Chapter 3

Landau Fluid Equations

The Navier–Stokes equations for neutral fluids are highly effective at describing the flows of many gases and liquids under typical conditions. The short mean free path and high collision rate in most cases ensures that, from a macroscopic point of view, the particle distributions are very nearly Maxwellian, and the departures are well modeled by simple viscosity terms. The very low collision rate in tokamak plasmas leads to a breakdown of the standard Chapman–Enskog method for deriving the plasma fluid equations of Braginskii (1965). Plasma waves, especially those driven by temperature-gradient instabilities, can experience significant collisionless damping due to Landau damping. The approach to deriving plasma fluid equation by Hammett and Perkins (1990) was to derive the fluid closures by matching the linear kinetic response of a collisionless fluid. The resulting equations conserve particles, momentum and energy and at the same time accurately model the dissipation of electrostatic waves due to Landau damping.

The relation between fluid moment closures and the statistical moment closures considered in the previous chapter is illustrated by considering the one-dimensional Vlasov equation. With an appropriate change of variables, the linearized Vlasov equation is equivalent to the stochastic oscillator problem. The background particle distribution in the Vlasov theory takes the place of the frequency distribution for the oscillator problem. The results concerning general linear closures of stochastic oscillator moments carry over to linear closures of fluid moments.

Orthogonal-polynomial fluid moments can be defined analogously to the orthogonal-polynomial statistical moments. The natural background particle distribution is Maxwellian, so Hermite fluid moments form the natural basis for studying the Vlasov equation. The Hermite fluid moments are equivalent, in fact, to the coefficients of the Hermite series expansion of the Vlasov equation that has traditionally been used for one-dimensional plasma simulation (Armstrong et al. 1970). The connection between Hermite moments and cumulants is demonstrated for fluid moments as well.

Several insights can be gained from the transformations among coefficients in the series expansion of the Vlasov equation, fluid moments of the Vlasov equation, and statistical moments of the stochastic oscillator. Observations concerning the limitations of truncated series expansions have implications for fluid moment systems that evolve a finite number of moments. On the other hand, the general dissipative closure derived for fluid moments may be useful in applications to kinetic series solutions of the Vlasov equation.

Deriving closures from linear theory does not fully determine the nonlinear form of the closure. Closures with significantly different nonlinear features can be linearly identical. Two such closures are introduced in this chapter. The optimal nonlinear form of fluid closures is an interesting question that is only starting to be addressed.

3.1 Fluid Moments

Consider the one-dimensional kinetic theory of a distribution of particles at location z with velocity v given by $f(z, v, t)$. The standard fluid moments, number density n, mean velocity u, pressure p, heat flux q, and general higher moments r_n are defined by

$$\begin{aligned}
n(z) &= \int_{-\infty}^{\infty} f(z,v)\,dv, & n(z)u(z) &= \int_{-\infty}^{\infty} v f(z,v)\,dv, \\
p(z) &= m\int_{-\infty}^{\infty} [v-u(z)]^2 f(z,v)\,dv, & q(z) &= m\int_{-\infty}^{\infty} [v-u(z)]^3 f(z,v)\,dv, \\
r(z) &= m\int_{-\infty}^{\infty} [v-u(z)]^4 f(z,v)\,dv, & &\ldots \\
r_n(z) &= m\int_{-\infty}^{\infty} [v-u(z)]^{4+n} f(z,v)\,dv, & & \qquad (3.1)
\end{aligned}$$

where m is the particle mass. These moments are nonlinear functions of the particle distribution as a result of their definition with respect to the mean velocity $u(z)$. To connect plasma kinetic problems with the stochastic oscillator moments, the v^n moments are defined by

$$M_n(z) = \int_{-\infty}^{\infty} v^n f(z,v)\,dv. \qquad (3.2)$$

The first n moments in either formulation contain exactly the same information. There is an algebraic transformation from either set of moments to the other given by

