An expression for the temperature gradient in chaotic fields

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(Received 27 October 2008; accepted 11 December 2008; published online 27 January 2009)

A coordinate system adapted to the invariant structures of chaotic magnetic fields is constructed. The coordinates are based on a set of ghost-surfaces, defined via an action-gradient flow between the minimax and minimizing periodic orbits. The construction of the chaotic coordinates allows an expression describing the temperature gradient across a chaotic magnetic field to be derived. The results are in close agreement with a numerical calculation. © *2009 American Institute of Physics*. [DOI: 10.1063/1.3063062]

We continue[1] the study of anisotropic heat transport across a chaotic magnetic field $\mathbf{B}$, where the heat flux vector is given as

$$\mathbf{q} = \kappa_\parallel \mathbf{b}\mathbf{b}\cdot\nabla T + \kappa_\perp \nabla T \tag{1}$$

for $\mathbf{b}=\mathbf{B}/|B|$, and T is the temperature. The parallel and perpendicular transport are characterized by the diffusion coefficients $\kappa_\parallel$ and $\kappa_\perp$, which we take to be constants. In a region with no significant sources or sinks, the steady-state temperature is determined by the second-order differential equation

$$\nabla\cdot\mathbf{q}=0. \tag{2}$$

For fusion relevant plasmas,[2] the heat transport is highly anisotropic: $\kappa_\perp/\kappa_\parallel \approx 10^{-10}$. It is instructive to consider the "ideal-limit," where the parallel transport is infinite compared to the perpendicular transport: $\kappa_\perp/\kappa_\parallel=0$. The condition $\nabla\cdot\mathbf{q}=0$ then requires that along each field-line $\mathbf{B}\cdot\nabla T=\alpha B^2$, where α is a constant, and the only acceptable value is $\alpha=0$. To see this, consider integrating along a field-line from some initial point, where $T=T(0)$, to obtain $T(\eta)=T(0)+\alpha\int_0^\eta B^2 d\eta$, where η parametrizes distance along a field-line, $\partial_\eta \equiv \mathbf{B}\cdot\nabla$. If the field-line returns to the initial point after a nonzero distance $\eta_{p/q}$, for the temperature to be a single valued position of space we require $T(\eta_{p/q})=T(0)$. Thus, for periodic orbits we must have $\alpha=0$. An irrational field-line that lies on a flux surface comes arbitrarily close to the initial point after an arbitrarily long distance, and irregular field-lines come arbitrarily close to any point in a finite volume, including the initial point. Thus, we must have $\alpha=0$ almost everywhere, and in the limit that $\kappa_\perp/\kappa_\parallel=0$, the temperature is invariant under the field-line-flow, $\mathbf{B}\cdot\nabla T=0$.

This paper will explore the hypothesis that if coordinates (s,θ,ϕ) can be adapted to the invariant structures of the magnetic field, the steady-state temperature will take the form $T=T(s)$. This is justified *a posteriori* by deriving an expression for the temperature gradient and showing that this expression leads to an accurate description of the temperature profile, as compared to a numerical solution.

Chaotic coordinates. Generally, the temperature is represented as a function of three-dimensional space. For example, in toroidal geometry $T=T(\psi,\theta,\phi)$, where ψ is an arbitrary radial coordinate (e.g., ψ labels flux surfaces of a nearby integrable field), and θ,ϕ are poloidal and toroidal angles, respectively. If the field possesses a smooth set of nested flux surfaces (i.e., the field is integrable), labeled with radial coordinate ψ, magnetic coordinates can be constructed globally so that $\mathbf{B}\cdot\nabla\psi=0$. The temperature is then constant on the flux surfaces: $T=T(\psi)$. For slightly chaotic fields, flux surfaces with sufficiently irrational rotational-transform are guaranteed to survive sufficiently small perturbation by virtue of the Kolmogorov–Arnold–Moser (KAM) theorem,[3,4] and these can be used as a framework for the radial coordinate. An irrational surface is the closure of an irrational field-line, so in the limit $\kappa_\perp/\kappa_\parallel=0$, the temperature must be constant on the KAM surfaces. In a region where no true invariant surfaces exist, a more esoteric construction of radial coordinate surface is required.

Previously,[1] numerical evidence was given suggesting that the steady-state temperature contours in a chaotic field will coincide with a set of so-called ghost-surfaces,[5] which are a class of almost-invariant surface.[6,7] Motivated by this result, here we extend the construction of magnetic coordinates to chaotic magnetic fields.

