Hamilton-Jacobi theory for continuation of magnetic field across a toroidal surface supporting a plasma pressure discontinuity

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The vanishing of the divergence of the total stress tensor for the magnetic field and pressure of a plasma in a neighborhood containing a surface of discontinuity gives boundary and jump conditions that strongly constrain allowable continuations of the magnetic field across the surface. The boundary conditions allow the magnetic fields on either side of the discontinuity surface to be described by surface magnetic potentials, reducing the continuation problem to that of solving a Hamilton-Jacobi equation. The characteristics of this equation obey Hamiltonian equations of motion, and a necessary condition for the existence of a continued field across a general toroidal surface is that there exist invariant tori in the phase space of this Hamiltonian system. It is argued from the Birkhoff theorem that existence of such an invariant torus is also, in general, sufficient for continuation to be possible. A method for determining the existence of invariant tori that will be used in further work on this problem is foreshadowed.

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I. INTRODUCTION

This paper is concerned with plasma confinement in a non-axisymmetric toroidal magnetic field. Pressure gradients cannot be sustained along field lines so determination of the structure of the magnetic field within the plasma is an important step in understanding confinement. In axisymmetric systems field lines draw out toroidally shaped *flux surfaces* and each surface sits nested between others filling the plasma volume. However, when the plasma is nonaxisymmetric (as is the case with stellarators and realistic tokamaks) the magnetic field lines that permeate the plasma become at least partially chaotic, so that some nested surfaces cease to exist, being replaced by new structures drawn from perturbed field lines.

Examples of field line structures in deformed plasmas include: flux surfaces that survive the deformation, though they will be nonaxisymmetric; cantori, toroidal structures drawn by a field lines that incompletely cover a surface, with gaps like a Cantor set; and chaotic regions where the field line no longer lies on a surface and wanders chaotically within an annular volume. Flux surfaces can act as barriers, separating these volumes of chaos from each other.

Chaotic field lines cannot feasibly support any pressure gradient within ideal MHD, so to achieve a non trivial pressure profile we assume finite changes, or *jumps* in pressure across each surviving flux surface.[1] Flux surfaces are infinitely thin so this results in a discontinuous pressure profile. Realization of magnetic surfaces sustaining finite pressure discontinuities arises naturally when applying Taylor relaxation to separate neighboring regions of the plasma,[1][2] but in this paper the condition is derived from basic MHD force balance.

This paper details a method for satisfying the force balance condition that applies when a finite pressure discontinuity occurs across an infinitely thin flux surface. We show that the force balance condition necessitates the magnetic field be derivable from potential functions defined on either side of the surface. The unknown surface potential can be determined by solving a Hamilton–Jacobi equation involving a Hamiltonian, the *pressure jump Hamiltonian* which is defined on a single flux surface, unlike the magnetic field line Hamiltonian described in Appendix B.

Earlier, Berk *et al* [3] introduced a Hamiltonian approach to investigate an individual flux surface's ability to withstand pressure, but their scope was limited to having a zero current field on one side of the surface and a zero field on the other. Kaiser and Salat[4] attempted to prove the existence of flux surfaces using only geometrical arguments, but this approach assumed the same field simplifications as Berk *et al* .

The geometrical solution (with no Hamiltonian basis) was implemented by Kaiser and Salat because of a worry that the phase space solutions achieved by a Hamiltonian method may not be valid when mapped back to the configuration space of the torus.[4] Kaiser and Salat were also doubtful of the application of the KAM theory, (which Berk *et al* alluded to) to ensure the existence

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of some flux surfaces under small deformations. Bruno and Laurence[5] investigate the existence of a global solution for the magnetic field that is Beltrami in neighboring annular toroidal volumes and satisfies force balance on the surface that separates the volumes. They appeal to the KAM theory to show that a solution to this problem can be found when departure from axisymmetry is sufficiently small. We concentrate on the local problem, the existence of invariant tori in the phase space of the Hamilton–Jacobi characteristics arising from the force balance condition. In contrast to Ref. 5 we are interested in highly perturbed configurations, so we discuss the use of tools other than the KAM theory to investigate the existence of invariant tori.

