# Relaxed MHD equilibria with current sheets

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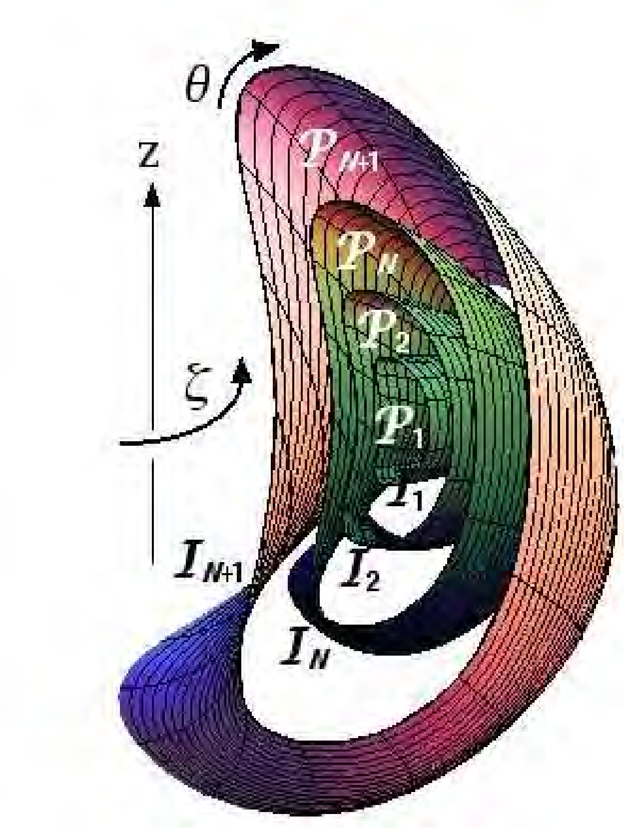
# Topics of talk

- MRXMHD: Multiregion Relaxed MHD
  - Partial MHD Relaxation
  - Current sheet equilibria inverse problem
  - Non-uniqueness of Hahm-Kulsrud current sheets
- SPEC: MRXMHD 3D equilibrium code
  - Reversed Field Pinch application
  - DIIID application

# Relaxed MHD equilibria

- Taylor relaxation
  - Minimize energy conserving *only* magnetic helicity and fluxes
  - Euler-Lagrange equation is Beltrami equation $\nabla\times\mathbf{B} = \mu\mathbf{B}$
- Partial Taylor relaxation
  - Minimize energy conserving helicity and fluxes + *subset* of ideal invariants [Casimir constraints: Z. Yoshida and R.L. Dewar, J. Phys. A: Math. Gen. **45**, 365502 (2012)]
  - Multiregion relaxed MHD (RXMHD) idea is to separate Taylor-relaxed states by toroidal interfaces consisting of arbitrarily thin layers of ideal-MHD fluid that act as "transport barriers." They are also *current sheets* because delta function currents must flow in them.

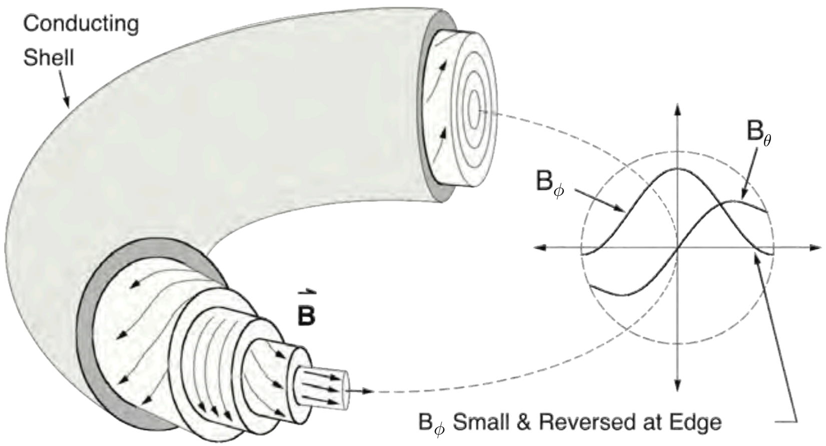
# MRXMHD 3D equilibrium code SPEC




- Relaxed regions $\mathcal{P}_i$ , separated by
- nested, ideal, toroidal barrier interfaces $\mathcal{I}_i$ , which
- freeze in flux and confine piecewise flat pressures (with pressure jumps ≡ $[[p]]_i$ across current sheets).
- Arbitrarily refinable *as long as magnetic surfaces* $\mathcal{I}_i$ *exist*
- Minimizes energy starting with an initial guess (e.g. a VMEC equilibrium)
- Allows islands and chaos between the toroidal current sheets

Two recent applications of SPEC follow ...

# The Reversed Field Pinch is a toroidal plasma confinement device (like a tokamak)




*Magnetic Field Structure of the RFP*

Poor confinement observed in "traditional" *axisymmetric* states

Better confinement now observed when spontaneous $m = 1, n = 7$ *helical* state forms in RFX-mod, Padua

Right figure source: David Shand, *Nature Physics Cover* **5**:8, August 2009

# Taylor's theory ($\mathbf{J} = \mu \mathbf{B}$) is a good description of 'typical' Reversed Field Pinches

Taylor's theory: Plasma quantities are only conserved *globally*

Ideal MHD: Plasma quantities conserved on *every flux surface*

Goal: *minimal* description of helical states in RFP

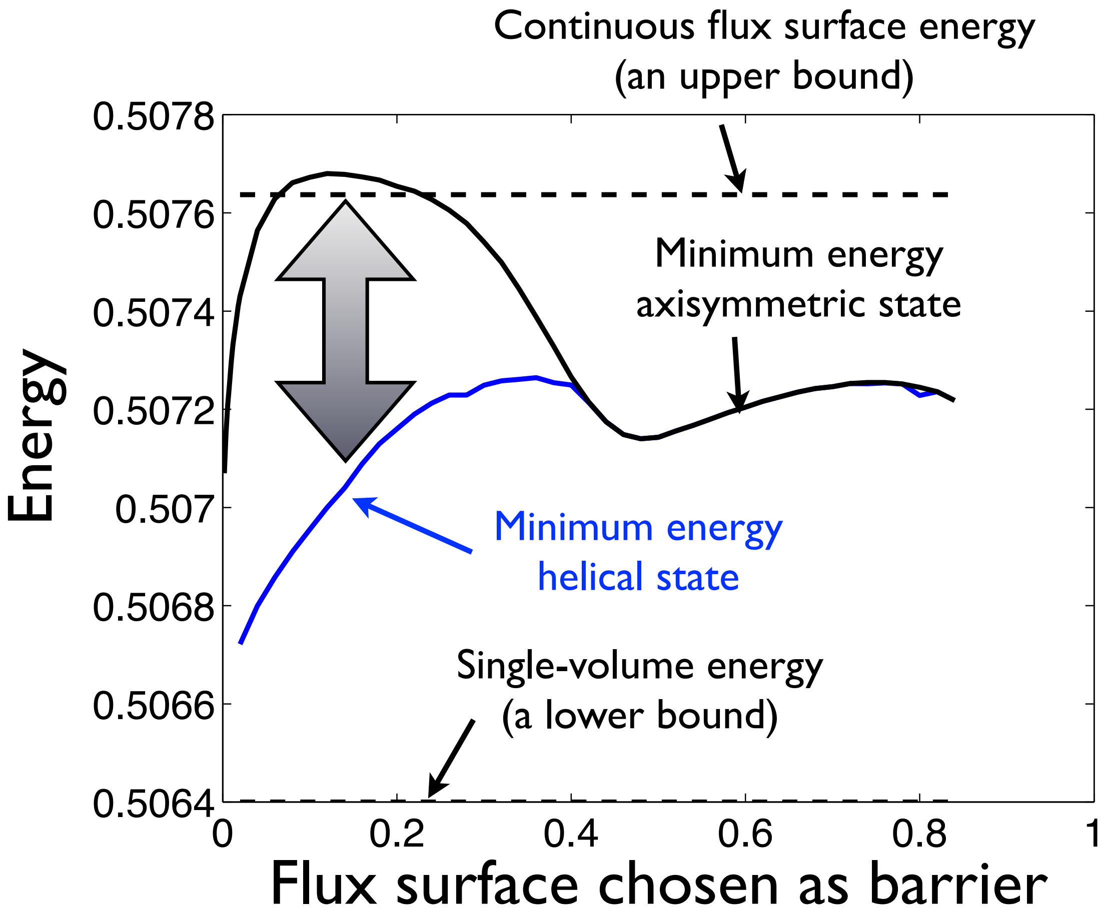
| $\mathbf{J} = \mu \mathbf{B}$ | $\mathbf{J} = \mu_i \mathbf{B}$ | $\mathbf{J} = \mu(\mathbf{x}) \mathbf{B}$ |
| --- | --- | --- |
| Taylor's theory (Linear force-free fields) ✗ | MRXMHD (Piecewise linear force-free fields) ? | Ideal MHD (Nonlinear force-free fields) ✓ |