The chaotic coordinates that we construct are adapted to structures invariant under the field-line flow; namely, the periodic orbits and the irrational field-lines. We consider fields with the so-called twist condition, so that the shear is nowhere zero. When an integrable field is destroyed by perturbation, the Poincaré–Birkhoff theorem[4] states that at least two periodic orbits will survive, which for small perturbation are the stable and unstable periodic orbits. Additionally, the Aubry–Mather theorem[4] tells us that the irrational field-lines will also survive perturbation. If the irrational field-line ergodically covers a surface, the surface is a KAM surface. If not, the irrational field-line is called an Aubry–Mather set[8,9] or a cantorus.[10] The irrational field-lines may be approximated arbitrarily closely by suitably chosen rational field-lines, so from practical perspective we need only consider adapting the chaotic coordinates to the periodic orbits. This is achieved by constructing a set of rational ghost-surfaces. (The importance of the irrational field-lines, i.e., the cantori, will be discussed below.)

Ghost-surfaces are defined using the Lagrangian formulation of magnetic field-line dynamics: magnetic field-lines are extremal curves $\mathcal{C}$ of the action integral[11] $S_{\mathcal{C}}=\int_{\mathcal{C}}\mathbf{A}\cdot\mathbf{dl}$. Constraining attention to (p,q) periodic curves, where θ

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$=\theta(\phi)$ satisfies $\theta(2\pi q)=\theta(0)+2\pi p$, the stable and unstable periodic field-lines are the minimax curve (a saddle point of the action) and the minimizing curve, respectively.[4] (Note that for sufficiently large perturbation, the minimax orbit also becomes unstable.) At the minimax curve, there exists a single direction in configuration space along which the action integral decreases. By perturbing the minimax orbit in this direction, then allowing the curve to flow down the action-gradient to the minimizing periodic orbit, the curve will trace out a surface, the (p,q) ghost surface.[1,6]

Numerical evidence indicates that different ghost-surfaces, as identified by their periodicity (p,q), do not intersect.[1,6] A selection of ghost-surfaces may be used as the framework for a radial coordinate. The ghost-surfaces are Fourier decomposed, and a piecewise linear interpolation of the Fourier harmonics ensures that the interpolated surfaces do not intersect. Note that each ghost-surface passes through its respective island chain and necessarily "captures" the minimax and minimizing periodic orbits. Furthermore, by selecting ghost-surfaces of sufficiently high periodicity, the chaotic coordinates may be adapted to the cantori.

The cantori are the action-minimizing irrational field-lines. They may be approximated arbitrarily closely by the action-minimizing rational field-lines. Near-critical cantori are particularly important for understanding transport in chaotic fields. By "near-critical" it is meant that perturbation slightly exceeds the value at which the irrational field-line no longer traces out a smooth surface; i.e., when the KAM surface is destroyed. Irregular (chaotic) field-lines may pass across the cantori. However, field-line transport across near-critical cantori can be extremely slow; thus, these cantori are effective partial barriers to transport.[12] Just as the most irrational (noble) KAM surfaces are most likely to survive perturbation,[13] the noble cantori typically have locally minimal field-line flux and present the most significant impediment to anisotropic heat transport in chaotic fields. Furthermore, the existence of near-critical cantori (and also the regions of regular trajectories near stable periodic orbits) violates the assumptions underpinning the random-walk, diffusive model of field-line transport in chaotic fields,[14] and one is led to a fractional-diffusion approach.[15] It is only when the field is "uniformly" chaotic, i.e., well above the stochastic threshold, that one may approximate field-line transport as a random process.

Comparison with numerical solution. To see that the ghost-surfaces coincide with isotherms, the steady-state solution to the anisotropic heat transport equation is solved numerically. A model magnetic field is considered, $\mathbf{B}=\nabla\times\mathbf{A}$, with vector potential $\mathbf{A}=\psi\nabla\theta-\chi\nabla\phi$, where $\chi(\psi,\theta,\phi)$ is the field-line Hamiltonian

$$\chi=\psi^2/2+\sum_{mn}\chi_{m,n}(\psi)\cos(m\theta-n\phi). \quad (3)$$

This magnetic field is stellarator symmetric, which allows several simplifications (e.g., periodic orbits lie on symmetry lines[16]), but does not alter the characteristic properties of the chaotic field. For nonzero $\chi_{m,n}$, magnetic islands form around the stable periodic orbit, and irregular field-lines emerge from near the unstable periodic orbits. To excite islands at