We argue that there is more information to be inferred form the Hamilton–Jacobi formulation of the force balance condition on its own in configurations far from axisymmetry, so we discuss the use of existence investigative tools other than the KAM theory.

This paper introduces a version of the Hamiltonian formulation with a more general applicability than those derived before. The paper clears up the analytical misunderstandings that have stopped the Hamiltonian formulation being totally accepted as a method to investigate the conditions for which force balance is satisfied. Section II explains the physical basis of the force balance condition and defines the pressure jump Hamiltonian. Section III proves the applicability of the Hamiltonian formulation, then discusses and dismisses the KAM theory as an avenue to investigate the existence of the surface potential, proposing the use of Greene's residue criterion instead.

Readers aware of the magnetic field line Hamiltonian, whose trajectories are the field lines that permeate and form the magnetic structure of the plasma, may want to refer to Appendix B for a comparison between the magnetic field line Hamiltonian and the pressure jump Hamiltonian introduced in this paper.

II. FORMULATION

A. Physical Derivation

For a plasma with a pressure profile $P(\mathbf{r})$ and magnetic field $\mathbf{B}(\mathbf{r})$, momentum conservation gives the force balance condition[6]

$$\nabla \cdot \left(P\mathbf{I} + \frac{1}{2}B^2\mathbf{I} - \mathbf{B}\mathbf{B} \right) = 0 \,, \tag{1}$$

where $\mathbf{I}$ is the unit dyadic and we use units in which $\mu_0 = 1$. The purely divergence nature of Equation (1) means the quantity $\left(P\mathbf{I} + \frac{1}{2}B^2\mathbf{I} - \mathbf{B}\mathbf{B}\right)$ is the stress tensor at a point in the plasma.

We are interested in weak solutions of Equation (1) about a given surface consistent with the divergence free nature of the field

$$\nabla \cdot \mathbf{B} = 0 \,. \tag{2}$$

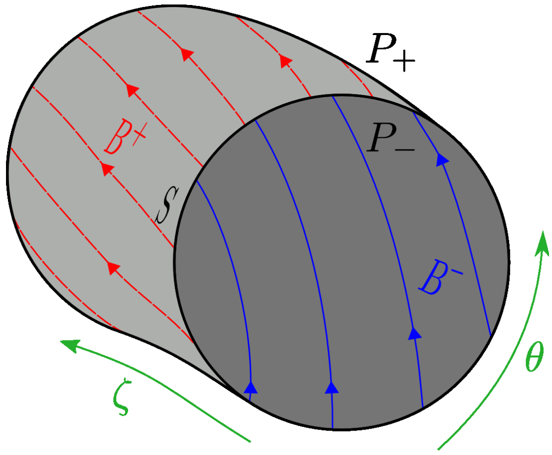

FIG. 1: (color online) Geometry of the problem, given the field one one side of the surface, find the field on the other when there is a pressure discontinuity along the surface.

Much can be inferred from Equation (1); the details are in Appendix A but here we report the important results. When there is a discontinuity in the pressure across a surface $\mathcal{S}$, and when the magnetic field is force free in the neighborhood of the surface, the following must apply

$$P = \text{const} \,, \tag{3a}$$

$$\mathbf{B} \cdot \mathbf{n} = 0 \,, \tag{3b}$$

$$(\nabla \times \mathbf{B}) \cdot \mathbf{n} = 0 \,, \tag{3c}$$

$$\left[\!\left[\frac{1}{2}B^2 + P \right]\!\right] = 0 \,. \tag{3d}$$

In Equations (3), $\mathbf{n}$ is the normal to the surface, and the notation $[\![x]\!]$ refers to the difference between x on one side of the surface and x on the other side. Equations (3a-3c) must hold on both sides of the surface. We refer to Equations (3) collectively as the *pressure jump conditions*. Note the magnetic field itself is discontinuous.

To describe toroidal fusion geometries we now consider the surface $\mathcal{S}$ to have a toroidal topology and use the coordinate system (θ, ζ). Here, θ and ζ are angle-like coordinates in the poloidal and toroidal directions respectively.