$$\begin{aligned}
n(z) &= M_0(z), \\
u(z) &= \frac{M_1(z)}{M_0(z)}, \\
p(z) &= m\left[M_2(z) - \frac{M_1(z)^2}{M_0(z)}\right], \\
q(z) &= m\left[M_3(z) - 3\frac{M_2(z)M_1(z)}{M_0(z)} + 2\frac{M_1(z)^3}{M_0(z)^2}\right], \\
r(z) &= m\left[M_4(z) - 4\frac{M_3(z)M_1(z)}{M_0(z)} + 6\frac{M_2(z)M_1(z)^2}{M_0(z)^2} - 3\frac{M_1(z)^4}{M_0(z)^3}\right], \\
&\vdots
\end{aligned} \tag{3.3}$$

which translates v^n moments into fluid moments and by

$$\begin{aligned}
M_0(z) &= n(z), \\
M_1(z) &= n(z)u(z), \\
M_2(z) &= \frac{p(z)}{m} + n(z)u(z)^2, \\
M_3(z) &= \frac{q(z) + 3u(z)p(z)}{m} + n(z)u(z)^3, \\
M_4(z) &= \frac{r(z) + 4u(z)q(z) + 6u(z)^2p(z)}{m} + n(z)u(z)^4, \\
&\vdots
\end{aligned} \tag{3.4}$$

which translates fluid moments into v^n moments. These transformations are nonlinear. A closure that is linear in fluid moments can therefore behave differently from a closure that is linear in v^n moments even if they are linearly equivalent. This difficulty is discussed in Section 3.4.

3.2 Vlasov Equation

The dynamics of particle motion along straight magnetic field lines is given by the one-dimensional kinetic equation,

$$\left[\frac{\partial}{\partial t} + v\frac{\partial}{\partial z} - \frac{q}{m}\frac{\partial\phi(z,t)}{\partial z}\frac{\partial}{\partial v}\right] f(z,v,t) = \nu\frac{\partial}{\partial v}\left(v + {v_t}^2\frac{\partial}{\partial v}\right) f(z,v,t). \tag{3.5}$$

The model collision operator of Lenard and Bernstein (1958) is included, which models collisions with a thermalizing background with collision frequency ν and thermal velocity v_t. In the limit of small potential perturbations, $q\phi \ll m{v_t}^2$, the collision term will dominate, so to lowest order the particles have a Maxwellian distribution,

$$f_0(v) = \frac{n_0}{\sqrt{2\pi {v_t}^2}} \exp(-v^2/2{v_t}^2). \tag{3.6}$$

The potential ϕ is related linearly to the distribution $f(z, v, t)$ through Poisson's equation, so the resulting system is quadratically nonlinear.

Taking fluid moments of the Vlasov equation (3.5) yields a hierarchy of fluid moment equations,

$$\begin{aligned}
&\frac{\partial n}{\partial t} + \frac{\partial}{\partial z}(un) = 0, && (3.7)\\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{1}{mn}\frac{\partial p}{\partial z} + \nu u = -\frac{q}{m}n\frac{\partial \phi}{\partial z}, && (3.8)\\
&\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial z} + 3p\frac{\partial u}{\partial z} + \frac{\partial q}{\partial z} + \nu\left[p - m(u^2 + {v_t}^2)n\right] = 0, && (3.9)\\
&\quad\vdots
\end{aligned}$$

These equations are nonlinear as a result of the moment definition with respect to the mean velocity. The v^n moment equations,

$$\begin{aligned}
&\frac{\partial M_0}{\partial t} + \frac{\partial M_1}{\partial z} = 0,\\
&\frac{\partial M_1}{\partial t} + \frac{\partial M_2}{\partial z} + \nu M_1 = -\frac{qM_0}{m}\frac{\partial \phi}{\partial z},\\
&\quad\vdots\\
&\frac{\partial M_j}{\partial t} + \frac{\partial M_{j+1}}{\partial z} + \nu j\left[M_j - (j-1)M_{j-2}\right] = -\frac{eM_{j-1}}{m}\frac{\partial \phi}{\partial z},
\end{aligned} \tag{3.10}$$

are linear equations and slightly simpler than the fluid moment equations. One very nice feature of the fluid moment equations, however, is that the potential only appears in one equation, the one for the velocity u, in contrast to v^n moment equations where it enters in every equation.