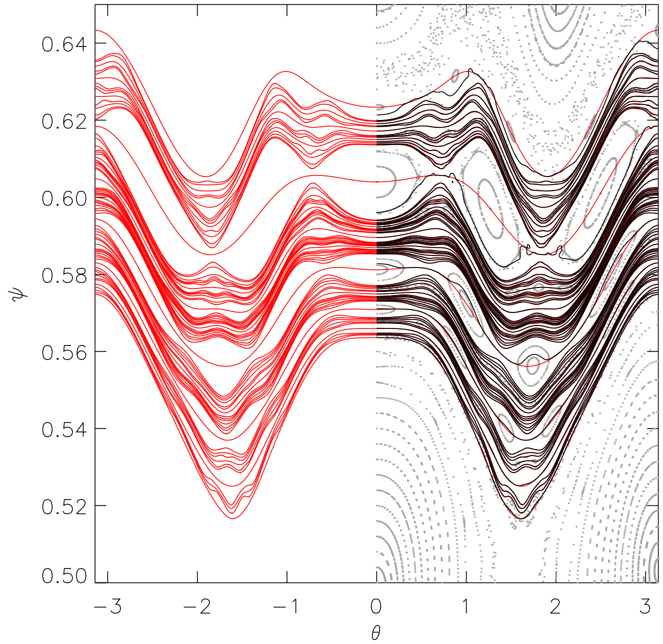

FIG. 1. (Color online) Ghost-surfaces (red lines) and corresponding isotherms (black lines, only for $\theta>0$) are shown for a near-critical chaotic field. A Poincaré plot is shown with gray dots. The ghost-surfaces and isotherms are almost indistinguishable.

the $\psi=1/2$ and $\psi=2/3$ rational surfaces, we set $2^2\chi_{2,1}=3^2\chi_{2,3}=k$, where k is a perturbation parameter. For large enough k, the region between the (1,2) and (2,3) island chains is dominated by irregular field-lines and island chains. A Poincaré plot of the field, with $k=4.5\times10^{-3}$, is shown in Fig. 1.

A temperature gradient across the chaotic field between the (1,2) and (2,3) islands is enforced by inhomogeneous boundary conditions, namely, that $T=1.0$ on $\psi=0.50$ and $T=0.0$ on $\psi=0.65$, and we study the case where the ratio of transport coefficients is given: $\kappa_\perp/\kappa_\parallel=10^{-10}$. The strong parallel transport is separated from the weak perpendicular transport by employing locally field aligned coordinates. The steady-state temperature is solved iteratively using finite differences on a high-resolution numerical grid. The numerical approach is identical to the approach used in Ref. 1.

In Fig. 1, 72 ghost-surfaces between the (1,2) and (2,3) islands are shown. This selection includes low-order rational ghost-surfaces, e.g., $(p,q)=(3,5)$, $(4,7)$, and $(5,8)$, which pass through the corresponding island chains. The low-order islands are typically larger (than the higher order islands), and if an island exceeds a critical width, $\Delta w\sim(\kappa_\perp/\kappa_\parallel)^{1/4}$, the temperature will tend to flatten inside the island.[2] In addition, ghost-surfaces with periodicities approximating various noble irrationals were selected; e.g., $(p,q)=(37,66)$, $(41,71)$, and $(44,75)$. These "irrational" ghost-surfaces form coordinate surfaces that "fill in the gaps" in the near-critical cantori. In addition, high-order ghost-surfaces that lie adjacent to the chaotic separatrices of the low-order islands are selected. In the strongly anisotropic limit, the temperature will flatten across the islands and will also adapt closely to the these "boundary" surfaces lying just outside the chaotic separatrix, giving the temperature a fractal struc-

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ture. (The term "boundary circle" was introduced to describe the closest KAM surface next to a chaotic separatrix.[17,18]) The near-fractal structure of coordinates matches the near-fractal structure of the temperature, and this allows a simple expression for the temperature gradient in chaotic coordinates to be derived.

Semi-analytic solution of temperature profile. To a remarkable degree, the ghost-surfaces coincide with isotherms, so we may use the approximation $T=T(s)$, where s labels the ghost-surfaces (and their interpolates). To derive an expression for the temperature gradient consider the following integral over a volume bounded by a surface s=const,

$$\frac{d}{ds}\int_V \nabla \cdot \mathbf{q} dV \equiv \frac{d}{ds}\int_{\partial V} \mathbf{q}\cdot \mathbf{dS} = 0, \tag{4}$$

where $\mathbf{dS}=\sqrt{g}\nabla s d\theta d\zeta$. An expression for the temperature gradient, $T' = dT/ds$, is derived using Eq. (1),

$$\frac{dT}{ds} = \frac{c}{\kappa_\parallel \varphi + \kappa_\perp G}, \tag{5}$$

where φ is the squared field-line flux across a coordinate surface and G is an averaged metric quantity,

$$\varphi = \int\int d\theta d\phi \sqrt{g} B_n^2, \tag{6}$$

$$G = \int\int d\theta d\phi \sqrt{g} g^{ss}, \tag{7}$$

for $B_n \equiv \mathbf{B}\cdot\nabla s/|B|$ and $g^{ss}=\nabla s\cdot\nabla s$. The integration constant c, and a second integration constant that appears when Eq. (5) is integrated to obtain $T(s)$, are determined from $T(a)$ and $T(b)$, the respective averages of the numerical solution on the lowermost and uppermost ghost-surfaces shown in Fig. 1. This allows the profile defined by Eq. (5) to be directly compared with the numerical solution, as shown in Fig. 2. Good agreement with the numerical solution is obtained.