The field on the inner side (side closest to the magnetic axis) of $\mathcal{S}$ is given by $\mathbf{B}^-$, and the field on the outer side is given by $\mathbf{B}^+$, as shown in Figure 1. Equation (3d) then becomes

$${B^+}^2 - {B^-}^2 = 2(P_- - P_+) \,. \tag{4}$$

When Equation (4) is written in terms of the covariant components of the magnetic fields, one finds that

$$2\Delta P = \sum_{i,j \in \{\theta,\zeta\}} g^{ij} \left[B_i^+ B_j^+ - B_i^- B_j^- \right] \,, \tag{5}$$

where $\Delta P = P_- - P_+$ and g^{ij} are the metric coefficients in the Riemannian space $\mathcal{S}$.[7]

In Equation (5), the angular metric components are,[5]

$$g^{\theta\theta} = g_{\zeta\zeta}/G \; , \quad g^{\theta\zeta} = -g_{\theta\zeta}/G \; , \quad g^{\zeta\zeta} = g_{\theta\theta}/G \; . \tag{6}$$

where $G = g_{\theta\theta} g_{\zeta\zeta} - g_{\theta\zeta}^2$. Also,

$$B^\theta = g^{\theta\theta} B_\theta + g^{\theta\zeta} B_\zeta \tag{7a}$$

$$B^\zeta = g^{\theta\zeta} B_\theta + g^{\zeta\zeta} B_\zeta \; . \tag{7b}$$

B. Hamiltonian Treatment

The keys to the Hamiltonian treatment are Equations (3b) and (3c), together, they imply

$$\partial_\theta B_\zeta^\pm - \partial_\zeta B_\theta^\pm = 0 \; , \tag{8}$$

so that $B_\zeta^\pm$ and $B_\theta^\pm$ can be found from a single function, referred to from now on as the *surface potential* $f^\pm(\theta,\zeta) = \int \mathbf{B} \cdot d\mathbf{l}$,

$$B_\theta^\pm = \partial_\theta f^\pm \; , \qquad\qquad B_\zeta^\pm = \partial_\zeta f^\pm \; . \tag{9}$$

Equations (3b) and (3c) being implicitly satisfied, the pressure jump conditions reduce to the single condition, Equation (3d).

To this end we substitute Equation (9) into Equation (5) to give

$$2\Delta P = \sum_{i,j \in \{\theta,\zeta\}} g^{ij} \left[\partial_\theta f^+ \partial_\zeta f^+ - \partial_\theta f^- \partial_\zeta f^- \right] \; . \tag{10}$$

The goal is, given a surface (which defines g^{ij}) and a field on one side of the surface (which defines, say f^-) the goal is to find the potential on the other side of the surface (f^+). If f^+ can be found that has continuous second partial derivatives, the magnetic field will satisfy force balance and lie on the surface, satisfying both Equations (3).

The problem is symmetric, we can either, given f^- find f^+ (work inside-out), or given f^+ find f^- (work outside-in). So we refer to the unknown surface field potential simply as f. Then the pressure jump condition becomes a problem of calculating f from the equation

$$H(\theta, \zeta, \partial_\theta f, \partial_\zeta f) = \Delta P \; . \tag{11}$$

Equation (11) is a partial differential equation for f. More specifically, Equation (11) is a time independent Hamilton–Jacobi Equation with $\partial_i f = p_i$ and Hamiltonian

$$H(\theta, \zeta, p_\theta, p_\zeta) = g^{ij} p_i p_j + V(\theta, \zeta) \; , \tag{12}$$

where there is an implicit sum of $i, j \;\; \in (\theta, \zeta)$, and $V(\theta,\zeta) = g^{ij} \partial_i f^- \partial_j f^-$ where f^- has been arbitrarily chosen as the given potential. The Hamiltonian in Equation (12) we refer to as the *pressure jump Hamiltonian*.