In Fourier space, the term $v\partial/\partial z\, f(z, v, t)$ becomes $ikvf(k, v, t)$, which is analogous to the $iby(t)$ in the stochastic oscillator. Fluid moments can therefore experience phase-mixing decay, since the frequencies kv are continuously distributed. Simple truncation schemes applied to either of these moment hierarchies leads to unphysical behavior. The truncation $M_n(z, t) = 0$ leads to a solution for the lower moments that is essentially a truncation of the Taylor series representing the true solution, and therefore blows up as $t \to \infty$. The fluid moment truncation $q(z, t) = 0$, on the other hand, leads to a set of equations that conserves the total density, momentum and energy and therefore has a solution that does not diverge at large times. Still, the system fails to exhibit phase-mixing decay.

	Stochastic Oscillator	Vlasov Equation
frequency	b	$\|k\|v$
moments	$g_n(t) = \langle b^n y(t) \rangle$	$M_n(k,t) = \int_{-\infty}^{\infty} v^n f(k,v,t)\,dv$
orthogonal functions	$P_n(b)w(b)$	$\frac{1}{\sqrt{2^n n!}} H_n(v/\sqrt{2}v_t) \frac{e^{-v^2/2{v_t}^2}}{\sqrt{2\pi {v_t}^2}}$
frequency distribution	$\rho(b)$	$\frac{ike}{m}\frac{\partial}{\partial v} f_0(v)$
oscillator variable	$y(t)\rho(b)$	$Y(k,v,t)$
Response	$\mathsf{R}_{01}(\omega)$	$R(\zeta)$

Table 3.1: Equivalent concepts for the stochastic oscillator problem and the linearized Vlasov equation.

Hammett and Perkins (1990) obtained a dissipative closure model by matching the response of the closed moment system with the linear response to a small potential assuming the near-equilibrium density, $f(z,v,t) = f_0(v) + \tilde{f}(z,v,t)$. The Fourier transformed, linearized Vlasov equation in the collisionless limit $\nu = 0$ is just

$$\left(\frac{\partial}{\partial t} + ikv \right) \tilde{f}(k,v,t) = \frac{q}{m} ik\phi(k,t) \frac{\partial}{\partial v} f_0(v). \tag{3.11}$$

The response for this linear problem with the background Maxwellian defined in Eq. (3.6) can be written simply as

$$\tilde{n}(k,\omega) = -\frac{qn_0}{m{v_t}^2} \phi(k,\omega) R(\zeta), \tag{3.12}$$

where the Laplace transform $t \to \omega$ has been taken, the response function is given by $R(\zeta) = 1 + \zeta Z(\zeta)$, and the normalized frequency

$$\zeta = \frac{\omega}{\sqrt{2}|k|v_t} \tag{3.13}$$

is used.

This system is very similar to the stochastic oscillator equation (2.6). To connect the two problems, note that the solution to the linearized Vlasov equation (3.11) can be written as $\tilde{f}(k,v,t) = \int_{-\infty}^{t} Y(k,v,t-t_1)\phi(k,t_1)\,dt_1$, where the Green's function $Y(k,v,\tau)$ is the solution to

$$\left(\frac{\partial}{\partial t} + ikv \right) Y(k,v,t) = 0, \qquad Y(k,v,0) = \frac{q}{m} ik \frac{\partial}{\partial v} f_0(v). \tag{3.14}$$

This initial-value problem is essentially the stochastic oscillator problem (2.6). The frequency for this system is kv instead of b. The Green's function $Y(k, v, t)$ corresponds to $y(t)\rho(b)$, the oscillator variable weighted by the frequency PDF. The driving term $(q/m)\, ik\phi\, (\partial/\partial v)\, f_0(v)$ in the Vlasov equation corresponds to the distribution of frequencies of the stochastic oscillator. For a Maxwellian, $(\partial/\partial v) f_0(v) \propto v f_0(v)$, so the response function $R(\zeta)$ corresponds to the $\mathsf{R}_{01}(\omega)$ response of the stochastic oscillator with a Gaussian distribution of frequencies. The correspondence between the two problems is summarized in Table 3.1.

