Chapter 1

Introduction

Moment closure problems appear in a wide range of applications. In statistical theories of turbulence, for example, equations for second-order statistics (energy) are coupled to equations for third-order statistics (transfers), which are coupled to higher-order statistics *ad infinitum*. A number of theories exist that attempt to close the system by estimating the third-order correlations as a function of the second-order terms. In fluid mechanics the equations for momentum, a first-order velocity moment of the particle distribution, contain terms involving second-order velocity moments (the pressure tensor). For common fluids, the high collision rate and short mean free path enable one to derive simple models for the pressure tensor that work very well. Viscosity is essentially a closure for some second-order particle-velocity moments (off-diagonal components of the pressure tensor) as functions of gradients of the first-order particle velocity moment (fluid momentum). In many plasma turbulence problems, the long mean free path has led to the need to consider new approaches to the fluid moment closure problem. Ion-temperature-gradient-driven (ITG) instabilities in collisionless plasmas, for example, experience strong collisionless damping from the interaction of resonant particles with waves (ion Landau damping). The correct choice of fluid moment closure can enable fluid equations to model these kinetic effects (Hammett and Perkins 1990; Chang and Callen 1992).

A simple illustration of the moment closure problem is given by the stochastic oscillator problem (Kraichnan 1961; Kubo 1963; Kraichnan 1968), a simple model that has served as a useful paradigm for studying statistical moment closures. The oscillator problem considers an ensemble of phases $y(t)$ that evolve according to

$$\frac{\partial y}{\partial t} + iby = 0, \qquad y|_{t=0} = 1, \tag{1.1}$$

where b is a constant-in-time random variable that sets the frequency at which each phase oscillates. The phases of different realizations in the stochastic oscillator or different velocity components of a given perturbation will eventually become decorrelated, causing the ensemble average $\langle y \rangle$ to decay in time. (Averages of the form $\langle b^n y \rangle$ will be called *moments* of y.) For this simple problem, the exact time behavior of this average is easy to compute directly by integrating the evolution equation (1.1) in time and then taking averages. Instead, if one takes the ensemble average of Eq. (1.1) first, one obtains the evolution equation

$$\frac{\partial\langle y\rangle}{\partial t}+i\langle by\rangle=0, \qquad \langle y\rangle|_{t=0}=1. \tag{1.2}$$

The initial condition for and time derivative of the moment $\langle y\rangle$ are given by this equation. This equation contains the unknown moment $\langle by\rangle$, however, so it cannot be integrated alone to give the time behavior of the moment $\langle y\rangle$. After multiplying the evolution equation (1.1) by b^n and taking the ensemble average, one obtains the moment hierarchy

$$\begin{aligned} \frac{\partial\langle by\rangle}{\partial t}+i\left\langle b^2y\right\rangle&=0, & \langle by\rangle|_{t=0}&=\langle b\rangle,\\ \frac{\partial\left\langle b^2y\right\rangle}{\partial t}+i\left\langle b^3y\right\rangle&=0, & \left\langle b^2y\right\rangle\Big|_{t=0}&=\left\langle b^2\right\rangle,\\ &\vdots & &\vdots \end{aligned} \tag{1.3}$$

Because the evolution equation for a given moment contains the next higher-order moment, no finite set of these equations can be integrated. Determining an appropriate estimate for the unresolved moment in terms of resolved moments is the moment closure problem.

The stochastic oscillator problem may appear overly simplified. However, the oscillator is a good model problem for the physical process of phase mixing, which is a fundamental process in many systems. With the substitutions $y(t)\to f(v,z,t)$ and $ib\to v\partial/\partial z$, the stochastic oscillator becomes the propagator of the Vlasov equation in plasma physics. (This transformation was pointed out to us by Dr. John Bowman and will be used in this thesis to map closures for the stochastic oscillator to fluid moment closures of the Vlasov equation.) With the substitutions $y(t)\to v(x,t)$ and $ib\to v\cdot\nabla$, the stochastic oscillator equation becomes the Euler equation describing fluid turbulence in the absence of dissipation. (The fluid advection term is nonlinear in the advected term, unlike the stochastic oscillator. There are limits, however, in which phase-mixing processes are important, such as the coherent straining of short-wavelength modes by long-wavelength modes.)

There are many approaches to obtaining closures for unresolved terms in moment hierarchies. This thesis focuses on a set of simple closures that model the phase-mixing decay process by linear dissipative terms that introduce damping proportional to the spread of frequencies. For fluid moments of the Vlasov equation, the spread of frequencies for a given wave number k is proportional to $|k|v_t$, where v_t is the average thermal velocity. The subgrid-scale closure problem for drift-wave turbulence considered in Chapter 6 can also be understood in terms of phase mixing. The phases of different points on an eddy with wave number $\mathbf{k}$ become decorrelated at a rate proportional to $|\Delta\mathbf{V}\cdot\mathbf{k}|$ where $\Delta\mathbf{V}$ is the difference in the advecting velocity at the two points. An estimate for the velocity difference is given by $\Delta\mathbf{V} \sim \Delta\mathbf{x}\cdot\boldsymbol{\nabla}\mathbf{V}$ where $\Delta\mathbf{x}$ is the distance between the two points. For isotropic turbulence, a rough scaling of the damping rate that should be introduced by the subgrid-scale closure is therefore given by the average magnitude of the shear tensor $\boldsymbol{\nabla}\mathbf{V}$, since $|\Delta\mathbf{x}| \sim 1/|\mathbf{k}|$. Extending the results to three-dimensional (3D) ITG turbulence, one must be slightly more careful to account for the elongated nature of the eddies. (The perpendicular wave number is much larger than the parallel wave number, so the eddy turnover rate is enhanced over the basic shearing rate $\nabla_{||}\mathbf{V}_{\perp}$ by a factor of $|\mathbf{k}_{\perp}|/k_{||}$.)

