The Ion Temperature Gradient (ITG) Instability

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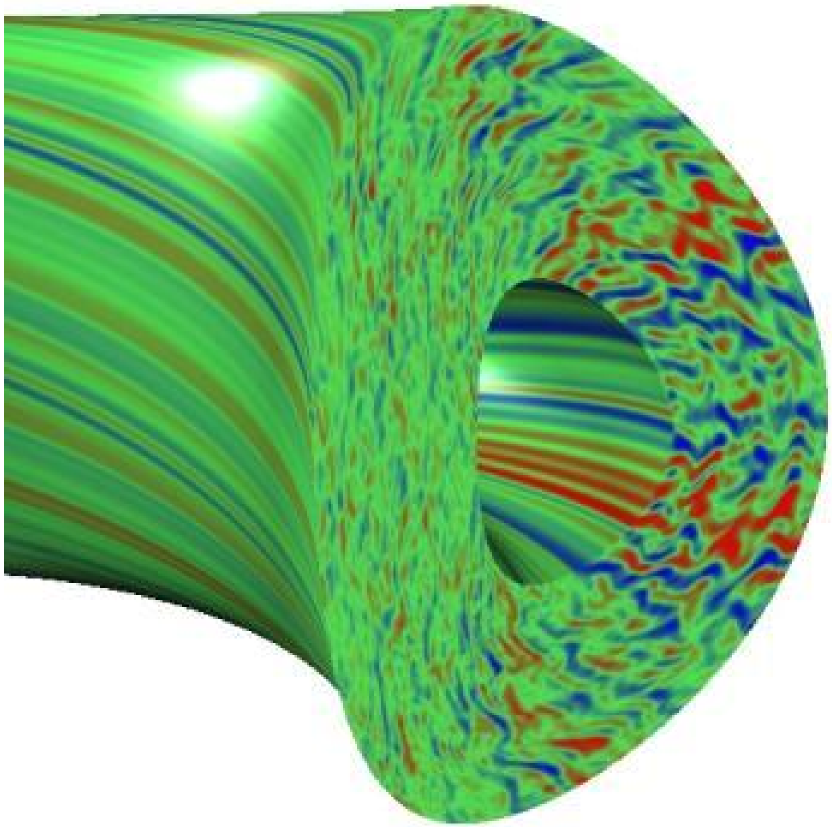
CMPD/CMSO Winter School, UCLA, 1/09/2007

1. Intuitive picture of the ITG instability -- based on analogy with Inverted pendulum / Rayleigh-Taylor instability

2. Rigorous derivation of ITG growth rate & threshold (in a simple limit) starting from the Gyrokinetic Eq.
(with sign errors in original lecture fixed.)



Candy, Waltz (General Atomics)

Acknowledgments:

Center for Multiscale Plasma Dynamics

& Plasma Microturbulence Project

(General Atomics, U. Maryland, LLNL, PPPL,
U. Colorado, UCLA, U. Texas)

DOE Scientific Discovery Through
Advanced Computing

http://fusion.gat.com/theory/pmp

J. Candy, R. Waltz (General Atomics)

W. Dorland (Maryland) W. Nevins (LLNL)

R. Nazikian, D. Meade, E. Synakowski (PPPL)

J. Ongena (JET)

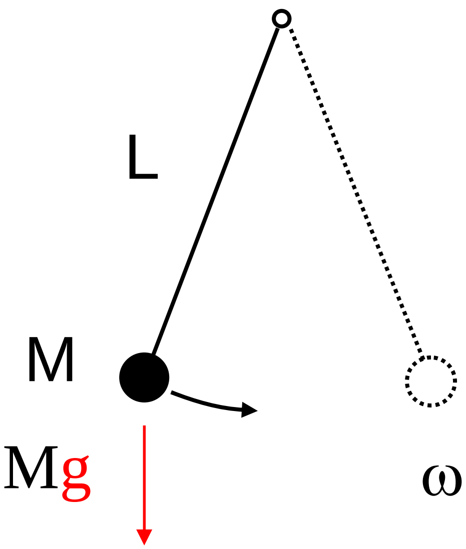
The Plasma Microturbulence Project

- A DOE, Office of Fusion Energy Sciences, SciDAC (Scientific Discovery Through Advanced Computing) project (~2001-2004)
- devoted to studying plasma microturbulence through direct numerical sumulation
- National Team (& four codes):
  - GA (Waltz, Candy)
  - U. MD (Dorland)
  - U. CO (Parker, Chen)
  - UCLA (Lebeouf, Decyk)
  - LLNL (Nevins P.I., Cohen, Dimits)
  - PPPL (Lee, Lewandowski, Ethier, Rewoldt, Hammett, …)
  - UCI (Lin)
- They've done all the hard work …