Note that for the stochastic oscillator problem the frequency variable b is the same variable that appears in the distribution and the orthogonal polynomial expansion, whereas the frequency variable kv in the Vlasov equation is different from the combination v/v_t that appears in the distribution and the orthogonal polynomials. This difference could be eliminated by transforming the time variable in the Vlasov equation by $\tau = kv_t t$, which results in the now dimensionless frequency v/v_t. This transformation does not quite work, however, since the phase-mixing decay should occur in the limit $t \to -\infty$, and negative wave numbers k effectively reverse the direction of time with this transformation. The transformation of variables

$$\tau = |k| v_t t, \qquad b = \frac{k}{|k|}\frac{v}{v_t} \tag{3.15}$$

preserves the direction of time. The transformed Green's function equation for the linearized Vlasov equation with background Maxwellian becomes

$$\left(\frac{\partial}{\partial \tau} + ib\right) Y(k, b, \tau) = 0, \qquad Y(k, b, 0) = \frac{q}{m{v_t}^2} i\phi(k, \tau) \frac{n_0}{\sqrt{2\pi {v_t}^2}} \frac{\partial}{\partial b} \exp(-b^2/2). \tag{3.16}$$

The solution to this equation is precisely the solution of the stochastic oscillator problem with underlying distribution $\rho(b) \propto b \exp(-b^2/2)$. (Strictly speaking this PDF is not a probability distribution, but the linear response theory developed in the preceding chapter does not depend on the positivity of the underlying distribution.) The transformation from the stochastic oscillator problem to the linearized Vlasov equation is summarized in Table 3.1.

With the transformation (3.15) it is possible to convert the stochastic oscillator closures to fluid moment closures. The linear frequency in the Vlasov equation is given by the combination kv, so an appropriate dissipative closure for fluid moments should introduce damping on the order of $|k| v_t$, which is a measure of the spread of frequencies. Landau-fluid closures involve operators in wave-number space that are proportional to the absolute value of the wave number (Hammett and Perkins 1990). The $|k|$ operator arises because the quantity $k/|k|$ enters the transformation of closures from the stochastic oscillator problem to fluid moments. The stochastic oscillator moments are polynomials in $b = kv/|k|v_t$, whereas the fluid moments involve polynomials in v.

3.3 Hermite Moments

Before relating the stochastic oscillator moment closures to fluid moment closures, the definition of orthogonal-polynomial moments for the Vlasov equation will be considered. Early studies of the Vlasov equation (Weissglas 1962; Engelmann et al. 1963; Armstrong 1967; Grant and Feix 1967; Armstrong et al. 1970) used a series expansion about the background Maxwellian in terms of Hermite polynomials in velocity,

$$f(z,v,t) = \frac{1}{\sqrt{2\pi {v_t}^2}} \sum_{n=0}^{\infty} a_n(z,t) \frac{1}{\sqrt{2^n n!}} H_n(v/\sqrt{2}v_t) e^{-v^2/2{v_t}^2}. \tag{3.17}$$

The thermal velocity v_t is taken to be a constant for now. This series expansion represents the background distribution $f_0(v)$ with one nonzero term, $a_0 = n_0$, and therefore works well for problems with near-Maxwellian distributions.