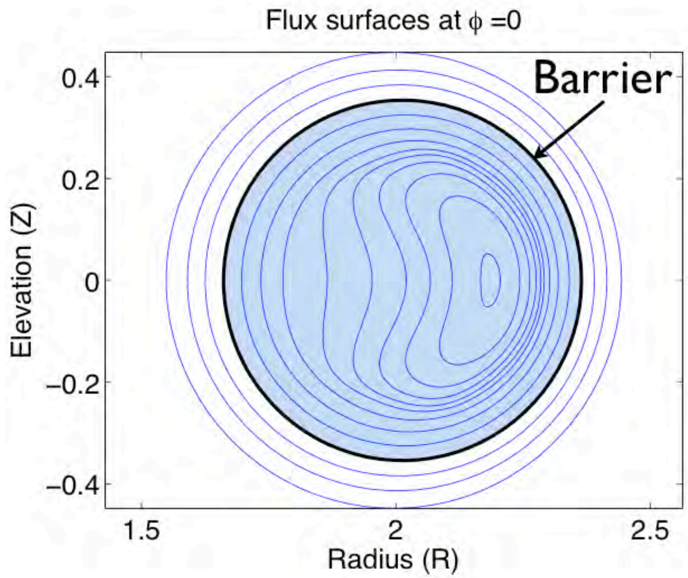
Fewer constraints ⟷ More constraints

Cannot explain RFX helical state

Can model RFX helical state *but* not double axis

# The plasma equilibrium is a minimum energy state




Initialize SPEC with a helical VMEC equilibrium. Use only *one* toroidal current sheet as a MRXMHD barrier.

# Comparison of VMEC and SPEC RFX-mod equilibria

G.R. Dennis, S.R. Hudson, D. Terranova, R.L. Dewar and M.J. Hole

http://arxiv.org/abs/1302.5458

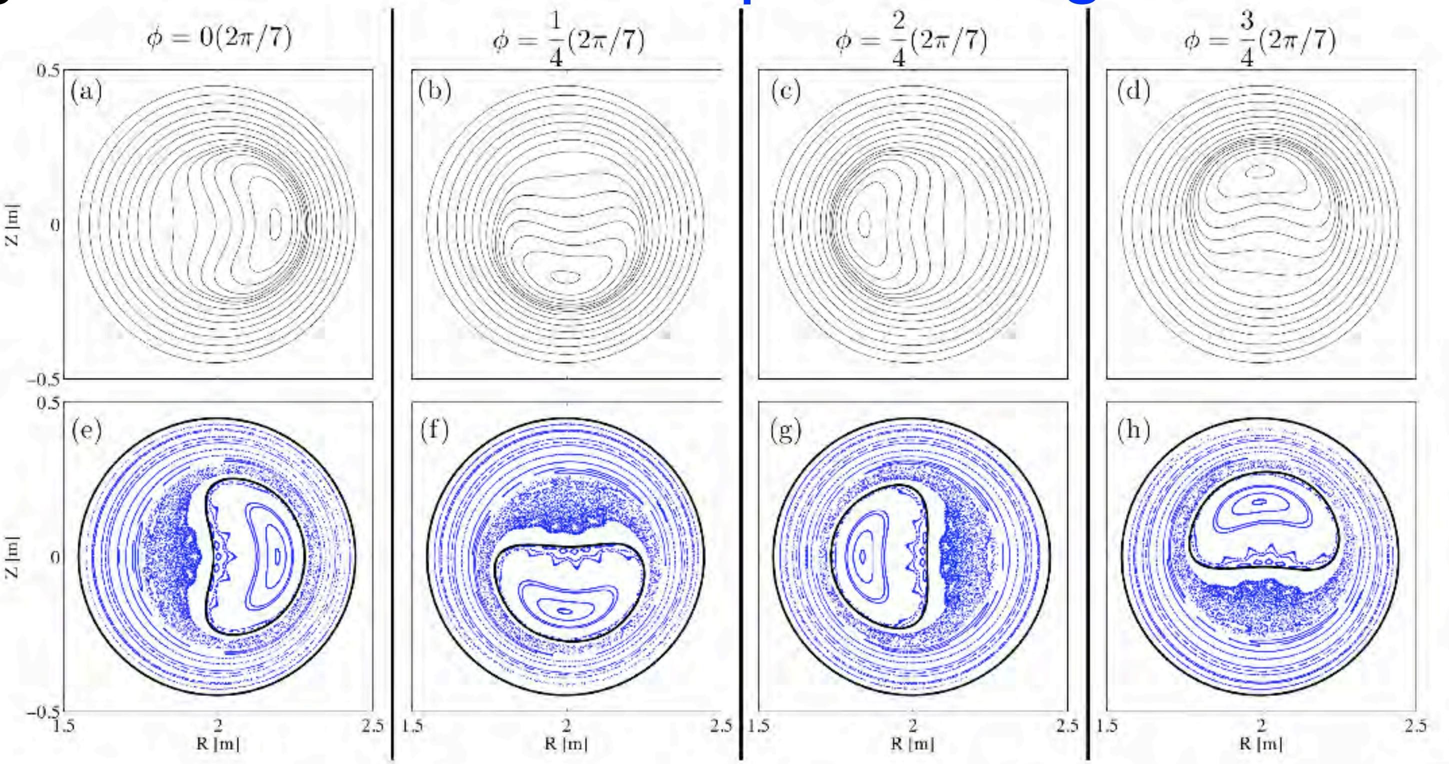
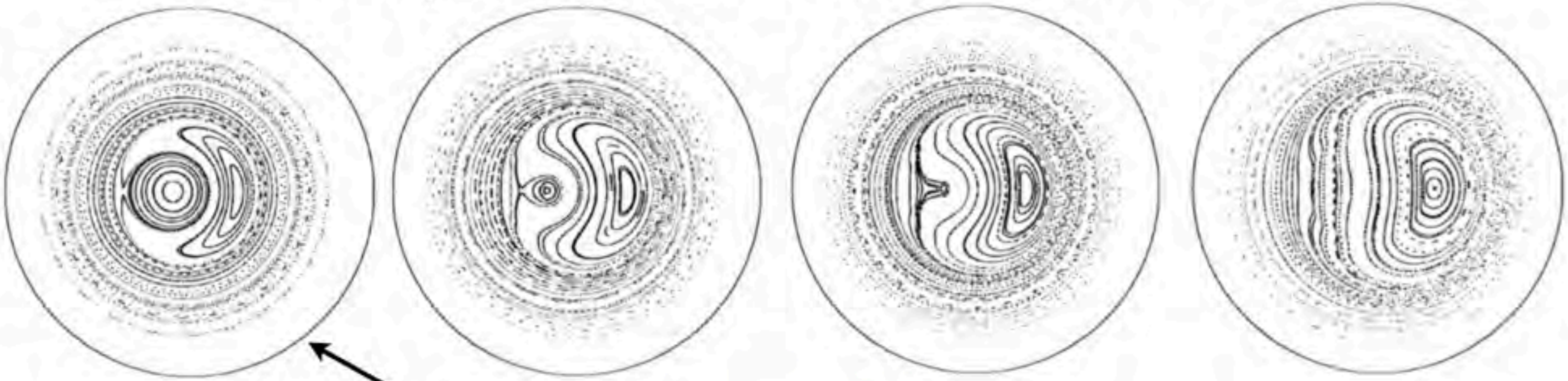
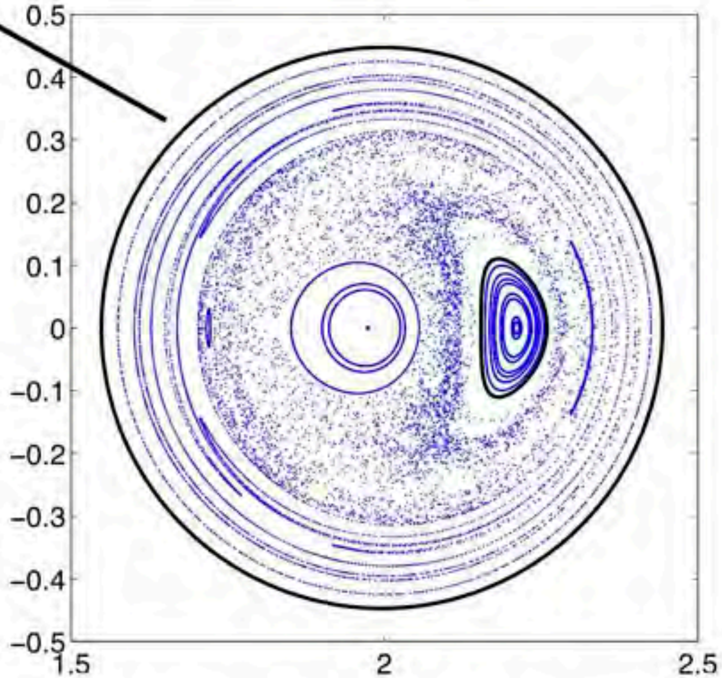



