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A new class of magnetic confinement device in the shape of a knot

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We describe a new class of magnetic confinement device, with the magnetic axis in the shape of a knot. We call such devices "knotatrons." An example is given that has a large volume filled with magnetic surfaces, with significant rotational-transform, and with the magnetic field produced entirely by external circular coils. © *2014 AIP Publishing LLC*.
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The principle of magnetically confining a plasma exploits the fact that the motion of charged particles in a strong magnetic field consists of a free-streaming motion parallel to the field combined with a small, perpendicular gyration. In a uniform field in cylindrical geometry, for example, the particles are well confined in the perpendicular direction: in the absence of electric fields and magnetic field inhomogeneities, the perpendicular loss of particles is due solely to collisions. However, a finite-length cylinder is not closed. To prevent so-called "end-losses," modern experiments, such as ITER[1] and W7X[2] presently under construction, join the ends of the cylinder and thereby construct a topological torus.

The simplest toroidal configuration has each magnetic fieldline making a circle about a given rotational axis of symmetry, e.g., the Z axis, where (R, ϕ, Z) are the usual cylindrical coordinates. Such a configuration, however, cannot support a plasma. Bending the magnetic field into a torus necessarily compresses the field on the inside of the torus. The vertical drift of particles caused by the inhomogeneity in the field strength produces an electric field, which in turn results in loss of confinement. To cancel the particle drifts, it is necessary that the magnetic fieldlines rotate the short way around the torus, the poloidal direction, while they rotate the long way around, the toroidal direction. The average pitch of this rotation is measured by the rotational-transform, defined as average number of poloidal rotations per toroidal rotation, $\iota \equiv \Delta\theta/\Delta\phi$ as $\Delta\phi \to \infty$, where θ measures the angle around the magnetic axis, which may be defined as follows.

Consider a magnetic field with at least one magnetic flux-surface, defined as a toroidal surface to which the field is everywhere tangential. Assuming a strong toroidal component of the field, so that $\mathbf{B} \cdot \hat{\phi} > 0$, we may take the $\phi = 0$ plane as a Poincaré section. Because $\nabla \cdot \mathbf{B} = 0$, the area of the Poincaré section bounded by the flux-surface is mapped to itself by the return map, which is defined by following along a fieldline around the machine back to the initial Poincaré section (or, if there is a q-fold symmetry in ϕ, its symmetric equivalent). The Brouwer fixed-point theorem[3] applies: there must be one (in fact, at least one) point on this area which is mapped to itself. The fieldline passing through this point is called the magnetic axis and closes upon itself after just one turn.

At the most primitive level, a magnetic confinement device must have a large volume of space occupied by magnetic fieldlines that wrap around on closed, toroidal surfaces.

The *tokamak* class of toroidal confinement device is rotationally symmetric. (Realistically, however, there will always be some "toroidal ripple" due to the finite number of external current coils that produce the required magnetic field.) In the axisymmetric case, the only possible way to produce a poloidal magnetic field needed for rotational-transform is to induce or drive a toroidal current in the plasma itself. Tokamak plasmas are, consequently, inherently prone to disruptions: anything that leads to a sufficient kinking or displacement of the plasma current can break the magnetic "bottle" confining the plasma.

Axisymmetry does provide some important advantages: toroidal magnetic fields are analogous with $1\frac{1}{2}$-dimensional Hamiltonian dynamical systems,[4] and the rotational symmetry *guarantees* that space is filled by magnetic surfaces, i.e., that the magnetic field is integrable. The particle trajectories are also governed by a Hamiltonian,[5] and the rotational symmetry means that the particle trajectories possess certain invariants so that particle transport is reduced.

There are methods other than driving a toroidal current by which a rotational-transform can be produced. For example, if the configuration is smoothly deformed to create a non-planar magnetic axis, which of course must destroy the rotational symmetry, the nontrivial torsion of the axis creates a rotational-transform.[6,7] This is the approach adopted by many[8] important examples of the *stellarator* class,[9] in which a magnetic field with non-zero rotational-transform is produced by currents external to the plasma. (This definition includes torsatrons,[10] heliotrons,[11] and heliacs,[12] all of which refer to a particular arrangement of the external currents and which are hereafter referred to as conventional stellarators.) The confining magnetic field in stellarators does not depend on the plasma current, and stellarators are thus far more stable than tokamaks.