Substituting this series expansion into the Vlasov equation (3.5) yields evolution equations for the coefficients,

$$\begin{aligned}
&\frac{\partial a_0}{\partial t} + v_t \frac{\partial a_1}{\partial z} = 0, \\
&\frac{\partial a_1}{\partial t} + v_t \left(\frac{\partial a_0}{\partial z} + \sqrt{2} \frac{\partial a_2}{\partial z} \right) + \nu a_1 = -\frac{q a_0}{v_t m} \frac{\partial \phi}{\partial z}, \\
&\ \vdots \\
&\frac{\partial a_j}{\partial t} + v_t \left(\sqrt{j} \frac{\partial a_{j-1}}{\partial z} + \sqrt{j+1} \frac{\partial a_{j+1}}{\partial z} \right) + \nu j a_j = -\frac{q\sqrt{j} a_{j-1}}{v_t m} \frac{\partial \phi}{\partial z}.
\end{aligned} \tag{3.18}$$

This 'kinetic' series solution of the Vlasov equation is in fact equivalent to the moment approach. Using the orthogonality of the Hermite polynomials, the series coefficients in Eq. (3.17) can be expressed as *Hermite moments*,

$$a_n(z,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2^n n!}} H_n(v/\sqrt{2}v_t) f(z,v,t)\, dv. \tag{3.19}$$

The first n Hermite moments and the first n fluid moments contain exactly the same information, since the polynomials $\{1, v, (v-u), \ldots, (v-u)^n\}$ span the same space as $\{H_0(v), H_1(v), \ldots, H_n(v)\}$. Each Hermite moment can be expressed in terms of fluid moments of equal or lower order by

$$
\begin{aligned}
a_0(z) &= n(z), \\
a_1(z) &= \frac{n(z)u(z)}{v_t}, \\
a_2(z) &= \frac{mu(z)^2 n(z) + \left[p(z) - m{v_t}^2\right] n(z)}{\sqrt{2} m{v_t}^2}, \\
a_3(z) &= \frac{q(z) + mu(z)^3 n(z) + 3u(z)\left[p(z) - m{v_t}^2 n(z)\right]}{\sqrt{6} m{v_t}^3}, \\
&\vdots
\end{aligned}
\tag{3.20}
$$

and each fluid moment can be expressed in terms of Hermite moments of equal or lower order by

$$
\begin{aligned}
n(z) &= a_0(z), \\
u(z) &= v_t \frac{a_1(z)}{a_0(z)}, \\
p(z) &= m{v_t}^2 \left[\sqrt{2} a_2(z) + a_0(z) - \frac{a_1(z)^2}{a_0(z)}\right], \\
q(z) &= m{v_t}^3 \left[\sqrt{6} a_3(z) - 3\frac{\sqrt{2} a_2(z) a_1(z)}{a_0(z)} + 2\frac{a_1(z)^3}{a_0(z)^2}\right], \\
&\vdots
\end{aligned}
\tag{3.21}
$$

Thus, there is a nonlinear transformation from fluid moments to v^n moments and from fluid moments to Hermite moments. The transformation from v^n moments to Hermite moments is just the linear transformation to orthogonal-polynomial moments discussed in Section 2.5.1. The theory from the previous chapter gives a systematic approach to developing linear closures that yield good approximations to the linear responses. As yet there is no theory to specify the nonlinear form of closures, leaving ambiguity in the choice of closure. Note that a linear closure in fluid moments translates to a nonlinear closure for Hermite moments and vice versa.

3.4 Nonlinear Aspect of Moment Closures

To demonstrate some of the nonlinear features of closures, two closures with the same linear behavior are presented here. Consider the heat-flux closure

$$
q(z) = -i\frac{2\sqrt{2}}{\sqrt{\pi}} v_t \frac{k}{|k|} \left[p(z) - m{v_t}^2 n(z)\right] \tag{3.22}
$$

of Hammett and Perkins (1990), where the $k/|k|$ operator is understood to represent the transformation to k space, multiplication by $k/|k|$, and transformation back to z space. Direct translation of the third Hermite moment closure

$$a_3(k,t) = -i\frac{k}{|k|}\frac{2\sqrt{2}}{\sqrt{3\pi}}a_2(k,t) \tag{3.23}$$

to fluid moments using Eq. (3.20) yields a significantly more complicated expression,

$$\begin{aligned} q(z) =& 3u(z)\left[m{v_t}^2 n(z) - p(z)\right] - mu(z)^3 n(z) \\ &- i\frac{2\sqrt{2}}{\sqrt{\pi}}v_t\frac{k}{|k|}\left\{p(z) + mn(z)\left[u(z)^2 - {v_t}^2\right]\right\}. \end{aligned} \tag{3.24}$$