FIG. 1. Comparison of the ideal MHD representation of the SHAx state in RFX-mod and the minimal model (MRXMHD) of this state presented in this work. Figures (a)–(d) show the (poloidal) magnetic flux contours of the ideal MHD plasma equilibrium at equally spaced toroidal angles covering one period of the helical solution. Figures (e)–(h) show Poincaré plots of the minimal model at the same toroidal locations as (a)–(d). The thick black lines mark the location of the transport barrier separating the two plasma volumes. The minimal model corresponds to the $s = 0.3$ configuration of Figure 3.

# Experimental Poincaré plots (RFX-mod)

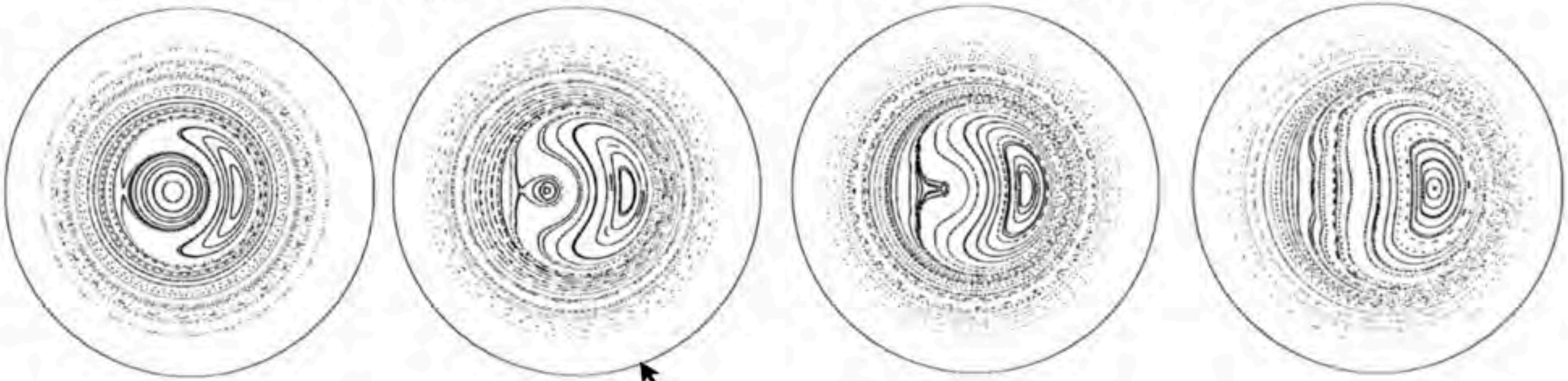


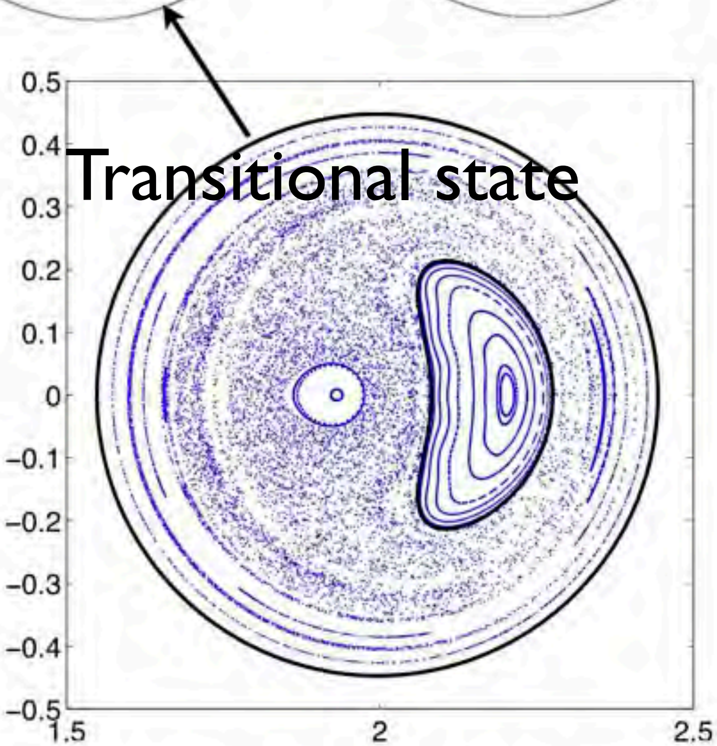
*Double Magnetic Axis* (DAx) state – cannot be described by VMEC but well described, at least qualitatively, by SPEC




# Theoretical Poincaré plots (SPEC)

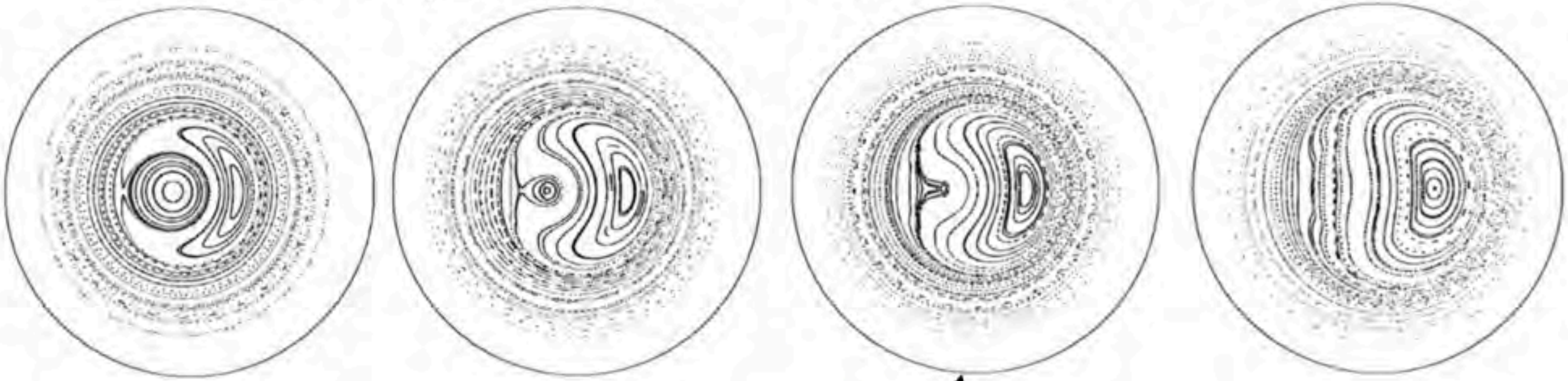
## Experimental Poincaré plots (RFX-mod)