To construct a set of external currents that provide vacuum magnetic surfaces with non-zero rotational-transform is, however, easier said than done. The lack of a continuous symmetry means that there will not, in general, be a nested family of flux-surfaces. Whenever there is a resonance between the geometry of the configuration and fieldlines with rational rotational-transform, $\iota = q/p$ for integers p and q, a magnetic island will form at the resonant surface. The

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size of the island depends on the strength of the geometrical resonance and the shear. In a large magnetic island there is a volume of "elliptic" flux-surfaces with fieldlines that rotate about a stable periodic orbit. A (p, q) periodic orbit is defined here as a fieldline that closes upon itself after completing q poloidal transits and p toroidal transits. When nearby islands are sufficiently large so as to overlap, irregular fieldlines emerge, which wander seemingly randomly over a volume. The existence of such chaotic volumes degrades confinement.

The challenge of stellarator design is, in part, to construct coil configurations that either avoid the resonances, by tailoring the configuration so that no strongly resonant, rational surfaces are present, or to ensure that strength of the geometrical resonance is sufficiently small so that the islands are negligible. This must be achieved by geometrical shaping. Magnetic islands also appear in tokamaks, but rather than being caused by geometrical resonances they result from resistive plasma instabilities associated with unstable pressure and current profiles and are usually called tearing modes.[13]

Breaking the rotational symmetry breaks the associated invariants of the particle motions. This introduces a variety of additional transport mechanisms by which particles may be lost. Historically, this has put stellarators at a disadvantage as compared to tokamaks.[14]

However, the lack of axisymmetry does not mean that stellarators *cannot* have a sufficiently large volume occupied by flux-surfaces. By careful geometrical design, both the vacuum field[15] and finite-pressure equilibria[16] in strongly non-axisymmetric stellarators can be made arbitrarily close to integrable. Similarly, the lack of axisymmetry does not mean that stellarators cannot have good particle trajectories, and modern stellarator designs are able to significantly reduce the level of neoclassical transport.[17]

The stellarator class allows for three-dimensional shaping and so a far greater variety of configurations is possible; but the theoretical analyzes are more complicated, numerical computation is more costly, and the configuration space of stellarator design takes longer to explore. We stress this point because the example *knotatron* configuration described below has not been geometrically optimized to obtain optimal plasma performance. This will be left for future work.

Joining the ends of a cylinder together to form a tokamak or conventional stellarator is not the only option for closing the magnetic field in order to prevent end-losses. The magnetic axes of rotationally symmetric tokamaks are circles, and those of conventional stellarators are smoothly deformable into circles. There is another class of plasma magnetic confinement device that (i) is closed, in the sense that magnetic fieldlines wrap around on flux-surfaces that enclose a finite volume; (ii) has significant rotational-transform, even in the absence of plasma current; and (iii) that has a magnetic axis that is *not* smoothly deformable into a circle. We may wonder if better plasma confinement can be obtained if the plasma is, instead, tied into a knot.

Mathematically, a knot is defined an embedding, $K : S^1 \hookrightarrow S^3$, of a 1-sphere, i.e., a circle, into the 3-sphere.[18,19] Or more simply, a knot is a closed, one-dimensional, non-intersecting curve in three-dimensional space, $\mathbb{R}^3$. This includes the trivial knot, the circle, which is also called the *unknot*. Both knots and unknots share the topology of the circle. Two knots, K and $\bar{K}$, are *ambient isotopic* if there exists a continuous one-parameter family, h_t, of homeomorphisms of S^3 such that h_0 is the identity map and $h_1 \circ K = \bar{K}$. In other words, two knots are equivalent if there is a continuous deformation of $\mathbb{R}^3$ that maps one knot into the other. A knot may be distinguished from the unknot, as it is not possible to deform a knot into the unknot without the knot passing through itself or it being cut. The theory of knots is relevant to a wide range of topics, including the study of DNA dynamics[20] and quantum field theory.[21]

A (p, q)-torus knot, where (p, q) is a pair of co-prime integers, is a special kind of knot that lies on the surface of a torus: it winds q times around the poloidal direction, and p times around the toroidal direction. (The torus that the knot winds around on need not have any particular symmetry, i.e., the torus may look like a conventional stellarator flux-surface.) An example parametrization is

$$\begin{aligned} x(\zeta) &= R(\zeta)\cos(p\,\zeta), \\ y(\zeta) &= R(\zeta)\sin(p\,\zeta), \\ z(\zeta) &= -r\sin(q\,\zeta), \end{aligned} \tag{1}$$

where $R(\zeta) = R_0 + r\cos(q\,\zeta)$, and R_0 and r are constants. Note that $R(\zeta + 2\pi/q) = R(\zeta)$ and $z(\zeta + 2\pi/q) = z(\zeta)$, so there is a q-fold rotational symmetry around the Z axis. The usual cylindrical angle, $\tan\phi = y/x$, is $\phi = p\,\zeta$. The simplest, nontrivial example, which will be considered in the following, is the (2, 3)-torus knot, which is also known as the trefoil knot.

