Penetration and amplification of resonant perturbations in 3D ideal-MHD equilibria

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I. *Importance of ideal-MHD*

Understanding three-dimensional (3D), ideal-MHD equilibria, as described by the ideal force-balance equation, $\nabla p = \mathbf{j} \times \mathbf{B}$, is fundamentally important for understanding the behavior of magnetically-confined plasmas. The potentially-dangerous instabilities that cause edge-localized modes, an important concern for ITER, are widely believed to be ideal, peeling-ballooning modes [1]; and a 'hot-topic' of current research is to discover how these modes may be suppressed by applying resonant magnetic perturbations (RMPs), i.e. by 3D effects [2]. Whilst there exist extended-MHD codes [3] with non-ideal physics, these codes come with significant computational cost; and so the plasma response to 3D perturbations is routinely determined perturbatively using codes such as IPEC [4], the Ideal, Perturbed Equilibrium Code.

However, there are two fundamental difficulties that are frequently over-looked.The first is the existence of *infinite* currents near resonant surfaces, and the second that ideal-MHD equilibria are *not* analytic functions of the 3D boundary.

II. *Unphysical, pressure-driven currents*

The infinite currents arise from enforcing charge conservation, $\nabla \cdot \mathbf{j} = 0$, in equilibria with smooth profiles. Wherever there are pressure-driven, perpendicular current-densities, $\mathbf{j}_\perp = \mathbf{B} \times \nabla p / B^2$, there must also be parallel current-densities that satisfy $\mathbf{B} \cdot \nabla u = -\nabla \cdot \mathbf{j}_\perp$, where $\mathbf{j} \equiv u\,\mathbf{B} + \mathbf{j}_\perp$. The solution for each Fourier harmonic, in straight-field line coordinates, is $u_{m,n} = i(\sqrt{g}\,\nabla \cdot \mathbf{j}_\perp)_{m,n}/(\iota\, m - n) + \Delta_{m,n}\,\delta(x)$, where $x \equiv \iota - n/m$ is the 'distance' in rotational-transform from the rational surface, and $\Delta_{m,n}$ is an as-yet undetermined constant.

MHD is a *macroscopic* model of plasma dynamics with no intrinsic length scale; and the δ-function density is just a mathematical idealization of localized currents. Singularities in the current-*density* are allowed, but the total current, $\int \mathbf{j} \cdot d\mathbf{s}$, passing through *each* and *every* surface must be finite for a physically-acceptable equilibrium.

The *net* current between adjacent flux-surfaces, e.g. $x = -\epsilon$ and $x = +\epsilon$ as $\epsilon \to 0$, resulting from the resonant δ-function density actually integrates to zero. However, the resonant Pfirsch-Schlüter current passing through the cross-sectional area enclosed by two flux-surfaces close to the resonant surface, e.g. $x = \epsilon$ and $x = \delta$ where ϵ and δ are small, between $\theta = 0$ and $\theta = \pi/m$ resulting from the $1/x$ current-density is proportional to $\int_\epsilon^\delta 1/x\, dx$, which logarithmically approaches infinity as ϵ approaches zero.

This is not physical: the ideal-MHD equilibrium model, with nested flux-surfaces, cannot admit pressure gradients in a small neighborhood of each rational surface. Because the rational surfaces are dense, this means that there can be no pressure at all, if the pressure is smooth; or the pressure must be fractal [5] and not amenable to standard numerical discretization; or the pressure must be discontinuous [6]; or the equilibrium boundary must be restricted to ensure that the appropriate resonant harmonics of the geometry vanish [7–9].

III. *Breakdown of perturbation theory*

The non-analyticity of ideal-MHD equilibria is encountered when computing the plasma displacement induced by an $\mathcal{O}(\epsilon)$ perturbation to the boundary of an equilibrium state. Assuming an expansion $\boldsymbol{\xi} \equiv \epsilon\,\boldsymbol{\xi}_1 + \epsilon^2\,\boldsymbol{\xi}_2 + \ldots$, the first-order displacement satisfies $L_0[\boldsymbol{\xi}_1] = 0$, where $L_0[\boldsymbol{\xi}] \equiv \nabla\delta p - \delta\mathbf{j} \times \mathbf{B} - \mathbf{j} \times \delta\mathbf{B}$, where the 'ideal' perturbations to the field and pressure are $\delta\mathbf{B}[\boldsymbol{\xi}] \equiv \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$ and $\delta p[\boldsymbol{\xi}] = (\gamma - 1)\,\boldsymbol{\xi} \cdot \nabla p - \gamma \nabla \cdot (p\boldsymbol{\xi})$.

The operator, $L_0[\boldsymbol{\xi}]$, is singular. To match a finite displacement at the boundary, and to ensure that the resonant component of the normal field vanishes at each rational surface (so that magnetic islands do not form), the solution for $\boldsymbol{\xi}_1$ must, generally, be discontinuous at each resonant surface. A discontinuous displacement is, however, inconsistent with nested flux-surfaces. The second order term satisfies $L_0[\boldsymbol{\xi}_2] = -\delta\mathbf{j}[\boldsymbol{\xi}_1] \times \delta\mathbf{B}[\boldsymbol{\xi}_1]$, and $\boldsymbol{\xi}_2$ is even more singular than $\boldsymbol{\xi}_1$. The breakdown of perturbation theory was known by Rosenbluth *et al.* [10], who wrote "we must abandon the perturbation theory approach" when computing ideal-MHD equilibria in 3D.

IV. *A new class of self-consistent solutions*

These difficulties are not fundamental flaws in ideal-MHD, which remains perhaps the most successful, relevant yet simplest model of plasma dynamics. It is just that, until recently, self-consistent solutions to the ideal-MHD equilibrium equation for arbitrary 3D geometry had not been discovered. There is, surprisingly, a class of solutions that eliminates both the infinite currents and the non-analyticity.