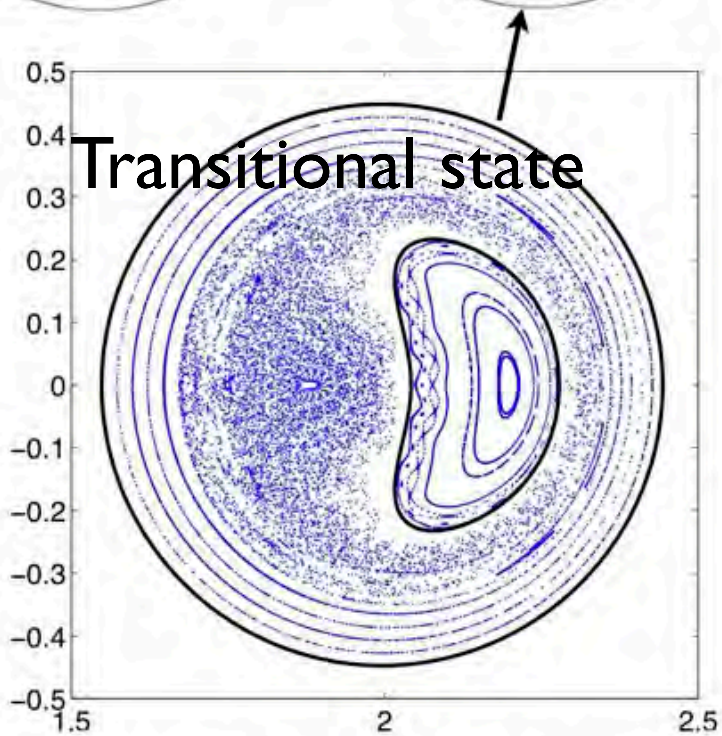





## Theoretical Poincaré plots (SPEC)

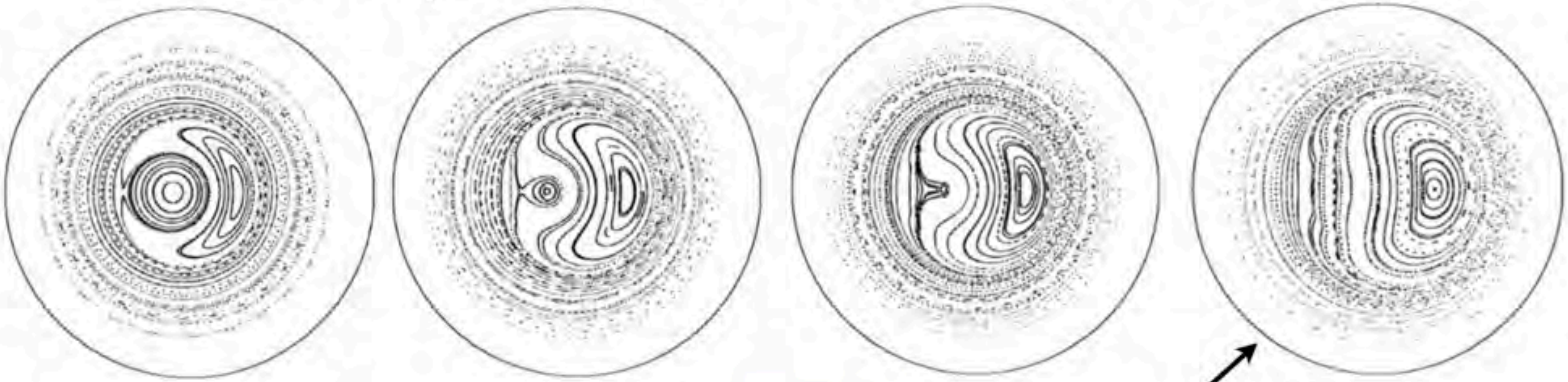
# Experimental Poincaré plots (RFX-mod)

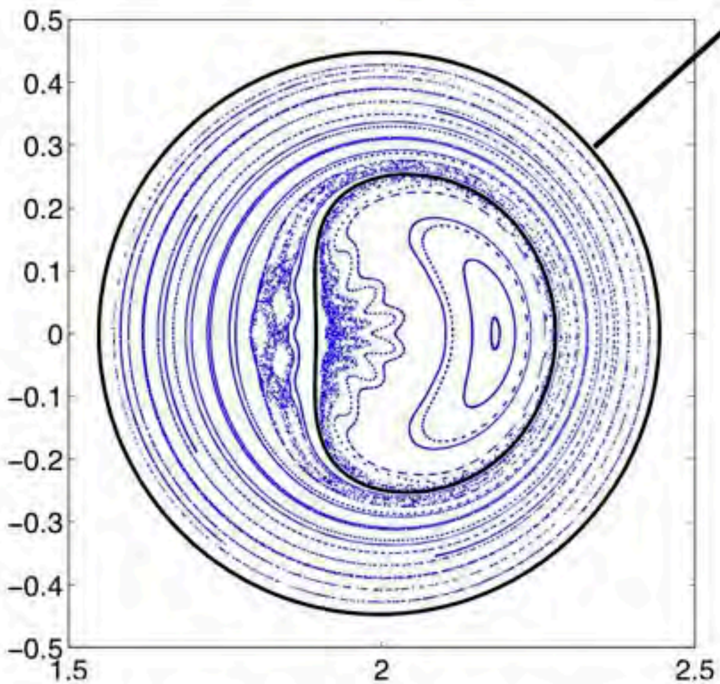





# Theoretical Poincaré plots (SPEC)

## Experimental Poincaré plots (RFX-mod)






*Single Helical Axis* (SHAx) state – SPEC describes full range of states

## Theoretical Poincaré plots (SPEC)

# Example calculation: DIIID with N=3 applied error field

→ axisymmetric boundary & pressure profile from experiment EFIT reconstruction, $\beta \approx 1.5\%$ ,

( Thanks to Ed Lazarus, Sam Lazerson . . .)

→ apply 3mm, n=3 boundary deformation, with broad m spectrum

***effect of RMP modelled by including (m,n)=(2,3), (3,3) & (4,3) boundary deformation,***
***(in spectrally condensed angle, so this corresponds to broad m spectrum in magnetic coordinates),***

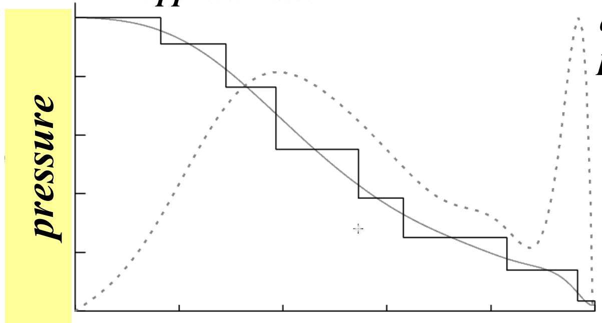
***at present : can only treat stellarator-symmetric configurations, in fixed boundary;***
***for future work : include up-down asymmetry; allow free boundary;***

→ strong pressure gradient near plasma edge

→ if $\mathbf{B}\cdot\nabla p \approx 0$, pressure gradients coincide with (irrational) flux surfaces

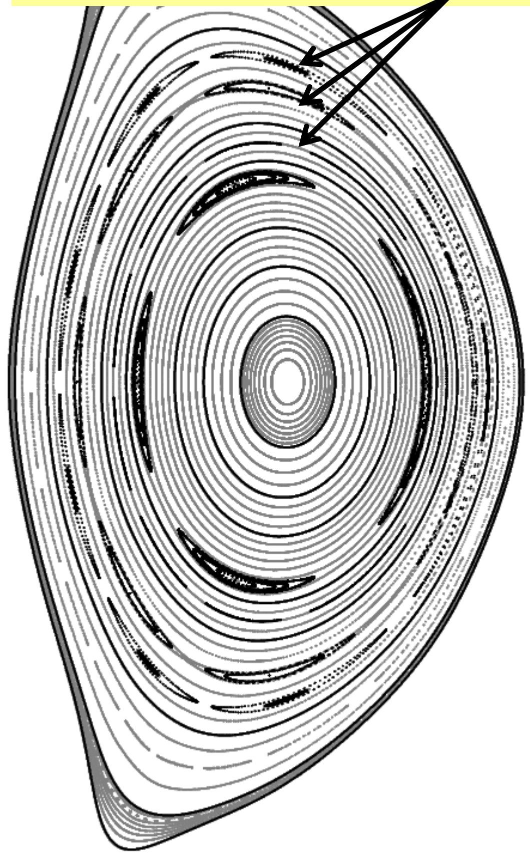
⇒ irrational intefaces chosen to coincide with pressure gradients

**smooth *EFIT pressure profile,***
***and* stepped *pressure profile***
***approximation***




***dotted curve is smooth***
***pressure* gradient**

***formation of magnetic islands***
***at rational surfaces***



→ relaxation, reconnection (i.e. island formation) is permitted,

→ no rational "shielding currents" included in calculation.