Part 1: Intuitive picture of the ITG instability

-- based on analogy with Inverted pendulum / Rayleigh-Taylor instability

Stable Pendulum

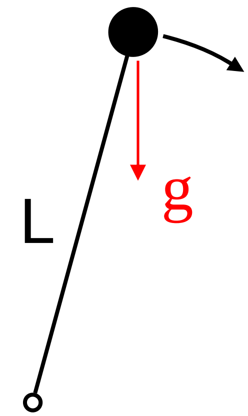


$F=Mg$

$\omega=(g/L)^{1/2}$

Unstable Inverted Pendulum

(rigid rod)

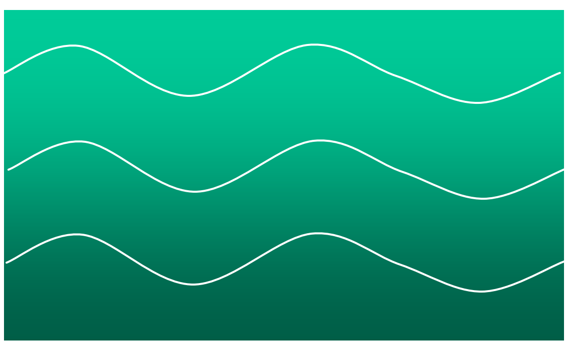


$\omega= (-g/|L|)^{1/2} = i(g/|L|)^{1/2} = i\gamma$

Instability

Density-stratified Fluid

$\rho=\exp(-y/L)$

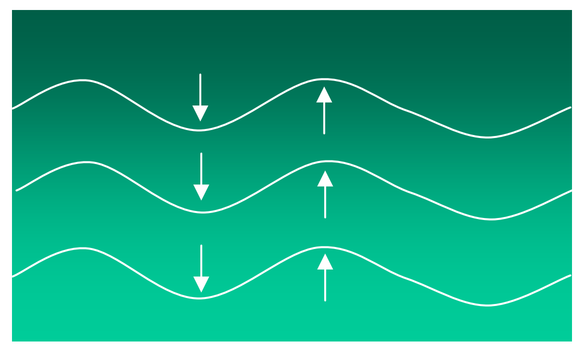


stable $\omega=(g/L)^{1/2}$

Inverted-density fluid

$\Rightarrow$Rayleigh-Taylor Instability

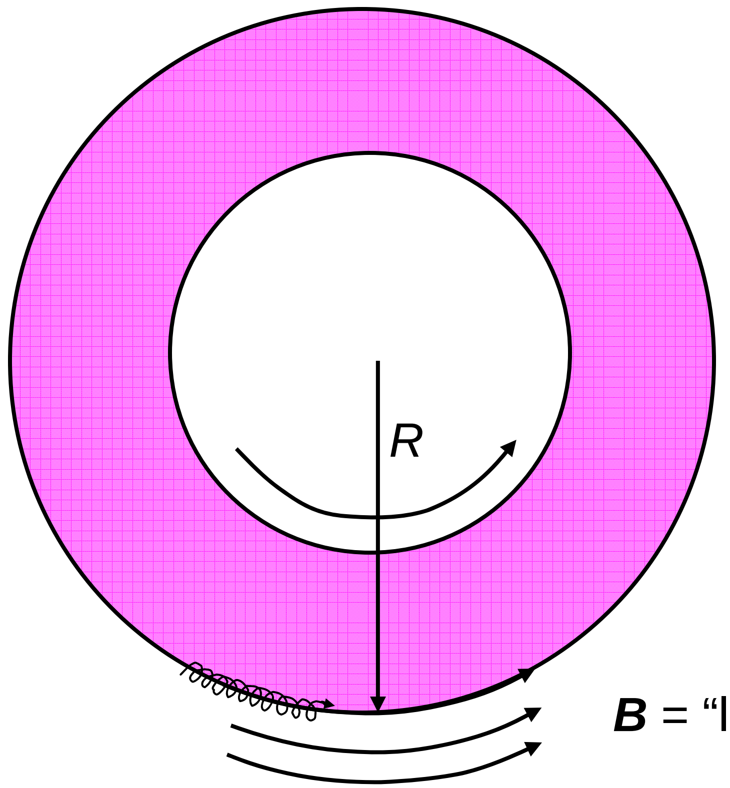
$\rho=\exp(y/L)$



Max growth rate $\gamma=(g/L)^{1/2}$

"Bad Curvature" instability in plasmas ≈ Inverted Pendulum / Rayleigh-Taylor Instability

Top view of toroidal plasma:



plasma = heavy fluid

B = "light fluid"

$g_{eff} = \frac{v^2}{R}$ centrifugal force

Growth rate:

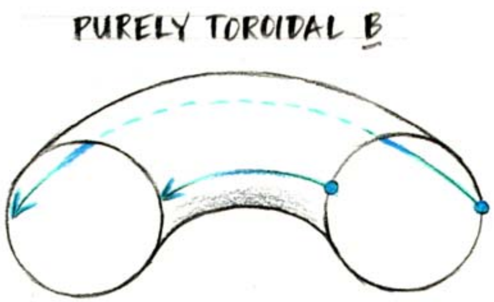
$$\gamma = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\mathbf{v}_t^2}{RL}} = \frac{\mathbf{v}_t}{\sqrt{RL}}$$

