COMMENTS

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Comment on ''A Lyapunov functional for stability of ideal magnetohydrodynamic systems in arbitrary motion'' [Phys. Plasmas 4, 1201 (1997)]

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A number of the results developed in this paper (referred to here as GB) are incorrect. In the paper the author purports to formulate a necessary and sufficient condition for linear stability of systems with mass flow, based on a Hamiltonian description of the dynamics. By mishandling some key integrals, the author misses the fact that the standard Hamiltonian is a constant of the motion, while ''proving'' that another integral is.

For simplicity, we only consider a plasma perturbed about a steady-state equilibrium with flow velocity $\mathbf{v}$, and boundary conditions of zero flow across a fixed boundary. The equation governing the (real) perturbation displacement vector $\boldsymbol{\xi}(\mathbf{r},t)$ is [Eq. (1) in GB]

$$\rho\boldsymbol{\xi}_{tt}+2\rho\mathbf{v}\cdot\boldsymbol{\nabla}\boldsymbol{\xi}_t=\mathbf{F}_a(\boldsymbol{\xi}),$$

where $\mathbf{F}_a$ is a self-adjoint operator. Multiplying the equation by $\boldsymbol{\xi}_t$, integrating over the volume, and using the fact that $\rho\mathbf{v}\cdot\nabla$ is anti-symmetric, yields

$$\frac{d}{dt}\int\left(\frac{1}{2}\rho|\boldsymbol{\xi}_t|^2-\frac{1}{2}\mathbf{F}_a(\boldsymbol{\xi})\cdot\boldsymbol{\xi}\right)d^3\mathbf{r}=0.$$

This equation expresses the conservation of energy. Indeed, the integral is identical with the Hamiltonian H defined in Eqs. (5) and (6) of GB, when the canonical momentum is expressed explicitly in terms of $\boldsymbol{\xi}$ and $\boldsymbol{\xi}_t$. The author attempts in his Appendix A to calculate the time-derivative of H when using the canonical variables, but reaches the surprising conclusion that ''it seems not possible to prove that $dH/dt=0$.'' The fault, of course, lies with an error in the calculation, similar to the one described next.

In his second, Lagrangian-coordinates formulation (Sec. III B), the author proceeds to define another Hamiltonian H [Eqs. (31) and (32)]. This formulation, it should be pointed out, is non-canonical, as can be seen from the fact that time derivatives in Eqs. (33), (34) have a factor ρ in front of them. There is no *a priori* reason for this H to be conserved since H depends explicitly on time when using moving (Lagrangian) independent variables. Expressing $H=\int\mathscr{H}(\mathbf{r};\boldsymbol{\xi},\boldsymbol{\pi})d^3\mathbf{r}$, where $\boldsymbol{\xi}=\boldsymbol{\xi}(\mathbf{r},t)$ and $\boldsymbol{\pi}=\boldsymbol{\pi}(\mathbf{r},t)$ are the author's so-called ''canonical'' variables, and $\mathbf{r}$ is again the Eulerian position variable, the author concludes correctly [Eq. (36)] that

$$\frac{dH}{dt}=\int\rho\mathbf{v}\cdot\nabla\left(\frac{\mathscr{H}}{\rho}\right)d^3\mathbf{r}.$$

However, the gradient in the integrand is applied only to the explicit $\mathbf{r}$-dependence of $\mathscr{H}$, while $\boldsymbol{\xi}$ and $\boldsymbol{\pi}$ are kept fixed. This prevents writing the integrand as a true divergence so that dH/dt does not generally vanish, contrary to the author's conclusion. With this non-conservation, the stability condition developed in the paper collapses.

That this condition is unlikely to be correct can also be seen from its explicit expression, Eq. (24). If we go to the limit of zero magnetic field ($\mathbf{B}\to 0$), zero external force ($\phi\to 0$), and incompressibility (div $\boldsymbol{\xi}\to 0$), the stability condition reduces to

$$-\int\boldsymbol{\xi}\,\nabla:[\boldsymbol{\xi}\rho(\mathbf{v}\cdot\nabla\mathbf{v})]d^3\mathbf{r}\geqslant 0.$$

The term $\mathbf{v}\cdot\nabla\mathbf{v}$ vanishes if, say, $\mathbf{v}=V(x)\hat{\mathbf{z}}$ which has the velocity vector perpendicular to the direction of the flow shear. Thus, the author's stability criterion is unable to pick out the Kelvin-Helmholtz instability present (generally) in such a configuration when Rayleigh's inflection-point criterion is violated.[1]

Some of the author's difficulties appear to be rooted in his understanding that the independent variables in the Lagrangian displacement vector $\boldsymbol{\xi}(\mathbf{r},t)$ are moving coordinates of fluid particles. This exhibits itself in the obviously mistaken Eq. (46). In fact, $\mathbf{r}$ and t are non-moving Eulerian variables, as can be seen in Newcomb's derivation.[2] In particular, all the derivatives in Eq. (1) of GB are Eulerian derivatives.

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Finally, we comment on the presence of instability when $\boldsymbol{\xi}$ is found to be growing linearly in t. Such growth does not always indicate instability, unlike the author's assumption. The relation between $\boldsymbol{\xi}$ and the perturbation of the velocity $\delta\mathbf{v}$ is given by [Eq. (4.6) of our Ref. 2]

$$\delta\mathbf{v}=\boldsymbol{\xi}_t+\mathbf{v}\cdot\nabla\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\nabla\mathbf{v}.$$

Choosing $\boldsymbol{\xi}=t\mathbf{v}$ results in $\delta\mathbf{v}=\mathbf{v}$, that is, $\delta\mathbf{v}$ is constant in time even though $\boldsymbol{\xi}$ is growing. It can also be shown that the other perturbed plasma profiles, δp, $\delta\rho$, and $\delta\mathbf{B}$, actually vanish. Another perturbation with similar properties is $\boldsymbol{\xi}=t\mathbf{B}/\rho$. Obviously, such $\boldsymbol{\xi}$'s should not be classified as instabilities. A more careful and detailed analysis of every linearly growing displacement $\boldsymbol{\xi}$ is required before the presence of an instability could be asserted.

[1]P. G. Drazin and W. H. Reid, *Hydrodynamic Stability* (Cambridge University Press, Cambridge, 1981), Sec. 22.
[2]W. A. Newcomb, Nucl. Fusion Suppl. **2**, 451 (1962).