# DIIID with Resonant Magnetic Perturbation (RMP) coils — comparison of SPEC pressure profile with a 3D equilibrium reconstruction from observational data

*SPEC steps at noble irrational qs. Flattening of SPEC pressure at low-order rational q agrees well with reconstruction*

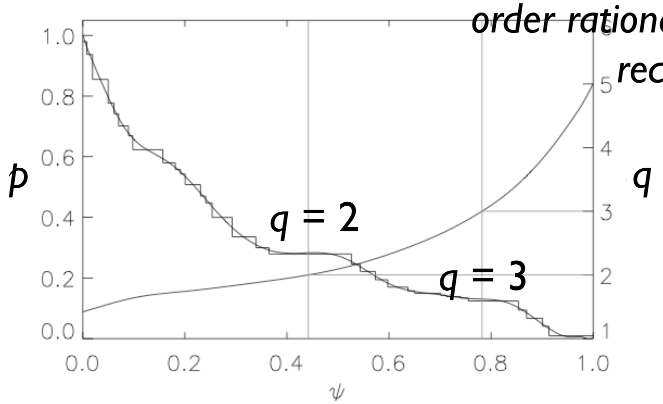



FIG. 7: Pressure profile (smooth) from a DIIID reconstruction using STELLOPT and stepped-pressure approximation. Also shown is the inverse rotational transform ≡ safety factor.

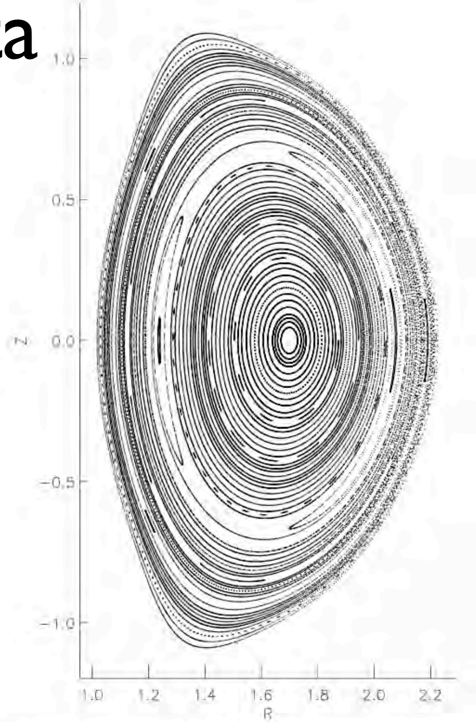



FIG. 8: Poincaré plot of a DIIID equilibrium with perturbed boundary, calculated using SPEC.

S.R. Hudson, R.L. Dewar, G. Dennis, M.J. Hole, M. McGann, G. von Nessi and S. Lazerson

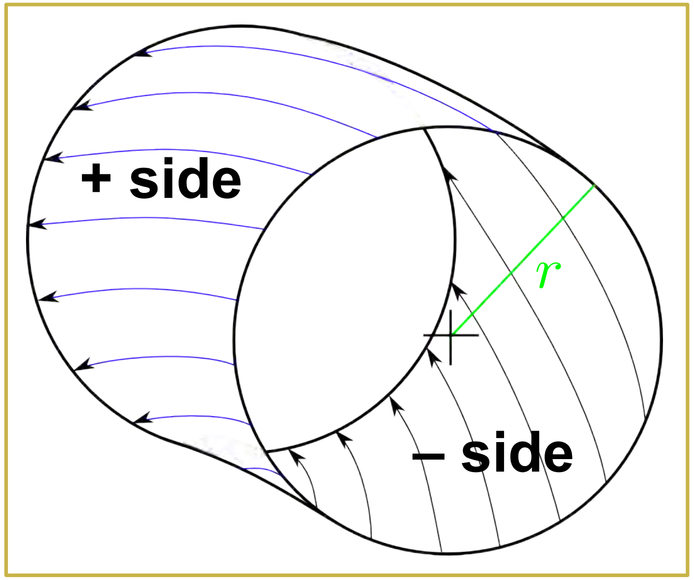
*Computation of multi-region relaxed magnetohydrodynamic equilibria*, Phys. Plasmas **19**, 112502 (2012)

# Existence and Stability of Equilibrium Current Sheets

- **Existence:** Hamilton-Jacobi method
- **Stability:** Local high *m*,*n* ideal-MHD stability test

# Existence criterion for equilibrium of current sheet

Toroidal SPEC interface: has *two* sides, inside and outside, denoted + and –




Details will be in Mathew McGann's PhD thesis, 2013

# Jump and boundary conditions on a current sheet

- SPEC interfaces must be *current sheets* so a delta function $\mathbf{J}\times\mathbf{B}$ force can balance the $\nabla p$ delta function
- *Force balance* criterion is simply $\left[\!\left[ p + \frac{B^2}{2} \right]\!\right] = 0$ where $[\![p]\!]$ denotes the *jump*, $p_+ - p_-$, between the two sides, $\pm$, of the interface
- In addition we have *tangentiality*, $\mathbf{B}\cdot\mathbf{n} = 0$, which implies the existence of two 2D *scalar* potentials $f_\pm(\theta, \zeta)$ such that $B_{\pm\theta} = \partial_\theta f_\pm$ , $B_{\pm\zeta} = \partial_\zeta f_\pm$. Here $\partial_i$ , $i = \theta, \zeta$, are the covariant derivatives on the interface, regarded as 2D Riemannian manifold with metric $g_{i,j}$

# Inverse problem on an equilibrium current sheet

- From pressure balance, $\widetilde{B^2}$ is the *same* on both sides of the interface because $\widetilde{p} \equiv p - \langle p \rangle = 0$, where $\langle \cdot \rangle$ denotes average over angles
- **Inverse problem:** Suppose angular variation of $B$ is specified on both sides of a current sheet, with metric tensor $g_{i,j}(\theta, \zeta)$, by a *scalar function* $V(\theta, \zeta)$ such that $\frac{1}{2}\widetilde{B^2} = V(\theta, \zeta)$, then find the possible tangent *vector* fields $\mathbf{B}_{\pm}$ allowed by pressure balance
- Solve using a *Hamilton-Jacobi* (HJ) equation (Berk et al Phys Fluids 1986, McGann et al Phys. Lett. A 2010), see next slide ...