The solution to this autonomous Hamiltonian system can be found by solving the corresponding characteristic equations[8]

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \; , \tag{13a}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} \; , \tag{13b}$$

$$\dot{\zeta} = \frac{\partial H}{\partial p_\zeta} \; , \tag{13c}$$

$$\dot{p_\zeta} = -\frac{\partial H}{\partial \zeta} \; . \tag{13d}$$

As a Hamiltonian system, ΔP is identified with the energy and f is a type two generating function (Hamilton's characteristic function) considered to generate the canonical pairs (θ, p_θ) and (ζ, p_ζ) through

$$p_\theta = \partial_\theta f = B_\theta \; , \qquad\qquad p_\zeta = \partial_\zeta f = B_\zeta \; . \tag{14}$$

If f exists, then the Hamiltonian orbit lies on an invariant torus in phase space. Such an orbit we refer to as *regular*.

Treating the problem as a Hamiltonian one allows utilization of the tools that have been developed for determination of integrability in Hamiltonian systems to investigate whether a solution (f) can be found that satisfies force balance. The intent of this paper is to provide a consistent explanation to prove that inferences from the Hamiltonian system are acceptable.

C. Reduction to a $1\frac{1}{2}$ DOF system

To simplify computation of Hamiltonian orbits, we condense these equations by dividing Equations (13) by Equation (13c) to trivialize the third of Hamilton's equations and make the toroidal angle-like coordinate the "time" variable. The division requires that $\dot{\zeta} \neq 0$, the implications of this being addressed in Section III A.

The reduced equations provide more physically relevant equations of motion. The first describes the path of the Hamiltonian trajectory through configuration space:

$$\frac{d\theta}{d\zeta} = \frac{g^{\theta\theta} p_\theta + g^{\theta\zeta} p_\zeta}{g^{\theta\zeta} p_\theta + g^{\zeta\zeta} p_\zeta} = \frac{B^\theta}{B^\zeta} \; . \tag{15}$$

where Equations (7) have been utilized. Equation (15) is the equation of a field line. Any solutions to this characteristic equation of the pressure jump Hamiltonian are thus potentially the field lines of the magnetic field on $\mathcal{S}$.

The second equation of motion is

$$\frac{dp_i}{d\zeta} = \frac{\partial_i g^{ij} p_i p_j - \partial_i V}{g^{\theta\zeta} p_\theta + g^{\zeta\zeta} p_\zeta} \; , \qquad (p_i = B_i) \; , \tag{16}$$

Physical Quantities and their Hamiltonian Equivalents

ΔP	Pressure jump	E	Energy
f	Surface potential	S	Action / Hamilton's principal function
(θ, ζ)	Curvilinear coordinates	$\mathbf{q}$	Generalized coordinates
(B_θ, B_ζ)	Covariant components of magnetic field	$\mathbf{p}$	Generalized momenta
(Θ, Φ)	Straight field line coordinates	$\mathbf{Q}$	(Action) Angle coordinates
$\iota\!\!\!-$	Rotational transform	w	Winding number (angular frequency)

TABLE I: A table summarizing the physical interpretations of Hamiltonian quantities in the problem.

for $i \in (\theta, \zeta)$. Equation (16) shows that the canonical momentum gives the covariant components of the magnetic field along a Hamiltonian trajectory for solutions with the required rotational transform.

The final equation of motion, Equation (13d), can be solved implicitly by the first two using the fact that the energy of this Hamiltonian system is conserved, i.e. the pressure jump is constant along the flux surface. This means p_ζ can be written as a function $p_\zeta = p_\zeta(\theta, p_\theta, \zeta; \Delta P)$. When this is substituted into Equations (13a) and (13b), the entire system can be solved within two differential equations. However, inversion of the Hamiltonian brings about an arbitrary sign, which for simplicity, we choose to be positive, discussing the choice further in Section III A.

The system is now condensed into the differential system

$$\frac{d\theta}{d\zeta} = u(\theta, \zeta, p_\theta; \Delta P) \tag{17a}$$

$$\frac{dp_\theta}{d\zeta} = v(\theta, \zeta, p_\theta; \Delta P) \tag{17b}$$

One method of identifying the existence of f is to calculate the Hamiltonian trajectories. If a solution to this $1\frac{1}{2}$ degree of freedom system can be found that lies on an invariant torus in $(\theta, p_\theta, \zeta)$ space with the correct winding number, the trajectory, when projected onto the 3D geometric torus, coincides with the field lines that lie on that surface.

