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Stepped pressure profile equilibria in cylindrical plasmas via partial Taylor relaxation

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Abstract. We develop a multiple interface variational model, comprising multiple Taylor-relaxed plasma regions separated by ideal magnetohydrodynamic (MHD) barriers. A principal motivation is the development of a mathematically rigorous ideal MHD model to describe intrinsically three-dimensional equilibria, with non-zero internal pressure. A second application is the description of transport barriers as constrained minimum energy states. As a first example, we calculate the plasma solution in a periodic cylinder, generalizing the analysis of the treatment of Kaiser and Uecker (2004 *Q. J. Mech. Appl. Math.* **57**, 1–17), who treated the single interface in cylindrical geometry. Expressions for the equilibrium field are generated, and equilibrium states computed. Unlike other Taylor relaxed equilibria, for the equilibria investigated here, only the plasma core necessarily has reverse magnetic shear. We show the existence of tokamak-like equilibria, with increasing safety factor and stepped-pressure profiles.

1. Introduction

The existence of three-dimensional (3D) equilibria with smooth pressure gradients has been a long-standing and unresolved issue [1]. The problem occurs because, in general, flux surfaces form only at irrational rotational transform ι, where no magnetic islands exist. As such, $\nabla p = 0$ in all regions except at irrational ι. Across this surface a pressure difference can be supported. Taylor relaxation (see [2] and references therein) describes a plasma that has passed through a phase where strong global overlap of magnetic islands (e.g. due to tearing modes) has allowed the magnetic field to evolve to a minimum energy state, subject to the conservation of magnetic helicity and toroidal flux, and the presence of a perfectly conducting wall. In such states the pressure gradient is zero, and the magnetic field $\mathbf{B}$ satisfies the Beltrami equation [2]

$$\nabla \times \mathbf{B} = \mu \mathbf{B} \tag{1.1}$$

with the Lagrange multiplier μ below some critical value μ_T, which depends only on the vessel. Taylor's model was developed to describe a strongly turbulent reverse field pinch. In more quiescent systems such as tokamaks and stellarators, global island overlap does not occur. Thus, relaxation is, at worst, local. In regions where the rotational transform is strongly irrational, island overlap is suppressed. Such regions can act as robust ideal MHD barriers between different Taylor relaxed states, thus leading to equilibria with stepped pressure profiles. A principal motivation is the development of a rigorous solution to a 3D magnetic equilibrium problem, which remains an unsolved magnetic containment theory problem [1]. Such a stepped pressure-profile equilibrium is also apparent, however, in a partially relaxed or locally minimum energy state and so may explain *ab initio* the existence of transport barriers in toroidal magnetic confinement experiments. Other theories exist to describe the subsequent formation of transport barriers (e.g. shear flow suppression of turbulence [3] or chaotic magnetic field line dynamics [4]). Importantly, the model proposed here is not offered as an accurate physical picture, but rather as one which is well-posed within ideal MHD, and thus worth examining before attempting a more elaborate study.

Our working builds principally upon a variational model developed by Spies et al. [5], which comprised a plasma/vacuum/conducting wall system. In [5] the theory is applied to a plasma slab equilibrium, with boundary conditions designed to simulate a torus. Later analysis by Spies [6] extended the plasma model to include finite pressure. More recently, Kaiser and Uecker [7] analyzed the finite pressure model in cylindrical geometry.

2. Multiple interface plasma vacuum model

We generalize the analysis of Kaiser and Uecker [7] to an arbitrary number N of Taylor relaxed states, each separated by an ideal MHD barrier. The system is enclosed by a vacuum, and encased in a perfectly conducting wall. The energy functional can be written

$$W = U - \sum_{i=1}^{N} \mu_i H_i/2 - \sum_{i=1}^{N} \nu_i M_i. \tag{2.1}$$

Setting the first variation to zero yields the following set of equations

$$\mathrm{P}_i : \nabla \times \mathbf{B} = \mu_i \mathbf{B}, \quad p_i = \text{constant}, \tag{2.2}$$

$$\mathrm{I}_i : \quad \mathbf{n} \cdot \mathbf{B} = 0, \qquad \langle p_i + 1/2 B^2 \rangle = 0, \tag{2.3}$$

$$\mathrm{V} : \nabla \times \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{B} = 0 \tag{2.4}$$