With $\kappa_\parallel \gg \kappa_\perp$, the temperature gradient given by Eq. (5) is dominated by surfaces with minimal φ. In our construction of chaotic coordinates, local maximum temperature gradients will coincide with the noble cantori. In the ideal limit, infinite gradients are supported on any KAM surfaces that exist (where $\varphi=0$; compare to the ideal equilibrium model described by Dewar *et al.*[19]), and the profile will approach a devil's staircase. If $\kappa_\perp$ is nonzero, T' will everywhere be finite, so $T(s)$ will be smooth.

Comments. This paper has presented evidence suggesting that the numerically intensive task of solving highly anisotropic heat transport in chaotic fields may be reduced to the task of constructing chaotic-coordinates. Computationally, this is much simpler. There are, however, several questions that remain outstanding. For example, for a given chaotic field, what is the best selection of ghost-surfaces to serve as the coordinate framework? The selection of ghost-surfaces shown in Fig. 1 was empirical—a set of surfaces was chosen that resulted in a good fit to the temperature profile. To be of practical value however, it is required to *a priori* determine which set of ghost-surfaces is optimal. Implicit in the introductory discussion was that the perpendicular diffusion $\kappa_\perp$ is negligible, so the temperature exactly adapts to the fractal structure of the field, and thus also to the chaotic coordinates. However, the fine-scale structure of the temperature is smoothed out as $\kappa_\perp$ increases. The temperature will not completely flatten across islands less than the critical island width, and accordingly the temperature will not exactly coincide with boundary ghost-surfaces that are too close to these islands' separatrices. It would be beneficial to know how the optimal selection of ghost-surfaces depends on the ratio $\kappa_\perp/\kappa_\parallel$. These questions are the topic of ongoing investigation.

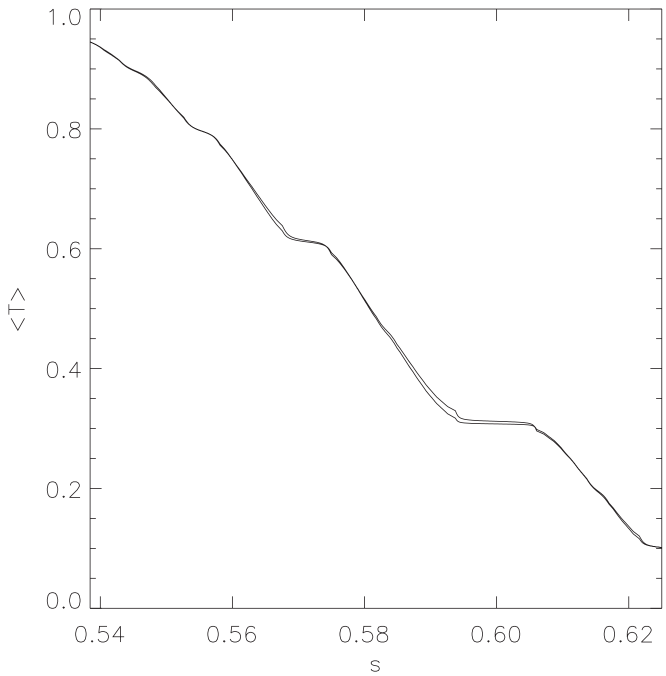

FIG. 2. Temperature profile $T(s)$ constructed from Eq. (5) is compared with the numerical profile with good agreement.

The agreement between the ghost-surfaces and the isotherms shown in Fig. 1, and the agreement between the numerical and reconstructed temperature profile (Fig. 2) is qualitative. A detailed quantitative comparison will be deferred until a systematic selection of an optimal set of ghost-surfaces has been derived. We expect to show that the error between the "exact" numerical profile and the reconstructed profile can be reliably and systematically reduced.

There exist other constructions of almost invariant surfaces that may be suitable for use as the radial framework for chaotic-coordinates. Ghost-surfaces have been chosen here as they fit neatly with Lagrangian integration methods, which provides a robust approach to the construction of cantori in strongly chaotic fields;[20] however, the quantity φ bears a striking resemblance to the quadratic-flux functional.[7,21] This suggests that quadratic-flux minimizing surfaces may be more suitable for organizing anisotropic heat transport.

Finally, we note that Eq. (5) is quite general and independent of the construction of chaotic-coordinates. If there is no local source, the isotherms form a set of nested surfaces which may themselves be used as coordinate surfaces. In this

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case, the approximation $T=T(s)$ is exact, and, therefore, so is Eq. (5).

This work was supported in part by U.S. Department of Energy Contract Nos. DE-AC02-76CH03073 and DE-FG02-99ER54546.

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