The solution of the pressure jump Hamiltonian is in general not unique. It may be that many Hamiltonian trajectories are regular, and so are physical candidates for a field to satisfy force balance. To make the solution unique one can require the corresponding field line have a certain *rotational transform*, defined as

$$\lim_{\Delta\zeta\to\infty} \frac{\Delta\theta}{\Delta\zeta} = \iota\!\!\!- \,. \tag{18}$$

Although the pressure jump conditions (Equations (3)) allow a pressure jump in $\iota\!\!\!-$, Mills *et al* [9] found it was difficult or impossible to construct a discontinuous pressure barrier as the limiting case of a sequence of MHD stable, finite width barriers unless $[\![\iota\!\!\!-]\!] = 0$. Therefore we require the field have the same rotational transform on either side of $\mathcal{S}$.

D. Action Angle Coordinates

In action angle coordinates the canonical momenta are constant, so that the equations of motion are trivial. The equations of motion are, after an appropriate choice of initial conditions,

$$\Theta = w_\Theta t \,, \qquad \Phi = w_\Phi t \,. \tag{19}$$

where w_Θ and w_Φ are constant. Thus,

$$\Theta = \frac{w_\Theta}{w_\Phi}\Phi \,. \tag{20}$$

This coordinate system is, in fusion research, known as straight field line coordinates, as the magnetic field appears as a straight line in these coordinates. Such coordinates are helpful because the rotational transform, usually a quantity that requires integration along the entire length of a field line (often infinitely long) is now explicit in the equations of motion:

$$\lim_{\Delta\zeta\to\infty} \frac{\Delta\theta}{\Delta\zeta} = \frac{d\Theta}{d\Phi} = \frac{w_\Theta}{w_\Phi} = \iota\!\!\!- \,. \tag{21}$$

Such coordinates can be found when f is separable in the configuration coordinates, in this case the corresponding Hamiltonian trajectory lies on an invariant torus [10].

III. ANALYTICAL CONCERNS

A. Ambiguity of Sign

The pressure jump Hamiltonian is a constraint on the square of the magnetic field, so it is expected that there are two magnetic fields that would satisfy force balance. This arbitrariness is made explicit when one inverts the Hamiltonian to find $p_\zeta = p_\zeta(\theta, p_\theta, \zeta; \Delta P)$ in an effort to reduce the phase space. When completing the square one has the expression

$$g^{\theta\zeta} p_\theta + g^{\zeta\zeta} p_\zeta = \pm\Delta \,, \tag{22}$$

where $\Delta = (g^{\zeta\zeta}\left(2\Delta P + V(\theta,\zeta) - g^{\theta\theta}p_\theta^2\right) + g^{\theta\zeta}p_\theta^2)^{1/2}$. The left hand side of Equation (22) is the result of Equation (13c), i.e.

$$\frac{d\zeta}{dt} = B^\zeta = \pm\Delta \,, \tag{23}$$

showing that the two solutions correspond to magnetic fields with opposite toroidal direction.

The information on the toroidal direction of the given field is similarly lost within the pressure jump Hamiltonian so the choice of sign must be made to reflect the initial conditions. It is expected that physical configurations would require a field to be in the same toroidal direction on either side of an infinitely thin flux surface, otherwise there would be a very strong current sheet, susceptible to a tearing instability.

The choice of sign has also been shown in Bruno and Laurence to be equivalent to choosing the sign of the rate of change of toroidal flux.[5]

When reducing the phase space of the Hamiltonian system, the condition in which $\dot{\zeta} = 0$ was necessarily lost. It can be seen now that this corresponds to a field configuration of infinite rotational transform–a situation we ignore as it will not ergodically cover the surface, and thus never act as a flux surface.