# Hamilton–Jacobi equation for *f*

- Is a PDE for $f_\pm(\theta, \zeta)$: $H(\theta, \zeta, \partial_\theta f, \partial_\zeta f) = E$ = const

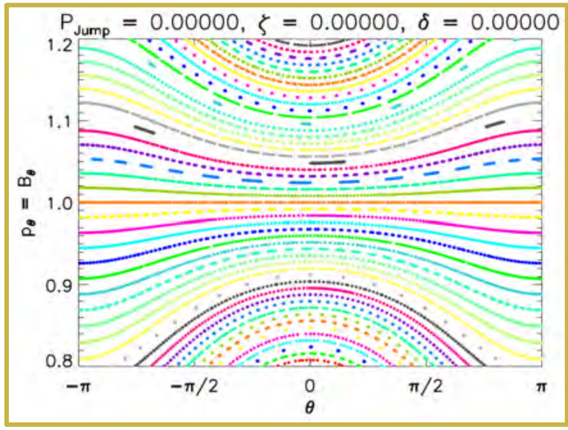
where "pressure jump Hamiltonian" is

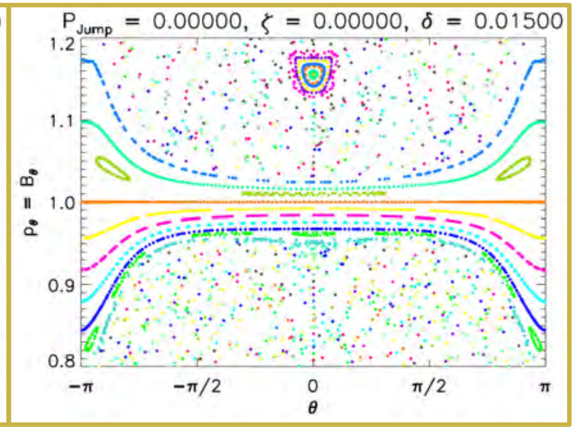
$$H(\theta, \zeta, p_\theta, p_\zeta) \equiv \sum_{i,j\in\{\theta,\zeta\}} \left[\tfrac{1}{2} g^{i,j}(\theta, \zeta) p_i p_j + V(\theta, \zeta)\right]$$

- NB This Hamiltonian has nothing to do with ordinary field-line Hamiltonian!
- Characteristics of HJ equation are Hamiltonian orbits in $(\theta, \zeta, p_\theta, p_\zeta)$ phase space (here $p_i$ are canonical momenta, not pressures!)
- Existence criterion is that orbits cover $(\theta, \zeta)$ configuration space ergodically, lie on *invariant tori* in phase space
- Greene's residue criterion for invariant tori existence
- Necessary: rotational transforms are *noble irrationals*

# Poincaré plots for 4D phase space zero pressure jump case

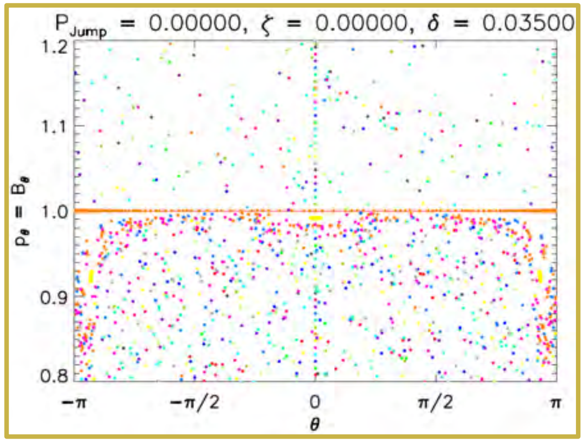
Case 1




Case 2

Case 3

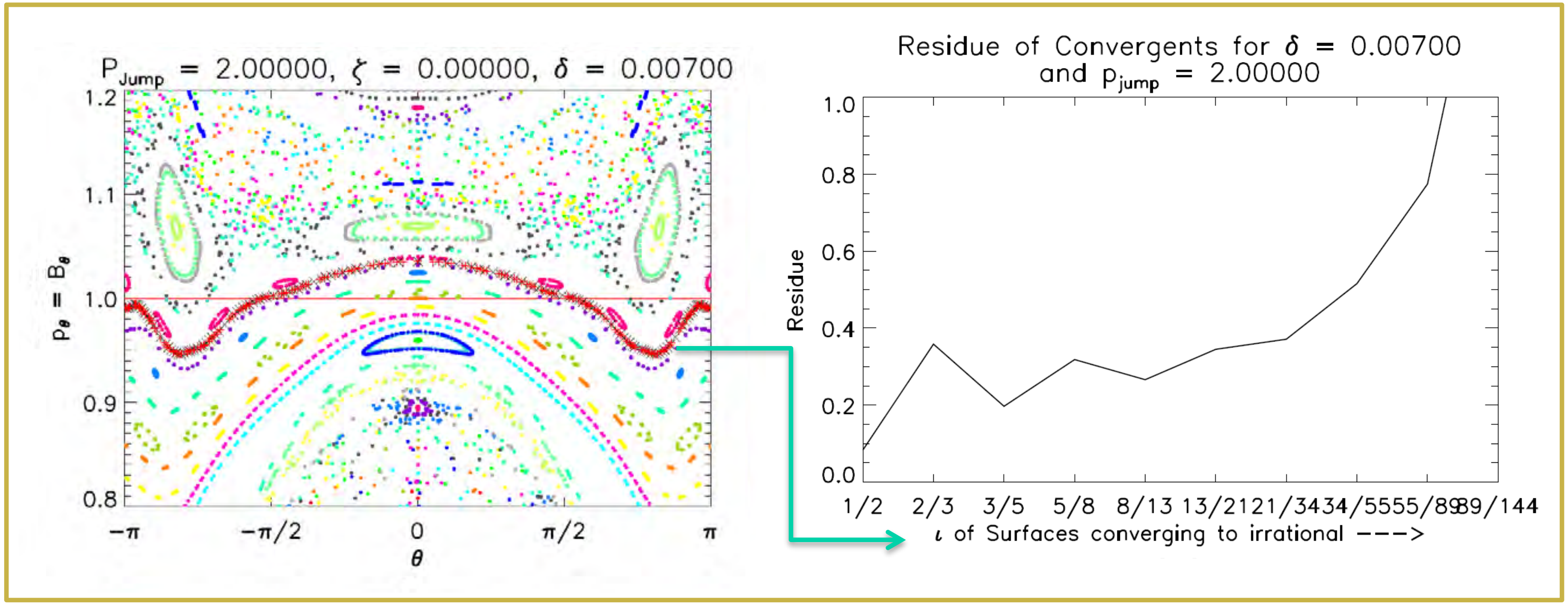



In these cases there is no current sheet unless outside (+) field has a different rotational transform, ie $\iota_+ \neq \iota_-$, giving a *tangential discontinuity* :

- Case 1 is *integrable*, so phase space foliated by invariant (KAM) tori.
- Case 2 indicates some KAM tori as candidates for $\iota_+ \neq \iota_-$ solution
- Case 3 indicates only feasible solution is trivial one, $\iota_+ = \iota_-$

# Finite pressure jump, no rotational transform jump

Test existence with Greene's residue criterion — this case is close to or just beyond the critical pressure jump:

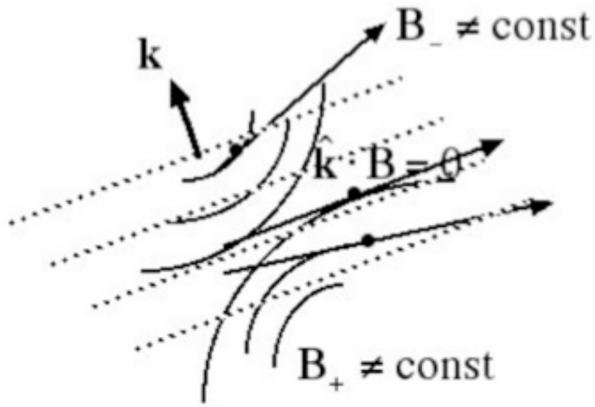


# High *m*,*n* ideal-MHD stability

David Barmaz' MSc thesis, 2011

- Generalizes Bernstein *et al.* 1958
- Sufficient condition for local stability:

*If all points of zero local shear are in regions of good curvature, then the interface is stable to perturbations localized to a neighbourhood around these points.*




Zero-local-shear point

Thus, a simple test (not yet implemented in SPEC) for high *m*, *n* stability is to plot the loci of zero-local-shear points and check they are in regions where normal curvature has the right sign.

