Homework 5

APAM 4990 (2010)

As we have discussed in the class, by changing $F(\mathbf{x}, \mathbf{v}, t)$ to $F(\mathbf{R}, \mu_B, U, \varphi, t)$, where $\mathbf{x} = \mathbf{R} + \boldsymbol{\rho}$, $\mu_B = v_\perp^2/2B$, $\mathbf{v} = \mathbf{v}_\perp + U\hat{\mathbf{b}}$, $\boldsymbol{\rho} = \hat{\mathbf{b}} \times \mathbf{v}_\perp/\Omega$, $\mathbf{v}_\perp = v_\perp(cos\varphi\hat{\mathbf{e}}_1 + sin\varphi\hat{\mathbf{e}}_2)$, $\hat{\mathbf{b}} = \hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2$, $\mathbf{v} = U\hat{\mathbf{b}} + \mathbf{v}_\perp$, and $\hat{\mathbf{b}}$ is the unit vector in the direction of the external magnetic field, we can write the Vlasov equation in slab geometry as

$$\frac{\partial F}{\partial t} + (U\hat{\mathbf{b}} + \frac{c}{B}\mathbf{E} \times \hat{\mathbf{b}}) \cdot \frac{\partial F}{\partial \mathbf{R}} + \frac{q}{m}\mathbf{E} \cdot \hat{\mathbf{b}}\frac{\partial F}{\partial U}$$

$$-\Omega\frac{\partial F}{\partial \varphi} + \frac{q}{m}\mathbf{E} \cdot (\frac{\mathbf{v}_\perp}{B}\frac{\partial F}{\partial \mu_B} + \frac{\hat{\mathbf{b}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial F}{\partial \varphi}) = 0, \tag{1}$$

by using $\partial F/\partial \mathbf{x} \approx \partial F/\partial \mathbf{R}$ and

$$\frac{\partial F}{\partial \mathbf{v}} = \frac{1}{\Omega}\hat{\mathbf{b}} \times \frac{\partial F}{\partial \mathbf{R}} + \frac{\mathbf{v}_\perp}{B}\frac{\partial F}{\partial \mu_B} + \hat{\mathbf{b}}\frac{\partial F}{\partial U} + \frac{\hat{\mathbf{b}} \times \mathbf{v}_\perp}{v_\perp^2}\frac{\partial F}{\partial \varphi}, \tag{2}$$

where $\mathbf{E}(\mathbf{x}) = -\nabla\Phi(\mathbf{x})$. From the ordering consideration of $\omega/\Omega \sim k_\parallel U/\Omega \sim e\Phi/T_e \sim o(\epsilon)$, where ϵ is a smallness parameter, we can conclude that

$$\Omega\frac{\partial F}{\partial \varphi} = 0$$

is the lowest order solution. This knowledge enables us to assume that

$$F = f + \epsilon g,$$

for which $f(\mathbf{R}, \mu_B, U, t) \neq f(\varphi)$, and the Vlasov equation to the order of ϵ then becomes

$$\frac{\partial f}{\partial t} + (U\hat{\mathbf{b}} + \frac{c\mathbf{E} \times \hat{\mathbf{b}}}{B}) \cdot \frac{\partial f}{\partial \mathbf{R}} + \frac{q}{m}\mathbf{E} \cdot \hat{\mathbf{b}}\frac{\partial f}{\partial U} - \Omega\frac{\partial}{\partial \varphi}(g - \frac{q\Phi}{mB}\frac{\partial f}{\partial \mu_B}) = 0. \tag{3}$$

Note that the last term in Eq. (1) is of $o(\epsilon^2)$ and, therefore, is negligible.

(1) Show that the last term in Eq. (3) comes from $\partial\Phi/\partial\mathbf{x} \approx \partial\Phi/\partial\mathbf{R}$ for $\boldsymbol{\rho} \approx const.$ and

$$\Omega\frac{\partial \Phi}{\partial \varphi} = -\mathbf{v}_\perp \cdot \frac{\partial \Phi}{\partial \mathbf{R}},$$

which can be obtained from Eq. (2) [hint: $\partial\Phi(\mathbf{x})/\partial\mathbf{v} = 0$, why?].