B. Birkhoff Theorem

Kaiser and Salat[4] solve the pressure jump discontinuity problem purely in configuration space, that is, a solution to force balance on the surface is sought that corresponds directly to geodesics covering a 3D torus. Such an approach is limited to situations where the field within the plasma volume is zero. However, Kaiser and Salat felt obliged to use this geodesic method because of concerns regarding the physical significance of Hamiltonian trajectories in phase space.

Kaiser and Salat's gravamen against the Hamiltonian formulation can be stated as the following: Suppose a solution to the pressure jump Hamiltonian system is found. This will correspond to a Hamiltonian orbit that lies in a four dimensional phase space. The actual field line however exists on the two dimensional torus embedded in Euclidean 3-space, i.e. configuration space. We must project a four dimensional phase space trajectory to the two dimensional configuration space–is it not possible that the projected trajectory intersects itself?

Assuming the magnetic field is nowhere zero, such intersections would make it impossible to interpret the projection as a physical field line.

However, we will now argue that such crossings do not occur, and as a result show that existence information drawn from the Hamiltonian formulation is sufficient to directly imply that the corresponding field line is consistent with Equations (3).

Our system is a $1\frac{1}{2}$ degree of freedom Hamiltonian whose trajectories define a 2D area preserving map by integrating Equations (17). The mappings of interest, those generated by a trajectory that lies on an invariant surface, are also twist maps as the metric is positive definite $[\det(\partial_{p_i}\partial_{p_j}H) > 0]$.[11]

Consider the phase space variables (q, p) in a 2 dimensional area preserving twist map, the Birkhoff Theorem states that, for a rotational invariant circle,[12]

$$p = Y(q) \tag{24}$$

where Y is a graph and is Lipschitz, meaning a line joining any two points on the graph of the function will never have a slope larger than a finite real number C, $(C \geq 0)$. In this case we can write the phase space mapping generated by the Hamiltonian as

$$(q', p') = T(q, Y(q)) \; . \tag{25}$$

Let us consider the operator π that is the projection of the phase space trajectory onto configuration space,

$$\pi(q, p) = q \tag{26}$$

$$q' = \pi(T[q, Y(q)]) = \alpha(q) \tag{27}$$

Thus, as T is a homeomorphism and the slope bounds on the Lipschitz Y are sufficient, α is also a homeomorphism.[12] The injective nature of a homeomorphism implies there will be no crossings under the mapping π.

Strictly, this is only true for a two dimensional system because the Birkhoff theorem applies only for a 2D phase space. Some limited higher dimensional results have been found.[13]

This means a homeomorphic mapping like Equation (24) can be generated to define completely the evolution of the system, and the above proof applies. Thus, when mapping the Hamiltonian trajectories to the 2D torus in configuration space no crossings are possible.

C. Existence of Invariant Tori

With the acceptance that solutions garnered from the pressure jump Hamiltonian map homeomorphically to field lines on the 3D toroidal surface, we can use the existence of the surface potential to declare that the corresponding field line satisfies Equations (3).

When a Hamiltonian trajectory can be transformed to action angle coordinates, the Hamiltonian orbit lies on an invariant torus; and after the mapping to configuration space the field line will lie on the surface and not intersect itself. Conversely, if the conditions are such that the Hamiltonian trajectory is chaotic, it does not lie on an invariant torus; after the mapping the field line will not lie on the given flux surface ($\mathbf{B} \cdot \mathbf{n} = 0$ is not satisfied) and no physically consistent field can exist.

Of great interest then, is the knowledge of whether an invariant torus of the Hamiltonian exists or not. There are various tools to use. Often one would appeal to the KAM theorem to be guaranteed the existence of invariant tori for systems slightly perturbed from axisymmetry. Kaiser and Salat found KAM like behavior in their treatment, but could not be sure of the precise conditions under which the KAM theory could apply. The invariant tori of interest are the most robust, the ones that require the greatest perturbation to destroy. It is unclear whether such perturbations lie within the domain of small and smooth perturbations that is the applicability of the KAM theorem. To avoid the problem one can follow other avenues for investigation, of special interest is Greene's residue criterion, which can determine to great accuracy whether an invariant torus exists or not.[14] Future papers will utilize Greene's residue criterion to investigate the conditions under which a flux surface may support pressure discontinuities, and how this depends on the deformation of the surface.