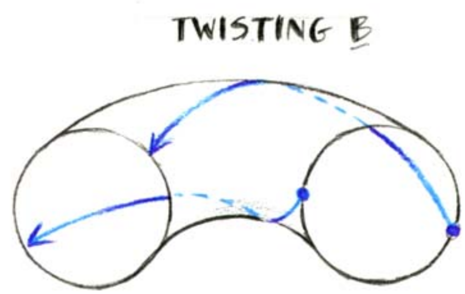
Similar instability mechanism in MHD & drift/microinstabilities

1/L = ∇p/p in MHD,
∝ combination of ∇n & ∇T in microinstabilities.

The Secret for Stabilizing Bad-Curvature Instabilities

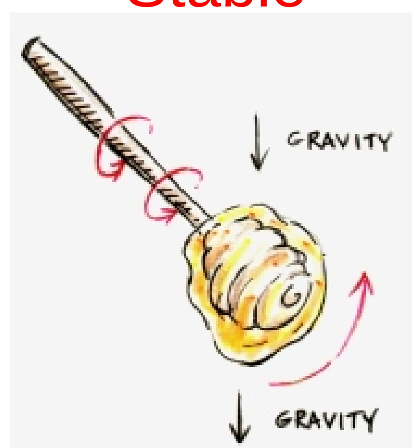
Twist in ***B*** carries plasma from bad curvature region to good curvature region:





Unstable

Stable



Similar to how twirling a honey dipper can prevent honey from dripping.

Twist in B stabilizes unless

$$\boxed{\begin{array}{c}\text{growth rate} \\ \text{in bad-curvature} \\ \text{region}\end{array} > \begin{array}{c}\text{propagation from bad-curvature} \\ \text{to good curvature regions}\end{array}}$$

MHD works well to lowest order in plasmas, so RHS $\Rightarrow$

$$\frac{V_t}{\sqrt{RL}} > k_{\parallel} V_A \sim \frac{V_A}{qR}$$

Square:

$$\frac{V_t^2 \; q^2 R^2}{V_A^2 \; RL} > 1$$

$$\text{LHS} = \frac{\beta}{2} \frac{q^2 R}{L} = \frac{1}{2} q^2 R \left| \frac{\partial \beta}{\partial r} \right| = \frac{1}{2} \alpha_{MHD}$$

While MHD works well to lowest order in plasmas, there are next-order FLR corrections that defrost the magnetic field & allow $E_{\parallel} \neq 0$ & allow the plasma to move separately from $\underset{\sim}{B}$.

Still have sound waves that can connect good & bad curvature region. Unstable if:

$$\gamma > \text{connection rate}$$

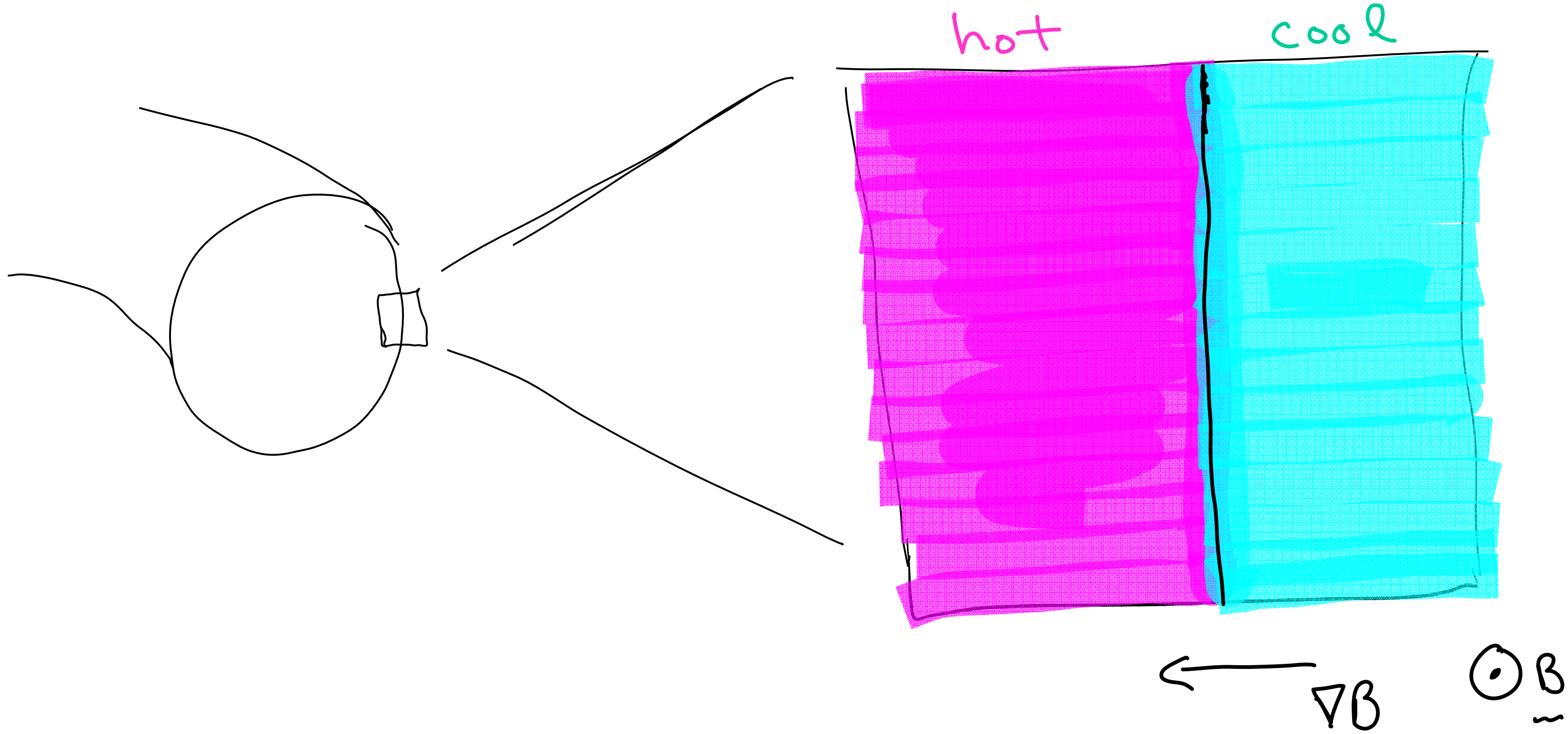
$$\frac{v_t}{\sqrt{RL}} > \frac{v_t}{qR}$$