1.1 Closures in One Dimension

Statistical moment closures for the stochastic oscillator problem are considered first. The procedure of Hammett and Perkins (1990) for deriving fluid moment closures is generalized and applied to the stochastic oscillator moment hierarchy. The n closure coefficients for the n-moment system are computed using conditions obtained by matching the response function for the closed moment system to the exact response function q times in the $\omega \to 0$ limit and $n-q$ times in the $\omega \to \infty$ limit. Results concerning the form of this closure are easily expressed for closures written in terms of polynomial moments where the polynomials are orthogonal with respect to the underlying distribution. Numerical evidence is presented indicating that the response function for the closed moment system converges to the true response for a Gaussian distribution of frequencies. The convergence rate is faster for larger q. A closure for n moments is shown to match the linear response for background distributions that can be written as a polynomial of order $n-q$ in the frequency b multiplied by the original background distribution. Therefore, a series of closures for which the number of moments n increases and q and $n-q$ increase will exhibit linear theory that simultaneously converges for a wide class of frequency distributions.

Transformations are presented to show that the stochastic oscillator problem is equivalent to the linearized Vlasov equation, so results concerning convergence of the response function carry over to Landau-fluid closures. There is a transformation from fluid moments to Hermite polynomial moments, which in turn are equivalent to coefficients in the Hermite series traditionally used in simulations of the Vlasov equation (Armstrong et al. 1970).

A simple nonlinear problem in one dimension, the electron plasma echo, is used to test the higher-order behavior of moment systems with closure. The number of moments retained turns out to be the primary factor in determining how well the second-order response is modeled. The precise form of the linear closure seems to be relatively unimportant for higher-order effects.

1.2 Subgrid-Scale Closures in Two Dimensions

A simple one-field two-dimensional (2D) drift-wave model is used to study the transfer of fluctuation energy to unresolved scales. This model equation contains the so-called "$i\delta$" term (Terry and Horton 1982; Liang et al. 1993), which produces linear drive, and a simple model for Landau damping, which provides the dominant source of dissipation. While this model cannot accurately reproduce the linear physics of ITG modes captured by more comprehensive fluid models, it does share the basic $\mathbf{E}\times\mathbf{B}$ advection nonlinearity responsible for the transfer of fluctuation energy to small scales. The advantages of studying the simplified 2D model are that simulations can be performed at high enough resolution to ensure convergence, and that a large number of simulations can be performed to parameterize the results.

Numerical simulations of drift-wave turbulence can be very expensive computationally. Typically one would like to obtain the saturated turbulent state for a large number of choices of parameters in order to study the scalings of macroscopic quantities such as the heat or particle flux. Performing the simulations at reduced spatial resolution saves a substantial amount of computation time. At sufficiently low resolutions, however, the absence of nonlinear coupling to smaller scales mode causes the simulations to fail to saturate or to yield wildly inaccurate results. It is possible to recover accurate results at low resolutions, through the use of simple dissipative terms that model the fluctuation energy transfer to smaller scales. A novel parameterized hyperviscosity proposed by Smith and Hammett (1997) is presented and applied to the simplified 2D model considered here.

The transfer of energy to small scales was calculated in direct numerical simulations for a wide range of parameters and cutoff wave numbers. The resulting subgrid damping rate was well parameterized by a hyperviscous dissipation term with magnitude that scaled with the mean rate of shear in the flow and exponent that scaled with the cutoff wavenumber. Traditionally hyperviscosities are applied to simulations with a fixed exponent that must be arbitrarily chosen. Expressing the exponent as a function of the simulation parameters eliminates this ambiguity.

The parameterization can be used as a nonlinear damping term in numerical simulations of drift waves. (The parameterization assigns a linear damping rate to each Fourier mode, but the magnitude of the damping rate is a function of the mean flow, so the damping is nonlinear.) Tests of the parameterization demonstrate that it performs better than using no additional damping term or than using a standard hyperviscosity.

The parameterized hyperviscosity contains no reference to the linear physics of the simple 2D drift-wave model studied here. The damping rate is a function of the large scale $\mathbf{E}\times\mathbf{B}$ flow only. One would therefore expect this nonlinear damping term to be applicable to more general fluid models that contain more accurate models of the linear physics.

1.3 Extension to Three Dimensions

Heuristic arguments are given to extend this hyperviscosity model to 3D drift-wave turbulence where eddies are highly elongated along the field line. Due to the computational cost of 3D turbulence simulations, a comprehensive study of the performance of this hyperviscosity has yet to be performed. Some preliminary results are presented, however, that demonstrate that this generalized 3D hyperviscosity is capable of reducing the resolution requirements for 3D gyrofluid turbulence simulations.