Determination of the existence of a surface potential is sufficient in the sense that it dictates definite existence criteria for the data supplied–surface shape, rotational transform, prescribed magnetic field on one side and pressure. However, on the scale of the entire plasma system it is only a necessary condition in that many other sources of destruction may be present throughout the plasma. These perturbations are generated outside the domain of the pressure jump Hamiltonian as it is only defined within an infinitesimal region of the surface.

IV. CONCLUSION

By applying force balance to neighboring regions of plasma of finitely different pressure separated by a infinitesimally thin flux surface, three general criteria were found that must be satisfied in order for the surface to be a flux surface. The three criteria were combined into one, referred to as the *pressure jump criterion* by condition of the existence of a *surface potential* f, a scalar function defined on the surface that is related to the magnetic field.

A Hamiltonian–Jacobi construction of the pressure jump criterion was introduced, and the surface potential was found to play the role of the generating function important in Hamiltonian dynamics.

The main analytical problem that faced the treatment was solved, namely the sufficiency of an existence criterion on f to determine one-to-one that the corresponding field line satisfied the pressure jump criterion. This was resolved by appealing to the Birkhoff theorem when the Hamiltonian system is integrated to form a two dimensional area preserving map. For investigating the existence of the surface potential a robust method such as Greene's residue criterion was argued to be preferable over the perturbative KAM theorem.

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Appendices

A. PRESSURE JUMP CONDITION

On a surface $\mathcal{S}$, defined as $n = 0$ where n is the distance from $\mathcal{S}$, the discontinuous pressure profile $P(\mathbf{r})$ and the corresponding discontinuous $\mathbf{B}(\mathbf{r})$ can be expressed as

$$P = P_-(\mathbf{r})h(-n) + P_+(\mathbf{r})h(n) \ , \tag{A1}$$

$$\mathbf{B} = \mathbf{B}_-(\mathbf{r})h(-n) + \mathbf{B}_+(\mathbf{r})h(n) \ , \tag{A2}$$

where h is the unit Heaviside step function and where $P_\pm$ and $\mathbf{B}_\pm$ are smooth (i.e. not discontinuous) functions. Such a form for P and $\mathbf{B}$ gives

$$\nabla P_\pm = \mathbf{j}_\pm \times \mathbf{B}_\pm \ , \tag{A3}$$

where $\mathbf{j}_\pm = \nabla \times \mathbf{B}_\pm$ are the associated currents. Dotting Equation (A3) with $\mathbf{B}_\pm$ gives $\mathbf{B}_\pm \cdot \nabla P_\pm = 0$, which implies the pressure is constant along a magnetic field line. As a flux surface is composed of a single magnetic field line, we have the condition that on both sides of the surface

$$P_\pm = \text{const} \ . \tag{A4}$$

Substitution of Equation (A2) into $\nabla \cdot \mathbf{B} = 0$ gives

$$\nabla \cdot \mathbf{B} = \tag{A5}$$

$$\mathbf{n} \cdot [\![\mathbf{B}]\!]\, \delta(n) + \nabla \cdot \mathbf{B}_- h(-n) + \nabla \cdot \mathbf{B}_+ h(n) \ , \tag{A6}$$

where $\mathbf{n}$ is the unit normal to $\mathcal{S}$ and where $[\![x]\!]$ is the jump of x across the interface, $[\![x]\!] = x_+ - x_-$. The divergence can only be zero if

$$[\![B_n]\!] \equiv \mathbf{n} \cdot [\![\mathbf{B}]\!] = 0 \ , \tag{A7}$$

i.e., the normal component of the magnetic field must be continuous.