$$\boxed{\frac{R}{L} > \frac{1}{q^2}}$$

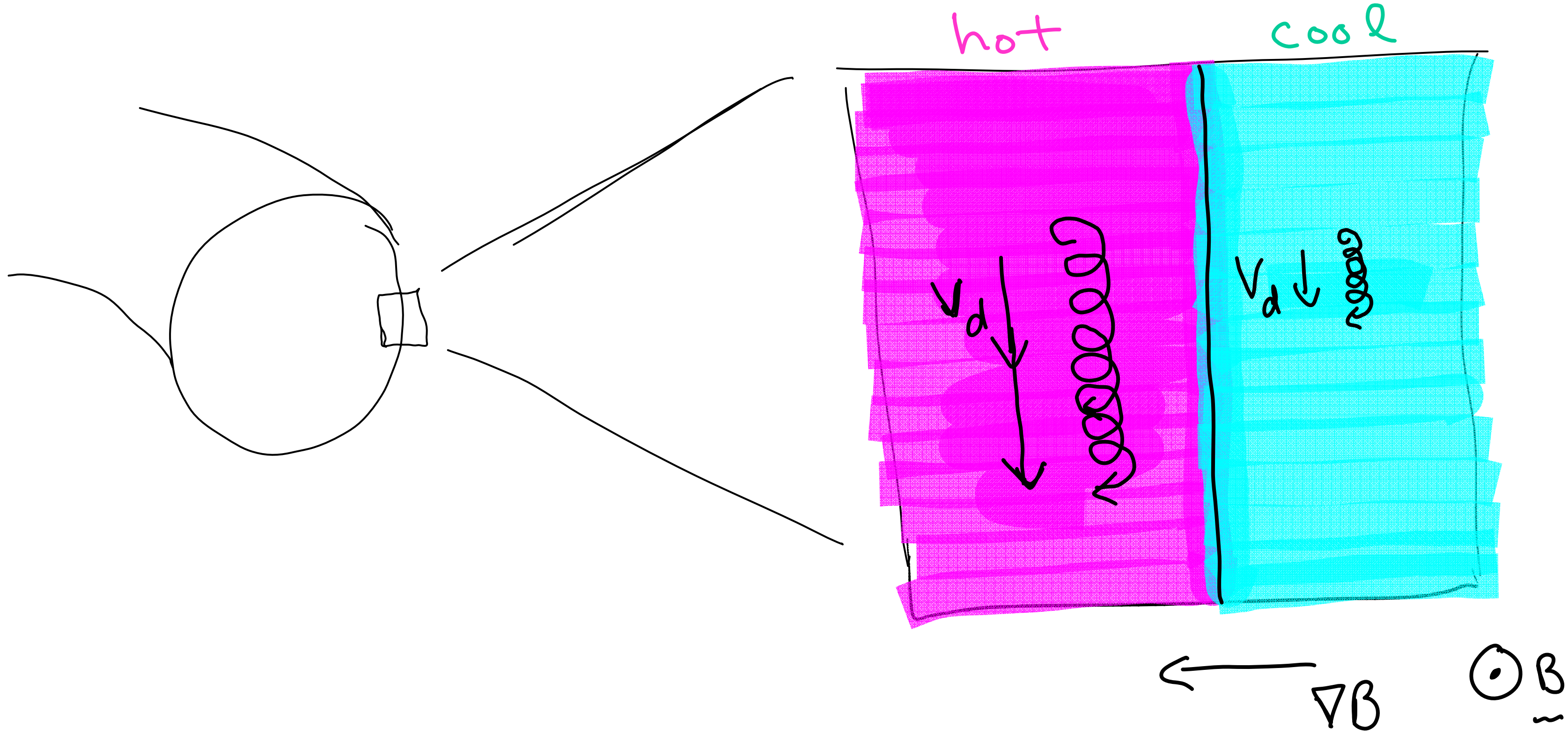
Rough, but tells us $\frac{R}{L}$ is important...