This article introduces a new type of plasma confinement device—the *knotatron*—a device that has a magnetic axis in the shape of a knot. A set of external coil currents that provides the required magnetic field is constructed as follows.

A reference curve, $\mathbf{x}(\zeta) \equiv x(\zeta)\,\mathbf{i} + y(\zeta)\,\mathbf{j} + z(\zeta)\,\mathbf{k}$, that has the shape of a (p, q)-torus knot is chosen as a proxy magnetic axis. The unit tangent to this curve is $\mathbf{t} \equiv \dot{\mathbf{x}}/|\dot{\mathbf{x}}|$, where the "dot" denotes derivative with respect to ζ.

A set of $i = 1,\ldots,N$ circular coils are positioned equally spaced along this reference curve, so that the center of each coil is located at $\mathbf{x}_i \equiv \mathbf{x}(\zeta_i)$, where $\zeta_i = 2\pi(i-1)/N$. Each coil is assumed to carry unit current and has radius a. The magnetic field produced by each coil, $\mathbf{B}_i$, is computed using the Biot-Savart law. The initial orientation is such that the i-th coil is described by

$$\mathbf{x}(\theta, \zeta_i) = \mathbf{x}(\zeta_i) + a\cos\theta\,\mathbf{n} + a\sin\theta\,\mathbf{b}, \tag{2}$$

where $\mathbf{n} \equiv \dot{\mathbf{t}}/\kappa$ is the principal normal, $\kappa = |\dot{\mathbf{t}}|$ is the curvature, and $\mathbf{b} \equiv \mathbf{t} \times \mathbf{n}$ is called the bi-normal. The set $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is called the Frenet-Serret frame. The torsion, τ, measures the speed of rotation of the bi-normal vector, $\dot{\mathbf{b}} = -\tau\mathbf{n}$.

By symmetry, $\mathbf{B}_i$ is tangential to the reference curve at the coil center, $\mathbf{B}_i(\mathbf{x}_i) \times \mathbf{t}_i = 0$; however, this initial orientation of the coils does not guarantee that the *total* magnetic field, $\mathbf{B} = \sum_i \mathbf{B}_i$, is tangential to the reference curve. To

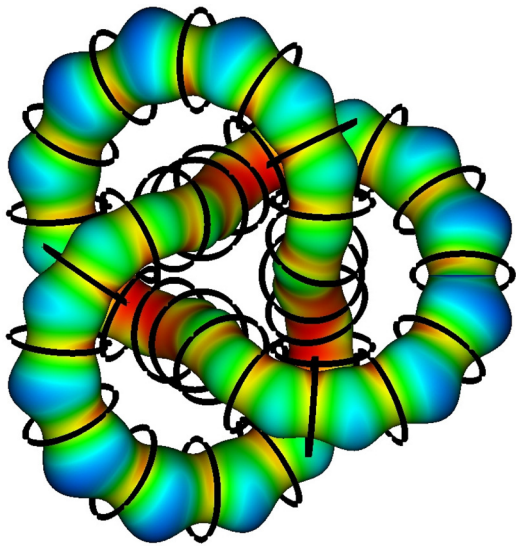

FIG. 1. A trefoil-knotatron: 36 circular coils, each with unit current, produce a magnetic field with an axis in the shape of a trefoil knot. The color indicates $|B|$ on a flux surface.

enforce the condition $\mathbf{B}(\mathbf{x}_i) \times \mathbf{t}_i = 0$, $\forall \mathbf{x}_i$, with a set of N circular coils centered at the $\mathbf{x}_i$, it is required to appropriately adjust their orientations. This is a simple numerical task: there are 2 degrees-of-freedom in the orientation of each coil, and the condition $\mathbf{B}(\mathbf{x}_i) \times \mathbf{t}_i = 0$, $\forall \mathbf{x}_i$, represents a total of $2N$ constraints. A solution may be obtained iteratively. With only a finite set of coils this algorithm cannot guarantee that $\mathbf{B}(\mathbf{x}) \times \mathbf{t} = 0$ everywhere along the reference curve, and so the magnetic axis of $\mathbf{B}$ will not exactly coincide with $\mathbf{x}(\zeta)$; but as N increases the agreement improves.

