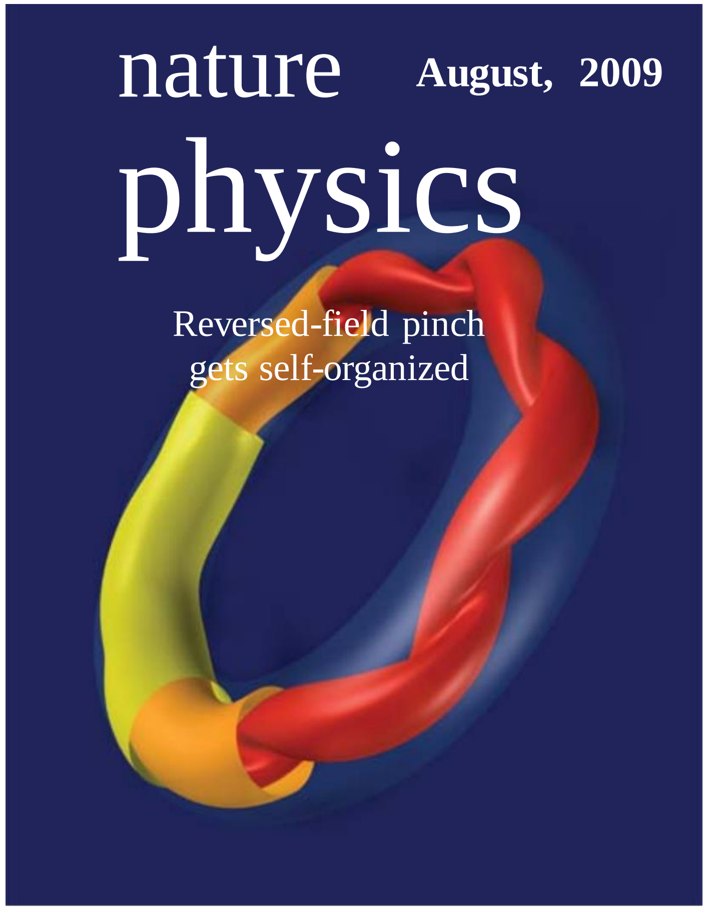
FIG. 5. Convergence: the error (Δ) between the continuous pressure (VMEC) and stepped pressure (SPEC) solutions are shown as a function of the number of plasma regions N for the $s = 1/4$ SPEC interface. The dotted line shows the zero-beta case ($p_0 = 0$), and the solid line shows the high-beta case ($p_0 = 16$). The grey line has a slope -2, the expected rate of convergence. These simulations were run on a single 3 GHz Intel Xeon 5450 CPU with the longest (the $N = 128$ case) taking 10.1 min using 20 poloidal Fourier harmonics and 768 fifth-order polynomial finite elements in the radial direction.

MRxMHD explains self-organization of Reversed Field Pinch into internal helical state

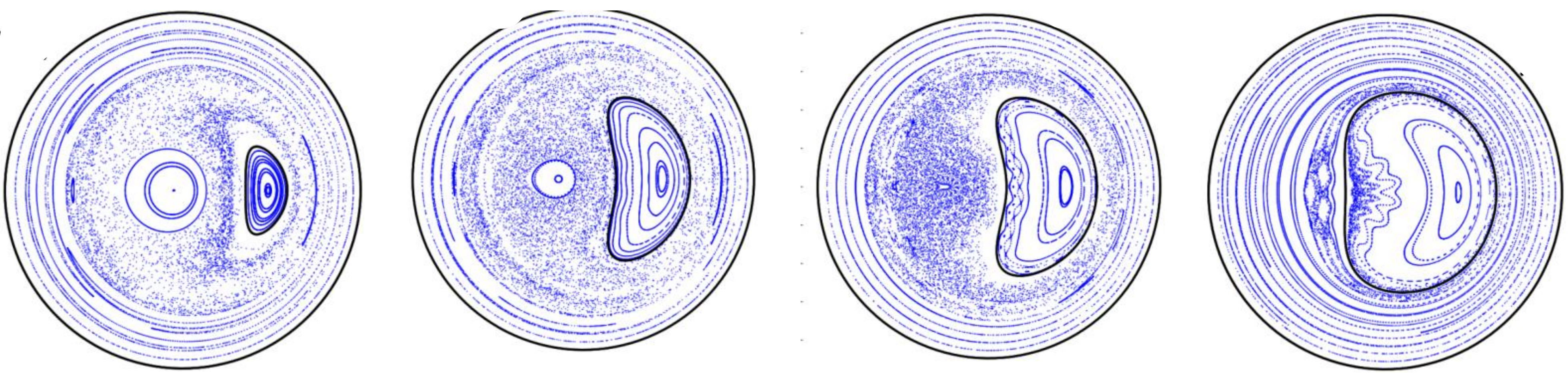
EXPERIMENTAL RESULTS
Overview of RFX-mod results
P. Martin et al., *Nuclear Fusion, 49 (2009) 104019*

Fig.6. Magnetic flux surfaces in the transition from a QSH state . . to a fully developed SHAx state . . The Poincaré plots are obtained considering <u>only the axisymmetric field and dominant perturbation</u>''





NUMERICAL CALCULATION USING STEPPED PRESSURE EQUILIBRIUM CODE
"Minimally Constrained Model of Self-Organized Helical States in Reversed-Field Pinches"
G. Dennis, S. Hudson, et al. PRL 111, 055003 (2013)]



Excellent Qualitative agreement between numerical calculation and experiment

→ this is first (and perhaps only?) equilibrium model able to explain internal helical state with two magnetic axes