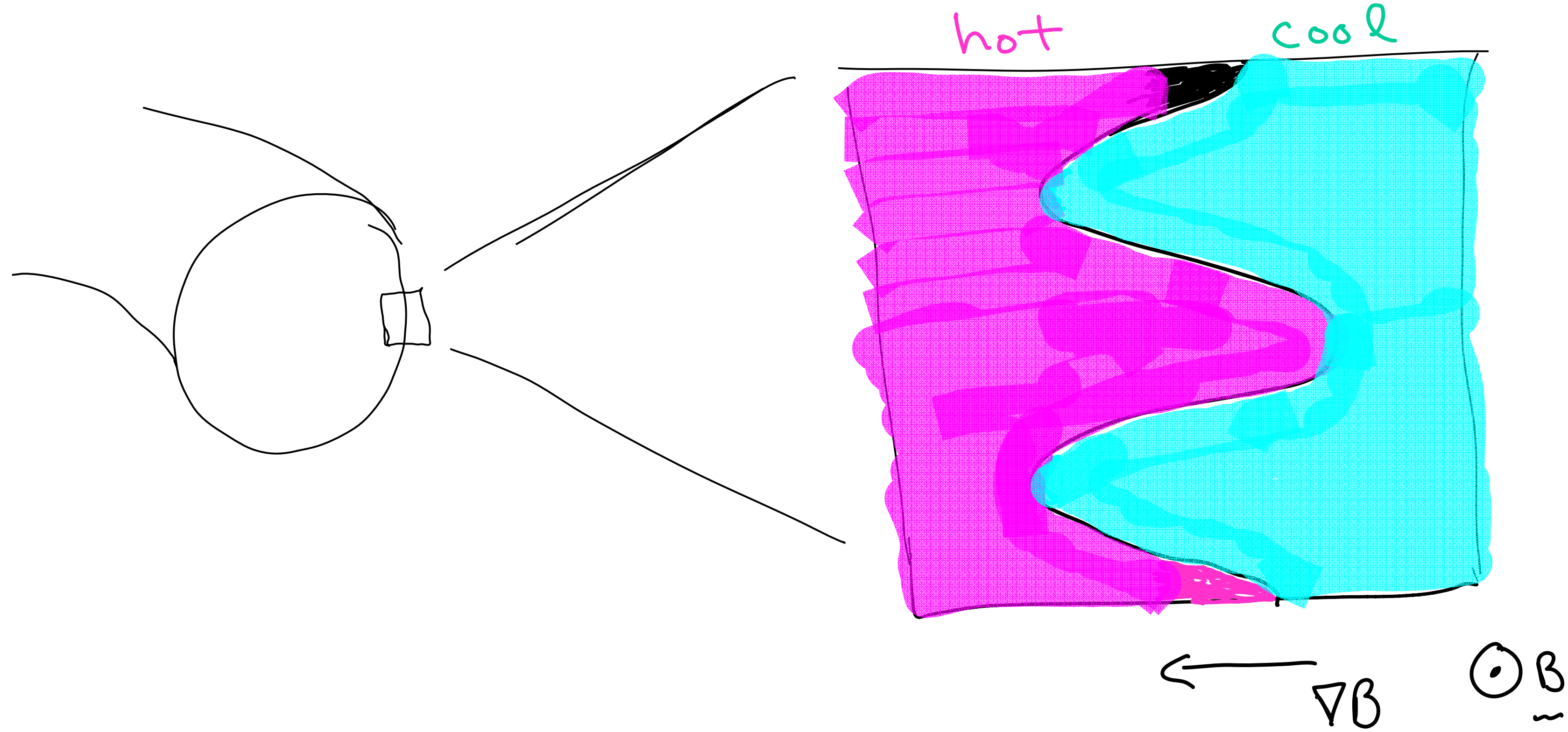
$\frac{1}{L} \sim \frac{\nabla T}{T}$

or $\sim \nabla p / p$