The gyrophase averaged equation can then be obtained from Eq. (3) as

$$\frac{\partial f}{\partial t} + \left(U\hat{\mathbf{b}} + \frac{c}{B}\bar{\mathbf{E}} \times \hat{\mathbf{b}}\right) \cdot \frac{\partial f}{\partial \mathbf{R}} + \frac{q}{m}\bar{\mathbf{E}} \cdot \hat{\mathbf{b}}\frac{\partial f}{\partial U} = 0, \tag{4}$$

and the gyrophase dependent part becomes

$$g = \frac{q}{mB}\frac{\partial f}{\partial \mu_B}(\Phi - \bar{\Phi}), \tag{5}$$

where $\bar{\mathbf{E}}(\mathbf{R}) = \langle \mathbf{E}(\mathbf{x})\rangle_\varphi$ and $\bar{\Phi}(\mathbf{R}) = \langle \Phi(\mathbf{x})\rangle_\varphi$ are the gyrophase-averaged electric field and potential, respectively, and $\langle \cdots \rangle_\varphi \equiv \int_0^{2\pi} d\varphi/2\pi$. From $\mathbf{E}(\mathbf{x}) = -\partial\Phi(\mathbf{x})/\partial\mathbf{x}$, where

$$\Phi(\mathbf{x}) = \sum_{\mathbf{k}} \Phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}} \Phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{R}} e^{i\mathbf{k}\cdot\boldsymbol{\rho}},$$

we arrive at $\bar{\mathbf{E}}(\mathbf{R}) = -\partial\bar{\Phi}(\mathbf{R})/\partial\mathbf{R}$ based on

$$\bar{\Phi}(\mathbf{R}) = \sum_{\mathbf{k}} \Phi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{R}} J_0(k_\perp v_\perp/\Omega),$$

by using

$$\langle exp(i\mathbf{k}\cdot\boldsymbol{\rho})\rangle_\varphi = J_0(k_\perp v_\perp/\Omega),$$

which comes from the identity of

$$e^{\pm ip sin\varphi} = \sum_{n=-\infty}^{+\infty} J_n(p) e^{\pm in\varphi}.$$

In the limit of $k_\perp v_\perp/\Omega \to 0$, $\bar{\Phi}(\mathbf{R}) \approx \Phi(\mathbf{x})$, which gives $\bar{\mathbf{E}}(\mathbf{R}) \approx \mathbf{E}(\mathbf{x})$. Thus, Eq. (4) becomes the usual drift kinetic equation in this limit, since there is no distinction between $\mathbf{R}$ and $\mathbf{x}$.

However, the different between Φ and $\bar{\Phi}$ needs to be maintained even in the limit of $k_\perp v_\perp/\Omega \to 0$ for Eq. (5), because

$$n = \int F d\mathbf{v} = \int f d\mathbf{v} + \int g d\mathbf{v},$$

and the latter is the all-important polarization density, i.e.,

$$n_p \approx \frac{q}{mB}\int \frac{\partial f_M}{\partial \mu_B}[\Phi(\mathbf{x}) - \bar{\boldsymbol{\Phi}}(\mathbf{R})]\mathbf{dv},$$

where the drift kinetic f in g, Eq. (5), can further be approximated by a Maxwellian.

(2) Show that, for $d\mathbf{v} = Bd\mu_B dU d\varphi/2\pi$, the corresponding gyrokinetic Poisson's equation becomes

$$\left[\nabla^2 + \frac{\omega_{pi}^2}{\Omega_i^2}\nabla_\perp^2\right]\Phi(\mathbf{x}) = -4\pi e\int (f_i - f_e) B d\mu_B dU, \tag{6}$$

by first transforming $\bar{\Phi}(\mathbf{R})$ in n_p back to the $\mathbf{x}$ coordinates through $e^{i\mathbf{k}\cdot\mathbf{R}} = e^{i\mathbf{k}\cdot\mathbf{x} - i\mathbf{k}\cdot\boldsymbol{\rho}}$ and then by using

$$J_0(k_\perp v_\perp/\Omega) \approx 1 - \frac{1}{4}\frac{k_\perp^2 v_\perp^2}{\Omega^2}.$$

Note that Eq. (4), in its drift kinetic form, and Eq. (6) agree with the *ad-hoc* equations that we derived in class earlier. The gyrokinetic Vlasov-Poisson system valid for $k_\perp v_\perp/\Omega \sim o(1)$ is included in the posted lecture and will be discussed later.