Recently, Loizu, *et al.* [11], for the first time, computed the $1/x$ and δ-function current-densities in 3D equilibria; and we realized that self-consistent solutions demand locally-infinite shear at the resonant surfaces. We then introduced [12] a new class of solutions that admit additional δ-function current-densities that *do* produce finite net currents between adjacent flux-surfaces, with a commensurate *discontinuity* in the rotational-transform that removes the singularities.

Most importantly, our solutions yield predictions that are in sharp contrast to previous predictions, with direct implications for understanding the penetration of RMPs in tokamaks: in ideal-MHD, a resonant perturbation penetrates past the rational surface and into the core of the plasma; and the perturbation is magnified by

pressure inside the resonant surface, increasingly so as stability limits are approached [13].

V. *Verification with analytic solution*

For illustration and verification, we consider the linear and nonlinear, ideal response to a resonant boundary perturbation in cylindrical geometry. The equilibrium is defined by an arbitrary, smooth pressure profile, $p(r)$, and a rotational-transform profile,

$$\iota(r) = \begin{cases} \iota_0(r) + \Delta\iota/2, & \text{for } r < r_s, \\ \iota_0(r) - \Delta\iota/2, & \text{for } r > r_s, \end{cases} \tag{1}$$

with $\iota_0(r)$ chosen so that $\iota(r)$ jumps across the rational $\iota_s \equiv n/m$, namely $\iota(r_s) = \iota_s \pm \Delta\iota/2$. The linearized equation, $L_0[\boldsymbol{\xi}] = 0$, reduces to Newcomb's equation [14],

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\,\xi = 0, \tag{2}$$

where ξ is the radial component of the resonant plasma displacement, and $f(r)$ and $g(r)$ are determined by the equilibrium. For $\Delta\iota = 0$, Newcomb's equation is singular where $\iota(r_s) = n/m$, because there is an order-two zero in $f(r)$ and an order-one zero in $g(r)$, and this is the cause of the non-analyticity. For $\Delta\iota > 0$ the singularity is removed; and for $\Delta\iota > \Delta\iota_{min}$ (the minimum required to ensure that the perturbed flux-surfaces do not overlap) the equilibrium equations comprise an analytic function of the 3D boundary and the perturbation expansion accurately approximates the true nonlinear solution. The value of $\Delta\iota_{min}$ may be estimated analytically [12].

VI. *Penetration and amplification of RMP*

For continuous transform, i.e. $\Delta\iota = 0$, the linear solution predicts complete 'shielding' of the perturbation at the resonant surface, and no penetration. For $\Delta\iota > 0$, the displacement penetrates past the resonant surface and into the origin. Fig.1 shows the result of numerical integration of Eq.(2), using smooth polynomials for $p(r)$ and $\iota_0(r)$, with $\Delta\iota > \Delta\iota_{min}$, and for increasing plasma pressure. The perturbation inside the resonant surface is magnified by plasma pressure, extremely so as ideal-stability limits are approached [13].

VII. *Benchmarking linear & nonlinear codes*

Our solution has resolved a confusion in efforts [15, 16] to benchmark the linearly-perturbed solutions with the nonlinear solutions provided by the widely-used equilibrium codes VMEC [17] and NSTAB [18]. Both employ a numerical representation for $\mathbf{B}$ that guarantees nested flux-surfaces; whereas, for $\Delta\iota = 0$, as was considered in Refs.[15, 16], the discontinuity in the linear solution means that the linearly-perturbed approximation gives overlapping flux-surfaces; and it is no surprise that the linear and nonlinear solutions disagree near the rational surfaces. For $\Delta\iota > \Delta\iota_{min}$, the linear solution gives non-overlapping flux-surfaces and the perturbation expansion converges: the linear and nonlinear codes should agree.

VMEC and NSTAB are, unfortunately, restricted to work with smooth profiles and cannot formally compute equilibria with discontinuous rotational-transform (though finite radial-resolution may imply an 'effective' $\Delta\iota$ [19]). The nonlinear equilibrium code SPEC [6] *does* allow for discontinuities. Fig.1 compares the SPEC solution to the analytic, linear solution, with excellent agreement. The nonlinear SPEC solution converges to the linear solution with error $\sim \mathcal{O}(\epsilon^2)$, as it should.

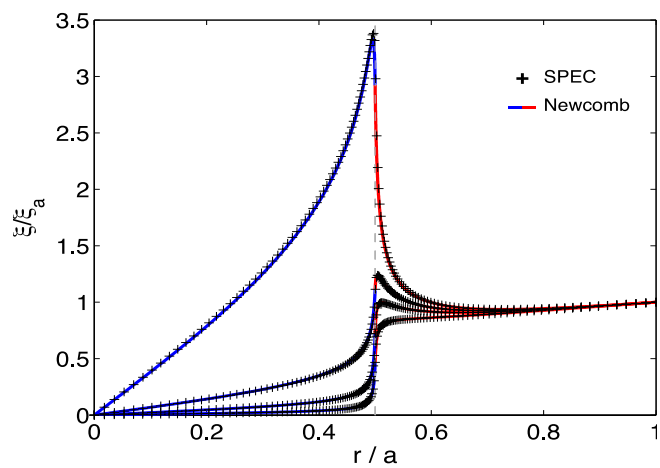

FIG. 1: Comparison of the analytic solution to the linearized equation, Eq.(2), and the SPEC solution for a perturbed cylindrical equilibrium, with $\Delta\iota > \Delta\iota_{min}$, and for increasing pressure. The RMP penetrates past the resonant surface and is amplified by plasma pressure.

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