Similarly, when Equations (A1 - A2) are substituted into the stress tensor in Equation (1),

$$\mathbf{n} \cdot \left[\!\!\left[P\mathbf{I} \left(\frac{1}{2}B^2 - \mathbf{BB} \right) \right]\!\!\right] = 0 \ . \tag{A8}$$

Substituting Equation (A7) into the above and dotting with $\mathbf{n}$ gives the condition

$$\left[\!\!\left[P + \frac{1}{2}B^2 \right]\!\!\right] = B_n\, [\![B_n]\!] = 0 \ , \tag{A9}$$

removing the middle equality gives the *pressure jump condition.*

As B_n is continuous, Equation (A9) can be written as

$$\left[\!\left[(\mathbf{n}\times\mathbf{B})^2\right]\!\right] = -2\,[\![P]\!]\ , \tag{A10}$$

so long as $[\![P]\!] \neq 0$,

$$[\![\mathbf{n}\times\mathbf{B}]\!] \neq 0\ . \tag{A11}$$

Crossing Equation (A8) with $\mathbf{n}$ gives

$$B_n\,[\![\mathbf{n}\times\mathbf{B}]\!] = 0\ . \tag{A12}$$

Combining Equation (A11) with Equation (A12) implies

$$\mathbf{n}\cdot\mathbf{B}_\pm = 0\ . \tag{A13}$$

Thus, if there is a pressure discontinuity across a surface, the field lines must lie on that surface.

The plasma in the neighborhoods either side of the surface is assumed to be force free, so that $\nabla P = 0$. Equation (A3) then implies that $\mathbf{j}_\pm$ is parallel to $\mathbf{B}_\pm$, which on comparison with Equation (A13) implies

$$\mathbf{n}\cdot\mathbf{j}_\pm = \mathbf{n}\cdot(\nabla\times\mathbf{B}_\pm) = 0\ . \tag{A14}$$

Thus, if there is a pressure discontinuity across a surface, the curl of the magnetic field must be parallel to the surface at all points on the surface.

B. COMPARISON OF HAMILTONIANS

In this section we compare the magnetic field line Hamiltonian system often used in the literature to the pressure jump Hamiltonian system introduced in this paper.

The magnetic field in the plasma volume be written as [15]

$$\mathbf{B} = \nabla\psi\times\nabla\theta + \nabla\zeta\times\nabla\chi\ . \tag{B1}$$

Using the equation of the field line $d\mathbf{r}/dt = \mathbf{B}(\mathbf{r})$, one finds [16]

$$\frac{d\theta}{d\zeta} = \frac{\partial\chi}{\partial\psi} \tag{B2}$$

$$\frac{d\psi}{d\zeta} = -\frac{\partial\chi}{\partial\theta}\ , \tag{B3}$$

which are of the form of Hamilton's equations. Thus each field line can be described as the solution of the magnetic field line Hamiltonian χ, in a phase space with the poloidal angle θ and toroidal flux function ψ as canonical variables.

Trajectories of this Hamiltonian correspond directly to magnetic field lines within the plasma. Trajectories that lie on invariant tori correspond to field lines that draw out flux surfaces. Trajectories that are chaotic correspond to chaotic field lines the compose the chaotic regions of the plasma.

The pressure enters *implicitly* into the magnetic field line Hamiltonian system, exciting currents that determine the flux functions. The field line dynamics feeds back into the determination of the pressure as a function of position.

In contrast, the pressure is explicit in the pressure jump Hamiltonian, but it only holds on a given toroidal surface. While the trajectories of this Hamiltonian system are not the field lines, the projection of the phase trajectories onto the 3D torus are field lines of the given rotational transform. If the surface potential can be found, then the surface is a flux surface.

A summary of the differences of the Hamiltonian systems in given in Table II.

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Magnetic field line Hamiltonian	Pressure jump Hamiltonian
P implicit	P explicit
Defined throughout plasma volume	Defined on a given magnetic surface
Phase space and time a representation of Euclidean 3-space	Phase space a combination of a Riemannian 2-space = magnetic surface (configuration space) and field components (momentum space)
All orbits are field lines	Orbits are not field lines, projections of regular orbits with specified $\iota\!\!\!-$ onto configuration space are field lines

TABLE II: A table summarizing the differences between the magnetic field line Hamiltonian and the pressure jump Hamiltonian.

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