Shown in Fig. 1 is an illustration of the resulting coil configuration for a (2, 3) torus-knot, where we have chosen $R_0 = 2$, $r = 1$, and $a = 2/3$, in units of meters, and $N = 36$. Shown in Fig. 2 is a Poincaré plot of the resulting magnetic field on the $\phi = 0$ plane, and Fig. 3 shows the rotational-transform. The $\phi = 0$ plane cuts a (p, q)-torus knotatron p times; and the magnetic axis is $2\pi p/q$ periodic in ϕ, rather than the usual $2\pi/q$, in a device with a q-fold symmetry. The rotational-transform on axis is $\iota \approx 0.261$. The colored surface shown in Fig. 1 is a flux-surface. The color indicates $|B|$, with blue showing areas of weak magnetic field and red showing strong. The intersection of this surface with the

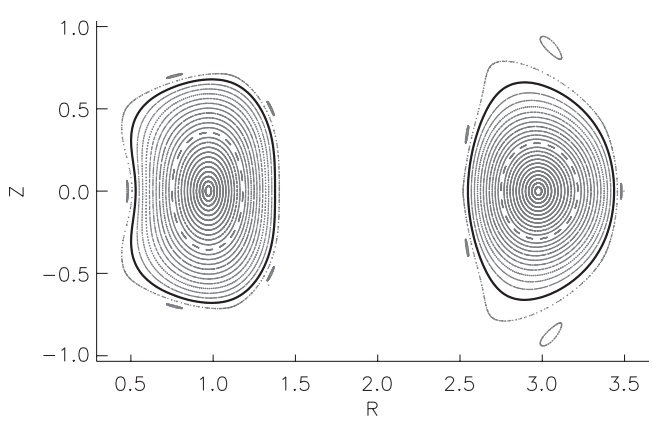

FIG. 2. Poincaré plot on the $\phi = 0$ plane.

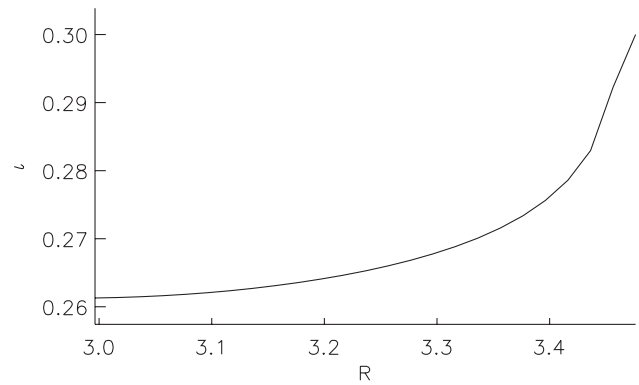

FIG. 3. Rotational-transform, ι, plotted against R.

Poincaré section is shown as the thick line on Fig. 2. There is a large volume of space, about $5.08\,\mathrm{m}^3$, filled with flux-surfaces with a significant rotational-transform.

The first conceptual obstacle to overcome in the investigation into whether knotatrons may provide advantages for confining plasmas is to overcome the perception that such devices may be awkward or even impossible to construct. To address this concern, rather than adjusting the coil geometry in order to shape the vacuum flux-surfaces to improve confinement, we have instead sought to construct a simple set of coils that do not intersect—the minimum coil-coil separation is 32 cm—and that produces the required magnetic axis. We have chosen a moderate number of coils, 12 coils per period, to illustrate that knotatrons should be easy, at least possible, to engineer; however, the fewer coils the greater the toroidal ripple, and some toroidal ripple is clearly seen in Fig. 1. (The surfaces closer to the magnetic axis are less rippled.)

This coil configuration is unlikely to be "flexible," in the sense that it may not be possible to construct many different magnetic configurations by simply varying the currents. Additional helical windings and/or vertical field coils, such as those used in conventional stellarators, will probably be required for an attractive experimental design.