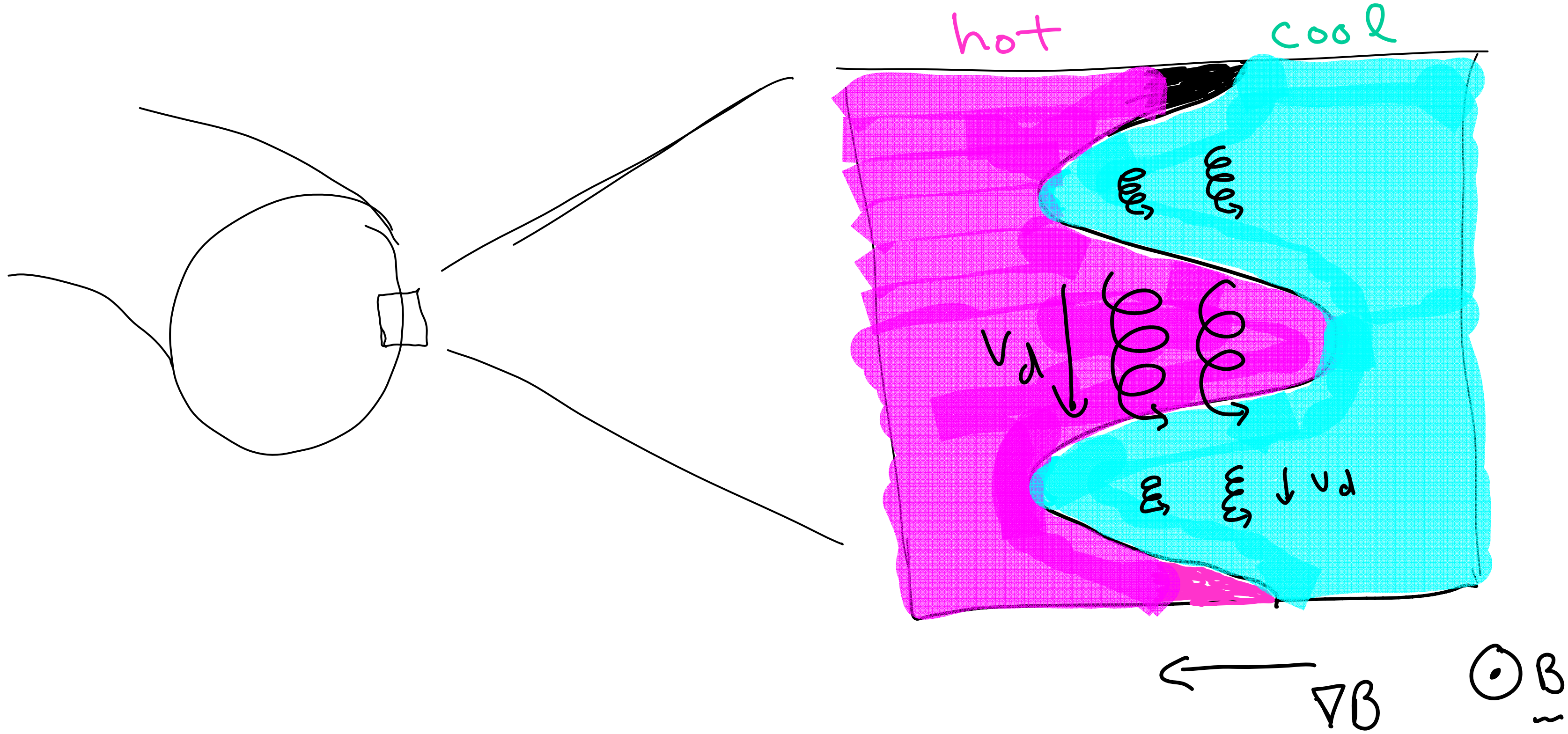


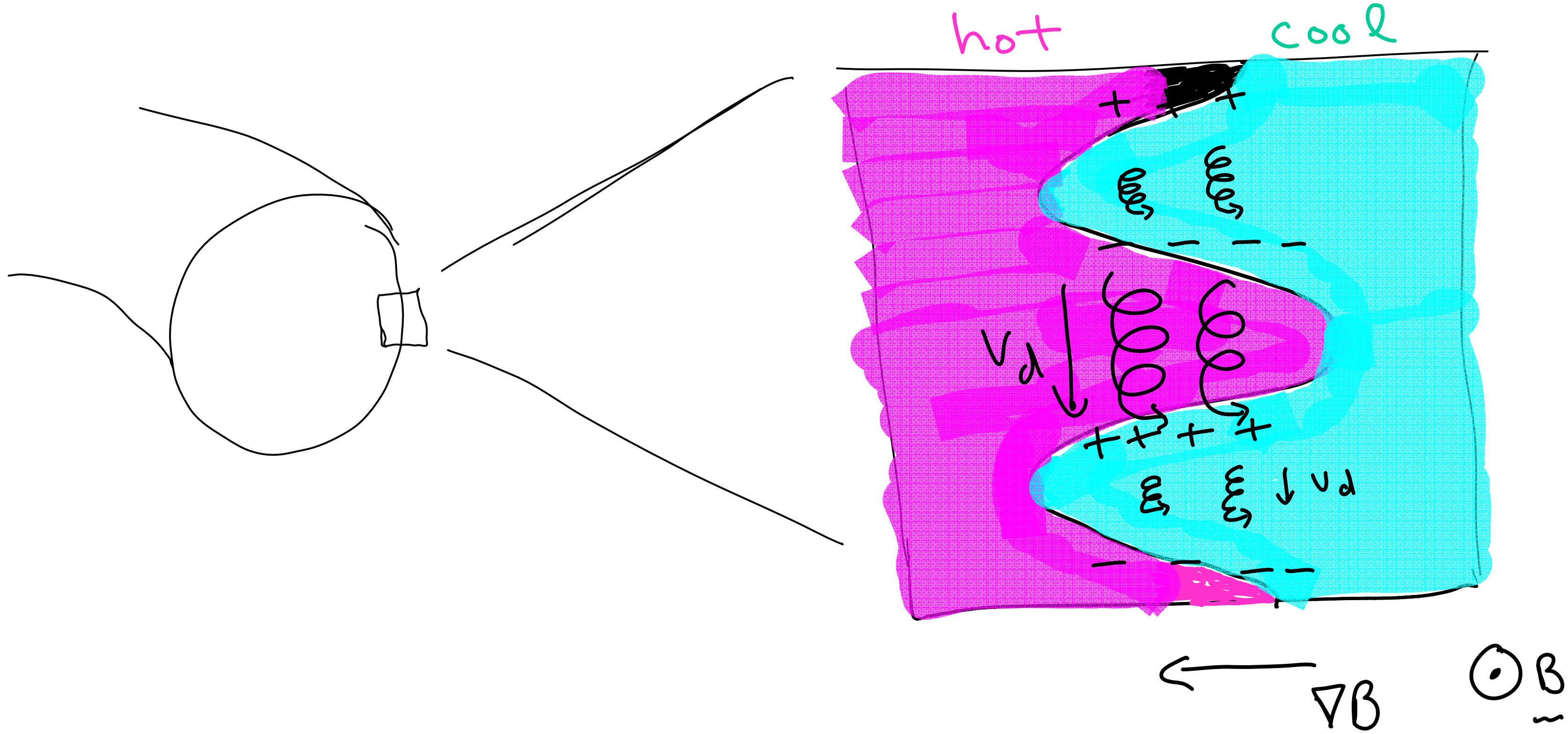
hot
cool
∇B
⊙B


hot
cool