Linear equivalence of these closures is demonstrated by splitting the moments into equilibrium and fluctuating components; for example $n(z) = n_0 + \tilde{n}(z)$ where $n_0 = \int_{-\infty}^{\infty} f_0(v)\,dv$ and $\tilde{n}(z) = \int_{-\infty}^{\infty} \tilde{f}(z,v)\,dv$. The odd equilibrium moments are zero ($u_0 = 0, q_0 = 0, \ldots$) and the even equilibrium moments can be expressed in terms of powers of v_t ($p_0 = m{v_t}^2 n_0, r_0 = 3m{v_t}^4 n_0, \ldots$). Assuming the fluctuating components are small, expanding both the Hammett–Perkins closure (3.22) and the transformed Hermite moment closure (3.24) give the same result to first order,

$$\tilde{q}(z) \sim -i\frac{2\sqrt{2}}{\sqrt{\pi}}v_t\frac{k}{|k|}\left[\tilde{p}(z) - m{v_t}^2\tilde{n}(z)\right], \tag{3.25}$$

but have different second-order terms. There is clearly great freedom in choosing closures that have the same linear behavior. For example, the linear combination $p(z) - m{v_t}^2 n(z)$, could be replaced by $n_0[T(z) - T_0(z)]$ where the temperature $T(z) = p(z)/n(z)$ is a nonlinear quantity that gives the same behavior to first order. Since $u_0 = 0$, any arbitrary expression multiplied by $u(z)^2$ may be added to the closure without affecting the linear behavior.

Note as well that both closures contain equilibrium constants, v_t and n_0. In simulations that assume a fixed background gradient, these parameters are constants that are input to the simulation. Extending closures to cases with a background equilibrium that vary in space or time may be nontrivial. In turbulence, the average density and energy along a field line vary slowly compared to the fluctuations themselves, so averaging the density $n(z)$ and the local thermal velocity $v_t(z) = \sqrt{p(z)/mn(z)}$ provides reasonable estimates for n_0 and v_t. The derivation of closure coefficients, however, assumes that these quantities are constant in time.

These questions have yet to be addressed satisfactorily. The following chapter presents a model nonlinear problem, the plasma echo, that can be accurately modeled provided enough moments are kept. Additional moments serve to approximate finer scales of the distribution function. It may be that the precise nonlinear form of the closure is irrelevant provided enough moments are kept. (Accurate simulations require a good linear closure, however, as is clear from the unphysical results obtained with the simplest moment discard approaches.)

3.5 Cumulants

The standard approach to deriving Landau-fluid closures expresses the fourth-moment closure in terms of the combination

$$\delta r(z) = r(z) - 3\frac{p(z)^2}{mn(z)}. \tag{3.26}$$

The term $3p(z)^2/mn(z)$ is often justified on the grounds that for a Maxwellian distribution, it is exactly equal to $r(z)$. For another point of view, one can examine the linear form of $\delta r(z)$. Formally, let us suppose that

$$f(z,v,t) = f_0(v) + \epsilon \tilde{f}(z,v,t), \tag{3.27}$$

where $f_0(v)$ is the background Maxwellian of Eq. (3.6) and ϵ is a formal ordering parameter assumed to be small. With this expansion we find that

$$\begin{aligned}\delta r(z) &= \epsilon\left[\tilde{r}(z) - 6\frac{p_0}{mn_0}\tilde{p}(z) + 3m{v_t}^4\tilde{n}(z)\right] + \mathcal{O}(\epsilon^2)\\ &= \epsilon m\int_{-\infty}^{\infty}\left(v^4 - 6{v_t}^2v^2 + 3{v_t}^4\right)\tilde{f}(z,v)\,dv + \mathcal{O}(\epsilon^2)\\ &= \epsilon m{v_t}^4\int_{-\infty}^{\infty} H_4(v/\sqrt{2}v_t)\tilde{f}(z,v)\,dv + \mathcal{O}(\epsilon^2).\end{aligned} \tag{3.28}$$

So to first order in the expansion parameter ϵ, we find that $\delta r(z)$ is proportional to the Hermite moment $a_4(z)$. In fact the moments that are typically used to express closures, $\tilde{n}(z)$, $\tilde{u}(z)$, $\tilde{T}(z) = [\tilde{p}(z) - T_0\tilde{n}(z)]/n_0$ and $\tilde{q}(z)$, are all proportional to Hermite moments. The effectiveness of this basis arises from the orthogonality property of the moments.