$$\mathrm{W} : \quad \mathbf{n} \cdot \mathbf{B} = 0 \tag{2.5}$$

where $\mathrm{P}_i, \mathrm{I}_i$ are the ith plasma region and interface (or ideal MHD barrier), and V, W are the vacuum region and wall, respectively. Also, μ_i is the Lagrange multiplier in each region, p_i the pressure in each region, $\mathbf{n}$ a unit vector normal to the plasma interface, and $\langle x \rangle = x_{i+1} - x_i$ denotes the change in quantity x across the interface I_i. The boundary conditions on $\mathbf{n} \cdot \mathbf{B}$ arise because each interface and the conducting wall is assumed to have infinite conductivity. In turn, these imply the flux constraints $\psi^{\mathrm{t}}_{\mathrm{R}} = \text{constant}$ and $\Psi^{\mathrm{p}}_{\mathrm{V}} = \text{constant}$ during Taylor relaxation, where the subscripts R are labels for each region and V denotes the vacuum only, and the superscripts p, t label the fluxes as poloidal and toroidal, respectively. Given the vessel with boundary W, the interfaces I_i, and the magnetic field $\mathbf{B}$, (2.2)–(2.5) constitute a free boundary problem for p_i. A stability assessment of this configuration is the subject of a future publication.

3. Cylindrical equilibria

In addition to providing an illustrative example of multiple-interface equilibria, cylindrical equilibria provide a simple template upon which to explore complicated stability limits. A similar approach was taken by Ho and Prager [8], in their exploration of current driven stability limits in partially relaxed reverse field pinch configurations. To compute solutions in cylindrical geometry, we use the co-ordinate system (r, θ, z), with equilibrium variations permitted only in the radial direction. Following Kaiser and Uecker we also use normalize the plasma–vacuum boundary to $r = 1$, and assume that the cylinder is periodic in the z direction, with periodicity L. In this system, solutions to (2.2)–(2.5) can be written in vector notation $\mathbf{B} = \{B_r(r), B_\theta(r), B_z(r)\}$ as

$$\mathrm{P}_1 : \mathbf{B} = \{0,\, k_1 J_1(\mu_1 r),\, k_1 J_0(\mu_1 r)\}, \tag{3.1}$$

$$\mathrm{P}_i : \mathbf{B} = \{0,\, k_i J_1(\mu_i r) + d_i Y_1(\mu_i r),\, k_i J_1(\mu_i r) + d_i Y_1(\mu_i r)\}, \tag{3.2}$$

$$\mathrm{V} : \mathbf{B} = \{0,\, B_\theta^V/r,\, B_z^V\}, \tag{3.3}$$

where $k_i, d_i \in \Re$, and J_0, J_1 and Y_0, Y_1 are Bessel functions of the first kind of order 0, 1, and second kind of order 0, 1, respectively. The terms B_θ^V and B_z^V are constants. The constant d_1 is zero in the plasma core P_1, because the Bessel functions $Y_0(\mu_1 r)$ and $Y_1(\mu_1 r)$ have a simple pole at $r = 0$ [9].

The equilibrium problem can be prescribed by the $4N + 2$ parameters describing the magnetic field profile and the radial position of the barriers. That is,

$$\{k_1, \ldots, k_N, d_2, \ldots, d_N, \mu_1, \ldots, \mu_N, r_1, \ldots, r_{N-1}, r_\mathrm{w}, B_\theta^V, B_z^V\} \tag{3.4}$$

where r_i are the radial positions of the N ideal MHD barriers, and r_w is the radial position of the conducting wall. Equivalently, the equilibrium can be constrained by the safety factors and magnetic fluxes. That is, the $4N + 2$ quantities

$$\{\Psi_1^\mathrm{t}, \ldots, \Psi_N^\mathrm{t}, \Psi_1^\mathrm{p}, \ldots, \Psi_N^\mathrm{p}, \Psi_V^\mathrm{t}, \Psi_V^\mathrm{p}, q_1^i, \ldots, q_N^i, q_1^o, \ldots, q_N^o\} \tag{3.5}$$

where q_i^i and q_i^o are the safety factor on the inside and outside of each interface. In cylindrical geometry the safety factor expands as

$$q_i^i = \frac{2\pi r_i}{L}\frac{B_{z,i}(r_i)}{B_{\theta,i}(r_i)}, \quad q_i^o = \frac{2\pi r_i}{L}\frac{B_{z,i+1}(r_i)}{B_{\theta,i+1}(r_i)}, \tag{3.6}$$

In the core, we note that the function $rJ_0(\mu_1 r)/J_1(\mu_1 r)$ has positive radial derivative regardless of the value of μ_1, and so the plasma core will necessarily exhibit reverse magnetic shear. The toroidal and poloidal fluxes compute as follows:

$$\Psi_i^\mathrm{t} = \int_{r_{i-1}}^{r_i} B_z(r) r\, d\theta\, dr = \frac{2\pi}{\mu_i}\left[k_i r J_1(r\mu_i) + d_i r Y_1(r\mu_i)\right]_{r_{i-1}}^{r_i}, \tag{3.7}$$

$$\Psi_i^\mathrm{p} = \int_{r_{i-1}}^{r_i} B_\theta(r) L\, dr = \frac{2\pi}{\mu_i}\left[k_i J_0(r\mu_i) + d_i Y_0(r\mu_i)\right]_{r_{i-1}}^{r_i}. \tag{3.8}$$

Finally, in the vacuum region, the fluxes compute as

$$\Psi_\mathrm{V}^\mathrm{t} = B_\theta^V L \ln(r_w), \quad \Psi_\mathrm{V}^\mathrm{p} = B_z^V \pi (r_w^2 - 1) \tag{3.9}$$

The plasma pressure can be expressed in terms of the field strength B at the barriers.