V_d
V_d
∇B
$\odot \underset{\sim}{B}$

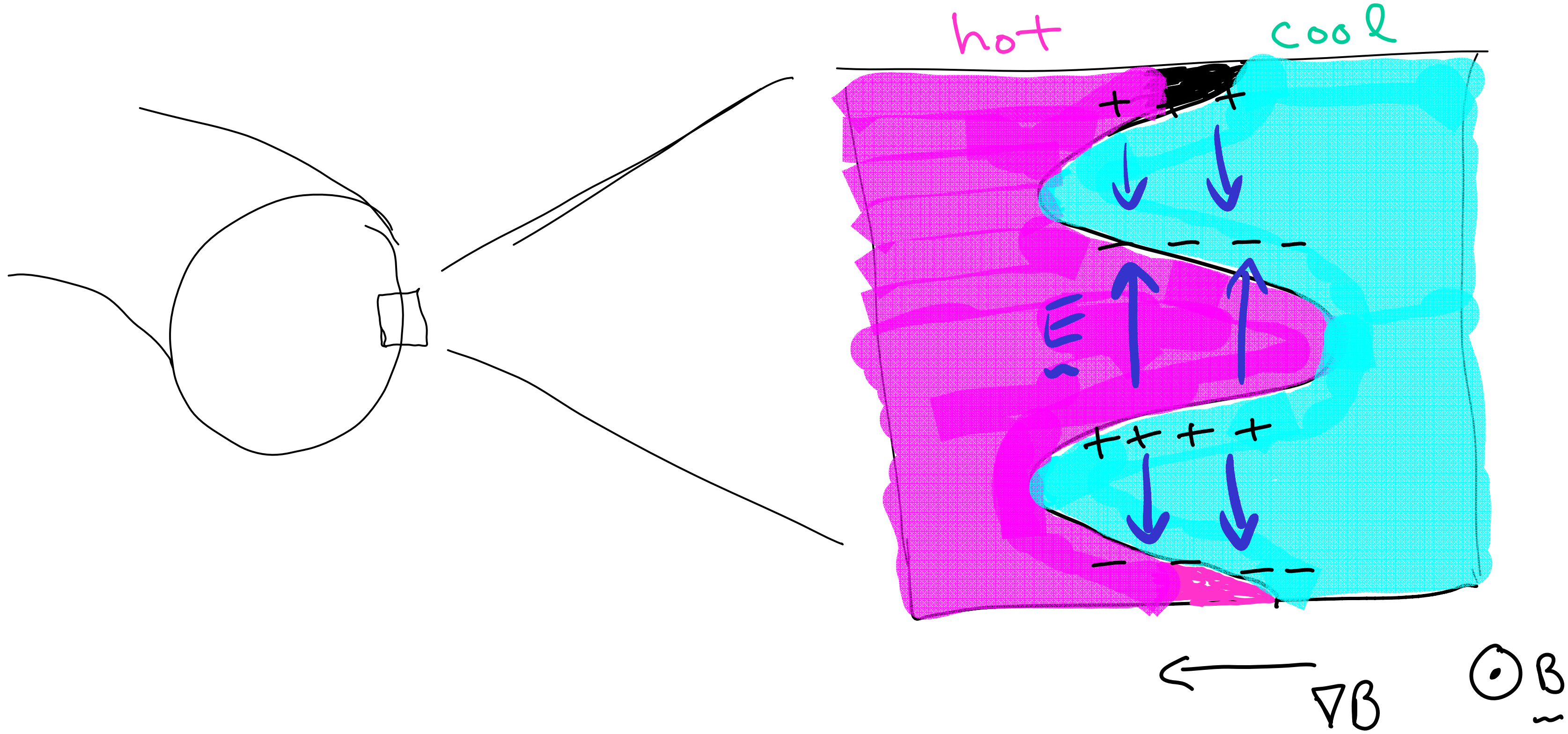


hot
cool
∇B