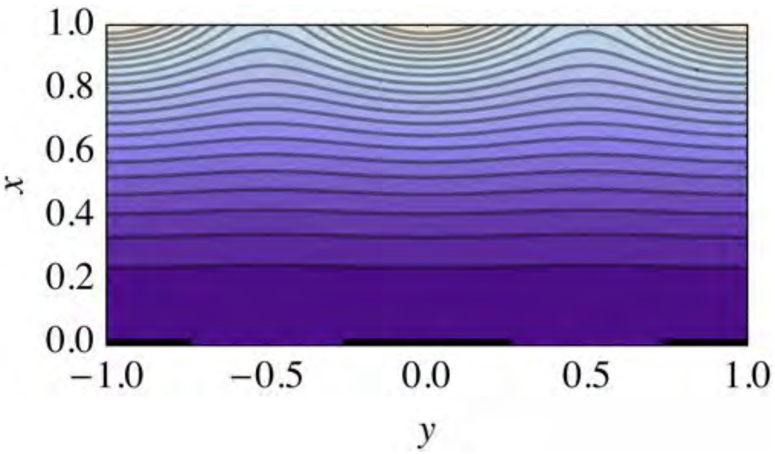
# Preliminary studies related to reconnection

- Motivation for studying current sheets with rational rotational transform in SPEC context
- Simple model showing lack of uniqueness of shielding current sheets

# Need to study rational transforms:

- During minimization of energy *with conserved helicities in each volume*, rotational transforms will change
- Thus only very special *initial* guesses will lead to minima with the required *noble irrationals* at the interfaces *after* energy minimization
- If SPEC is to be fully based on a generalized relaxation scenario, we need a mechanism whereby helicity can leak between barriers to adjust transforms
- Motivates the study of partial reconnection in current sheets with rational transforms
- Could plasmoid formation be the mechanism? Look at Hahm-Kulsrud-Taylor toy model ...

# Hahm-Kulsrud-Taylor equilibria




- Simple slab model for resonant current sheet formation near $x = 0$ in response to symmetrical periodic perturbation at boundaries $x = \pm a$
- Hahm & Kulsrud, Phys. Fluids 1985, found 2 solutions:

- shielding current sheet on $x = 0$

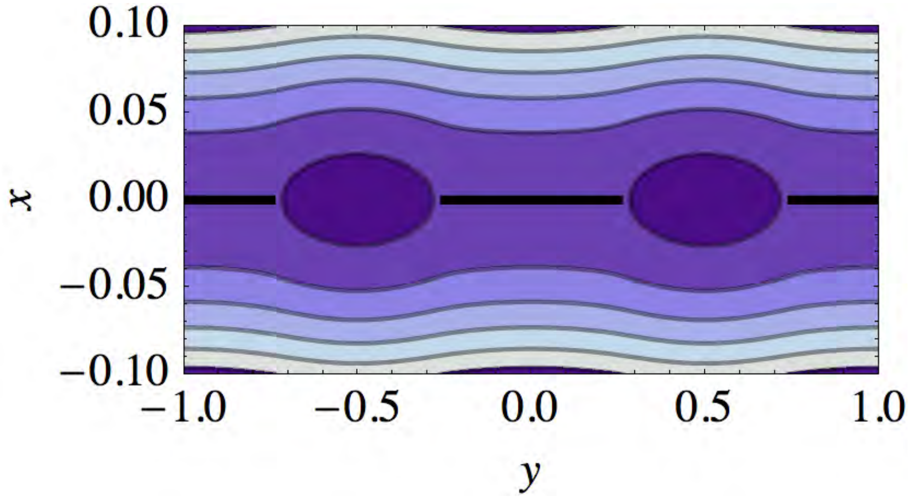
$$\psi = aB_y^a \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\sinh(ka)} |\sinh(kx)| \cos(ky) \right]$$

- island with no current sheet

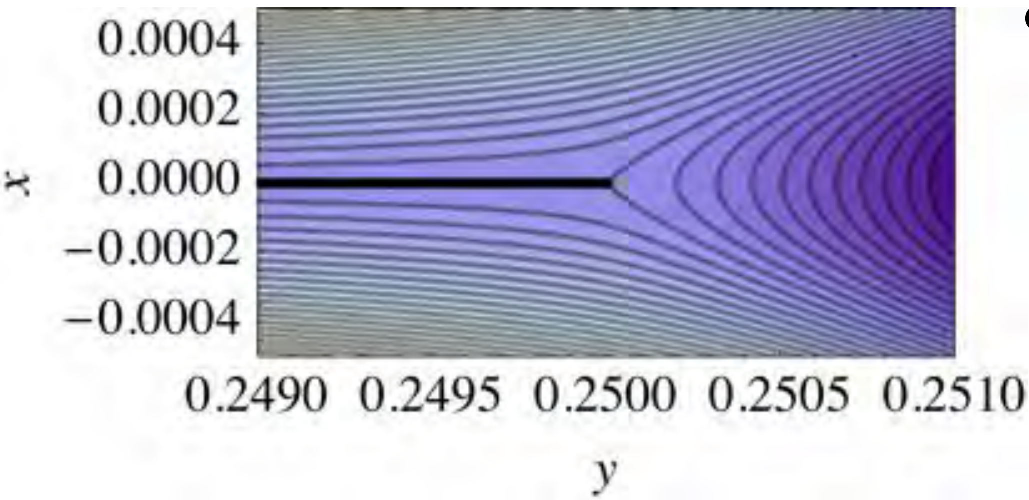
$$\psi = aB_y^a \left[ \frac{x^2}{2a^2} + \frac{\alpha}{\cosh(ka)} \cosh(kx) \cos(ky) \right]$$

where $B^a_y$ is |unperturbed poloidal field| at boundaries and $\alpha \ll 1$

# *New:* plasmoid solutions with partial current sheets




- partial current sheets between $x = \pm L + n\lambda$
- "plasmoids" between current sheets




- zoomed view of end of Sweet-Parker current sheet

# Construction is based on linear Grad-Shafranov equation (on *x*,*y* plane cut along current sheets)

$$\mathbf{B} = \nabla z \times \nabla \psi + F(\psi) \nabla z$$

Local force balance implies GS equation (except at cuts)

$$\nabla^2 \psi + \partial_\psi [\mu_0 p(\psi) + \tfrac{1}{2} F(\psi)^2] = 0$$

Choose *p* and *F* profiles such that

$$\mu_0 p(\psi) + \tfrac{1}{2} F(\psi)^2 = \text{const} - \frac{B_y^a}{a} \psi$$

then poloidal stream function $\psi$ obeys (except at cuts)

$$\nabla^2 \psi = \frac{B_y^a}{a}$$

with general solution

$$\psi = a B_y^a \left[ \frac{x^2}{2a^2} + \alpha \psi_1(x, y) \right]$$

where $\psi_1$ is any *harmonic function*, i.e. it obeys Laplace's equation

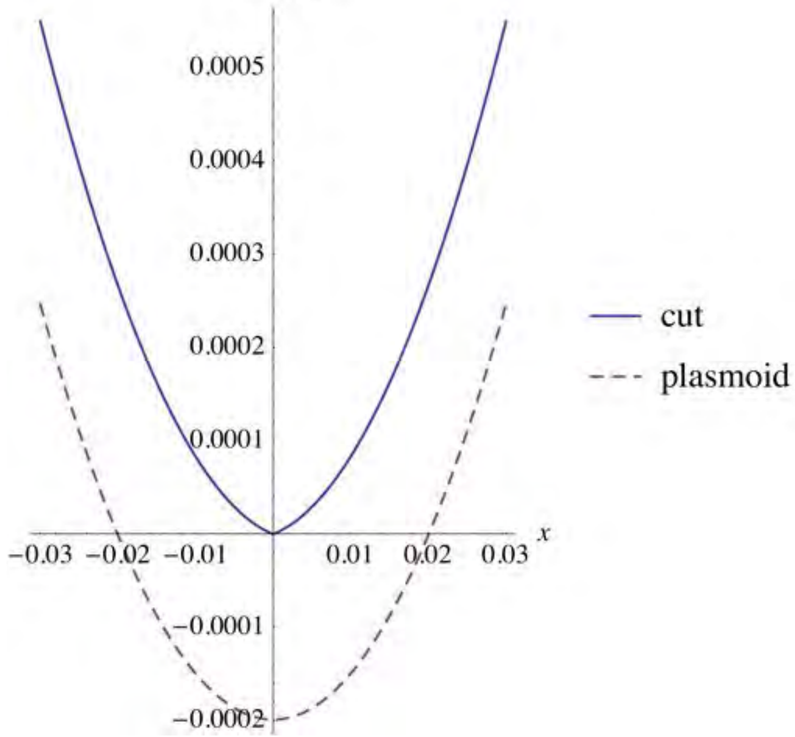
# Equilibrium condition on cuts

Cuts in *x*,*y* plane correspond to current sheets, so jump condition

$$\left[\!\left[ p + \frac{B^2}{2} \right]\!\right] = 0$$
$(\text{taking } \mu_0 = 1)$