Cumulant expansions can be used to systematically generate terms such as $\delta r(z)$ that vanish for a Maxwellian and linearly match the Hermite moments. Velocity averages are defined by

$$\langle g(v)\rangle = \frac{\int_{-\infty}^{\infty} g(v)f(z,v)\,dv}{\int_{-\infty}^{\infty} f(z,v)\,dv}, \tag{3.29}$$

and cumulant averages in this case are defined by identifying powers of x in the equation

$$\exp\left(\sum_{n=1}^{\infty}\langle v^n\rangle_c\frac{x^n}{n!}\right) = \langle\exp(vx)\rangle. \tag{3.30}$$

The cumulants can be formally expanded as $\langle v^n\rangle_c = \langle v^n\rangle_c^0 + \epsilon\langle v^n\rangle_c^1 + \cdots$. Upon substituting this expansion in the cumulant definition (3.30) and using the formal expansion of the distribution (3.27) in the equation for the averages (3.29), one finds the expression for the first-order part of the cumulant average,

$$\begin{aligned}\sum_{n=1}^{\infty} \langle v^n \rangle_c^1 \frac{x^n}{n!} &= \frac{1}{n_0}\left[\exp(-{v_t}^2 x^2/2)\int_{-\infty}^{\infty} \exp(xv)\tilde{f}(z,v)\,dv - \int_{-\infty}^{\infty} \tilde{f}(z,v)\,dv\right] \\ &= \frac{1}{n_0}\int_{-\infty}^{\infty} \sum_{n=1}^{\infty} \left(v_t/\sqrt{2}\right)^n H_n(v/\sqrt{2}v_t)\frac{x^n}{n!}\tilde{f}(z,v)\,dv. \end{aligned} \tag{3.31}$$

This result uses the Hermite polynomial generating function (2.25). The cumulant averages $\langle v^n \rangle_c$ are therefore linearly equivalent to multiples of the Hermite moments.

Cumulant averages therefore provide an alternative nonlinear basis of moments with which to form closures that are linearly equivalent to those formed from Hermite moments. The first few cumulant moments are

$$\begin{aligned} \langle v \rangle_c = u(z), \qquad & \left\langle v^2 \right\rangle_c = \frac{p(z)}{mn(z)}, \\ \left\langle v^3 \right\rangle_c = \frac{q(z)}{mn(z)}, \qquad & \left\langle v^4 \right\rangle_c = \frac{mn(z)r(z) - 3p(z)^2}{m^2 n(z)^2}. \end{aligned} \tag{3.32}$$

3.6 Summary

Transformations have been presented here that enable several problems involving closures to be identified with one another. The linearized Vlasov equation has been shown to be equivalent to the stochastic oscillator problem. The fluid moment closure problem in this case is therefore equivalent to a statistical moment closure problem. It has also been shown that the fluid moments of the Vlasov equation are equivalent to Hermite polynomial moments, which in turn are equivalent to coefficients in the Hermite series that has traditionally been used in simulations of the Vlasov equation. Fluid moment cumulants are linearly equivalent to Hermite moments. This equivalence is responsible for the nice properties of closures expressed in terms of cumulants.

Note that the transformation from fluid moments to Hermite moments is nonlinear, so linear closures with the same first-order behavior can produce different results at higher order. The correct nonlinear form of fluid moment closures is an issue that has yet to be addressed. In the following chapter, results are presented that indicate that higher-order behavior is more sensitive to the number of fluid moments retained in a model than to the precise form of the closure.