Equations (3.6)–(3.9) form a mapping from the magnetic field profile factors and interface positions, constraints (3.4), to the safety factors and magnetic fluxes, (3.5).

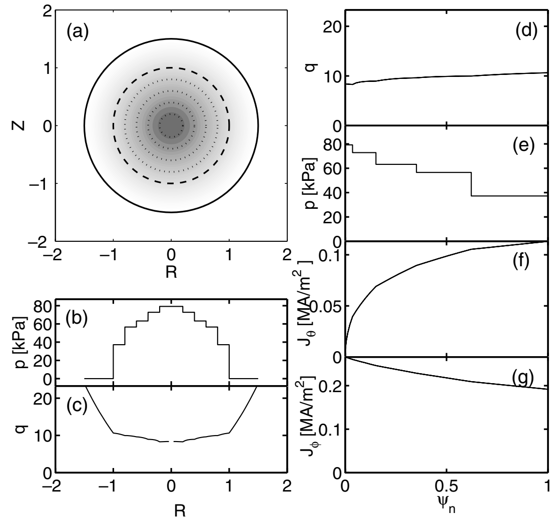

Figure 1. Example of a stepped-pressure plasma profile, with five ideal MHD barriers, showing : (a) a shaded contour plot of the poloidal flux ψ_p, (b) the pressure profile p, (c) the safety factor q versus position. Panels (d)-(g) show p, q, and toroidal, and poloidal current densities J_θ, J_ϕ as a function of normalized poloidal flux, where ψ_n is zero at the core, and unity at the plasma-vacuum interface. In panel (a), plasma-plasma interfaces are shown as dotted lines, the plasma-vacuum interface is dashed, whilst the plasma-wall boundary is solid.

An analytic form for the inverse mapping is not simply available. To compute the **B** field coefficients and interface positions, given the safety factor and magnetic flux, we have used the method of least squares. Starting in the core, $\Psi_1^{\rm p}, \Psi_1^{\rm t}$ and q_1^i can be solved for k_1, μ_1, r_1. In the plasma body, $\Psi_i^{\rm p}, \Psi_i^{\rm t}$ and q_i^i, q_i^o can be solved for k_i, d_i, μ_i, r_i. Finally, the position of the conducting wall $r_{\rm w}$ is found by solving the equation

$$r_{\rm w}^2 - 1 - \left(\frac{\Psi_{\rm V}^{\rm t}}{\Psi_{\rm V}^{\rm p}} \frac{B_\theta^V}{B_z^V} \right) L \ln r_W = 0. \tag{3.10}$$

Figure 1 shows an example with five ideal barriers. This particular example has been chosen with no change in q across the interfaces, and hence no surface currents. The equilibrium is described by the constraints: $r_{\rm w} = 1.5, B_{{\rm V},\theta} = 0.24$ T, $B_{{\rm V},z} = 0.40$ T, $r_i = \{0.2, 0.4, 0.6, 0.8, 1.0\}$, $k_i = \{0.22, 0.25, 0.29, 0.31, 0.35\}$, and $d_i = \{0.0, -0.010, -0.019, -0.050, -0.060\}$. This example demonstrates the existence of multi-interface, tokamak-like solutions, which do not require the existence of surface currents. By increasing the number of interfaces, the pressure can be approximated arbitrarily close to an experimental profile.

4. Conclusions

We have formulated a model for equilibria that comprise multiple Taylor-relaxed plasma regions, each of which is separated by an ideal MHD barrier of zero width. The system is enclosed by a vacuum region, and encased by a perfectly conducting wall. For these equilibria, the safety factor in the core necessarily decreases monotonically. For regions outside of the innermost ideal barrier, solutions can be constructed with increasing safety factors, and decreasing pressures. A tokamak like example of a multiple-interface equilibria is provided. These equilibria exhibit many of the same qualities observed in high-performance H-mode discharges. In future work, the stability of the multiple interface equilibria presented here will be studied. We will also generalize our equilibria model to three dimensions, in which the shape of all surfaces are free.

Acknowledgments

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