applies. Using $B^2 = |\nabla\psi|^2 + F^2$ and linear GS assumption: $p(\psi) + \frac{1}{2}F(\psi)^2 = \text{const} - \frac{B_y^a}{a}\psi$

we get simplified jump condition




$$[\![(\nabla\psi)^2]\!] = 0$$

which is automatically satisfied in HKT model because of assumed *symmetry* about *x* = 0

# Conformal mapping method

Relies on fact that real (or imaginary) part of an analytic complex function obeys Laplace's equation:

$$\nabla^2 \, \mathrm{Re} f(x+iy) = 0$$

Also $$\nabla^2 \, \mathrm{Re}\, g(f(x+iy)) = 0$$

So one solution can be mapped (conformally) into another simply by composing with a suitably chosen function of a complex variable.

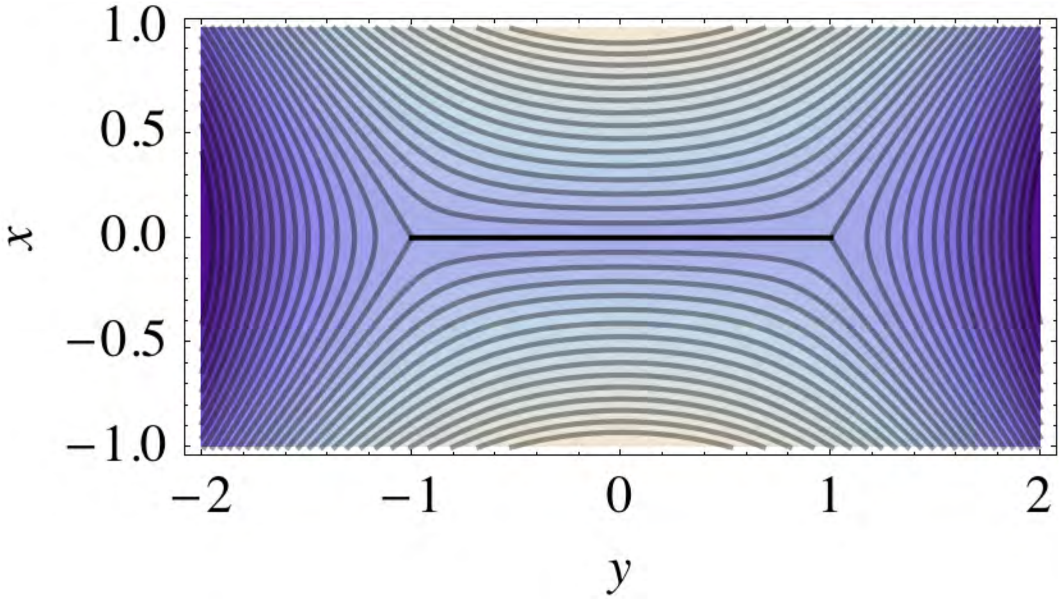
*Thus we can solve the slab Grad-Shafranov equation by finding a complex function whose branch cuts may be identified with current-sheet cuts.*

# Syrovatsky current sheets

Syrovatsky found a family of complex functions

$$F_{\mathrm{S}}(\zeta') = \mathrm{sgn}(\mathrm{Re}\,\zeta')\left\{\zeta'(\zeta'^2+1)^{1/2} + \gamma_S \ln[\zeta' + (\zeta'^2+1)^{1/2}]\right\}$$

with adjustable parameter $\gamma_S$ that may be chosen to model a Sweet-Parker current sheet:




$\gamma_S = 1$

makes sheet current vanish at end points and gives Y points shown

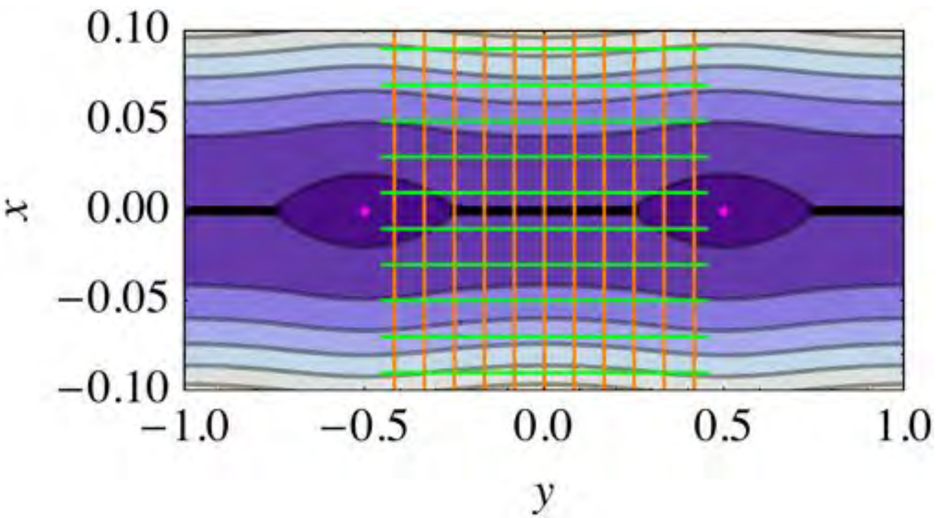
# Periodic conformal mapping of Syrovatsky current sheet

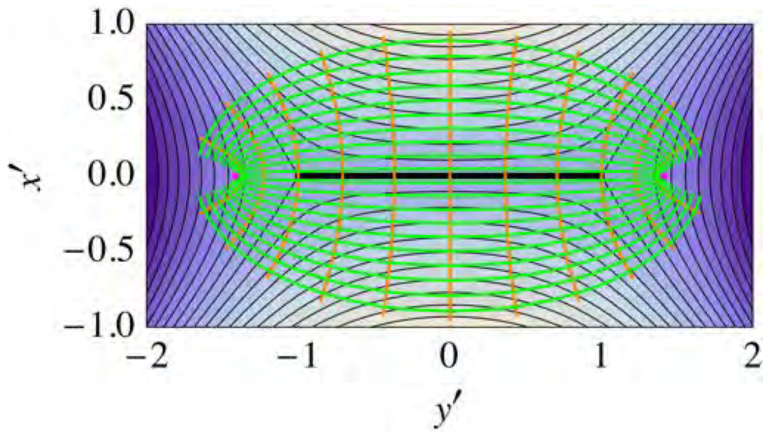
Define $f(\zeta) \equiv \frac{\sinh(\frac{1}{2}k\zeta)}{\sin(\frac{1}{2}kL)}$ and $\zeta' = f(x+iy)$

and use to map Syrovatsky solution so as to give the harmonic component of the GS solution:

$$\psi_1(x,y) = aB_0\,\frac{c+\mathrm{Re}F_{\mathrm{S}}(\zeta')}{d}$$

where *c* & *d* are chosen to match Hahm-Kulsrud boundary conditions



# Conclusions

- Multi-region generalization of Taylor relaxation to include current sheets has achieved practical realization in the SPEC code
- *But* SPEC must *ad hoc* violate helicity conservation in each sub-region if it is to satisfy side constraint of irrational rotational transform in order to be consistent with the Hamilton-Jacobi existence criterion
- We are exploring whether the use of more topologically complex "plasmoid" interfaces, combined with minimal reconnection, might provide a more natural self-organization scenario