It is natural to ask, what advantages do knotatrons provide? Common sense would suggest that the construction and operation of knotatrons would be far more complex than that of conventional stellarators, but this is not necessarily the case. Modern experiments are already quite complex, and there is no obvious reason why an experimental knotatron would need to be as compact as the example shown in Fig. 1.

The knotatron example presented here could perhaps be compared to an early stellarator design, e.g., the model-A stellarator, which had the magnetic axis in the shape of a figure-8.[9] The early stellarators are not attractive experimental designs by today's standards. To thoroughly investigate the potential of knotatrons, and thus determine what knotatrons may have to offer, it is required to perform extensive equilibrium, stability and transport studies, and to employ optimization algorithms to find an attractive design, as is required for all modern devices.

The rotational-transform is produced by the torsion, which measures the extent to which the magnetic axis deviates from lying in a plane. As non-trivial knots *cannot* lie on

a plane, one may speculate that knotatrons are capable of producing greater rotational-transform than a conventional stellarator, but this is unclear. The Fary-Milnor theorem[22] states that three-dimensional smooth curves with small total curvature must be unknotted: if $\oint \kappa(\zeta)\, d\zeta \leq 4\pi$, then the curve is an unknot; but it does not follow from this that curves with large total curvature must be knots.

Just as there is a much greater variety of conventional stellarators than tokamaks, there is a much greater variety of knotatrons than conventional stellarators. There is a seemingly infinite variety of knots: not all knots are torus knots, there is, for example, also the class of Lissajous knots;[23] and given two non-trivial knots an additional *composite* knot may be constructed by cutting each knot and joining the ends. Within each class of knot family it is possible to vary the geometry of the configuration to improve the confinement. This begs the question: is there a class of knot that provides optimal confinement? The curvature and torsion of the magnetic fieldlines are important for magnetohydrodynamic stability, and it would seem that knotatrons can provide a greater variety of curvatures and torsions than tokamaks or conventional stellarators.

As both tokamaks and conventional stellarators have magnetic axes that are ambient isotopic to the unknot, these devices may be called *unknotatrons*. As the confining magnetic field in a knotatron is produced by external currents, and the rotational-transform is produced by a non-planar magnetic axis, knotatrons are a new example of the stellarator class.

To our knowledge, this is first time that a magnetic confinement device in the shape of a knot has been considered. However, knotatrons have already been realized experimentally in a sense, albeit perhaps unintentionally and as yet unrecognized as knotted confinement regions. A (p, q) periodic orbit, for (p, q) relatively prime, is a torus-knot. The region of elliptic flux-surfaces in a large magnetic island, with fieldlines that rotate about a stable periodic orbit, forms a confinement region ambient isotopic to that of a knotatron.

Closely related to the theory of knots is the theory of links: given two closed curves, $\mathbf{x}(\zeta)$ and $\mathbf{y}(\zeta')$, the number of times one links the other is called the linking number and is given[24,25] by

$$-\frac{1}{4\pi}\oint\oint \frac{\mathbf{r}}{|r|^3} \times d\mathbf{y} \cdot d\mathbf{x}, \tag{3}$$

where $\mathbf{r}(\zeta, \zeta') \equiv \mathbf{y}(\zeta') - \mathbf{x}(\zeta)$, and $d\mathbf{x}$ and $d\mathbf{y}$ are infinitesimal line segments along each curves. Generalizing this to continuum of curves in a volume we obtain the helicity integral[24,25]

$$H \equiv -\frac{1}{4\pi}\oint\oint \frac{\mathbf{r}}{|r|^3} \times \mathbf{B}(\mathbf{y}) \cdot \mathbf{B}(\mathbf{x}) \cdot d^3y \cdot d^3x, \tag{4}$$

which, on using the vector potential in the Coulomb gauge,[24,25] reduces to $H = \int \mathbf{A} \cdot \mathbf{B} \cdot dv$.

The helicity integral has received a lot of attention in plasma physics: the principle of Taylor relaxation[26] has been successful in the predicting the behavior of reversed field pinch (RFP) experiments by postulating that weakly resistive plasmas will evolve to minimize the total energy, thermal-+magnetic, of the plasma under the constraint of conserved helicity. Seen as the generalization of the Gauss linking integral, the constraint of conserved helicity is equivalent to the constraint of constant "linked-ness." Recently,[27] the Taylor relaxation model has been extended to explain the experimentally observed self-organization of the RFP into helical states,[28] but the physical mechanism of Taylor relaxation remains a topic of ongoing investigation.[29–31]

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