$\odot \underset{\sim}{B}$

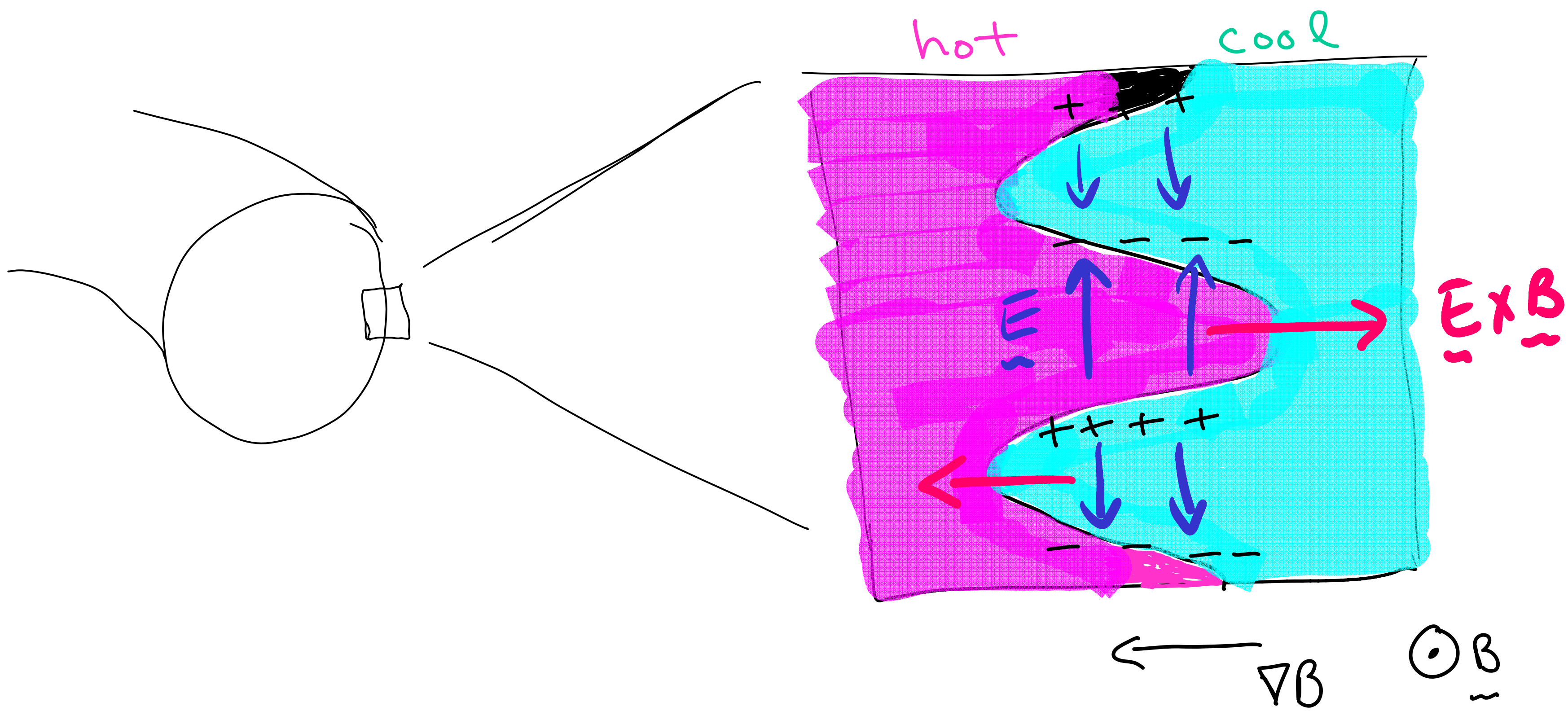

hot
cool
v_d
v_d
∇B
$\odot \underset{\sim}{B}$



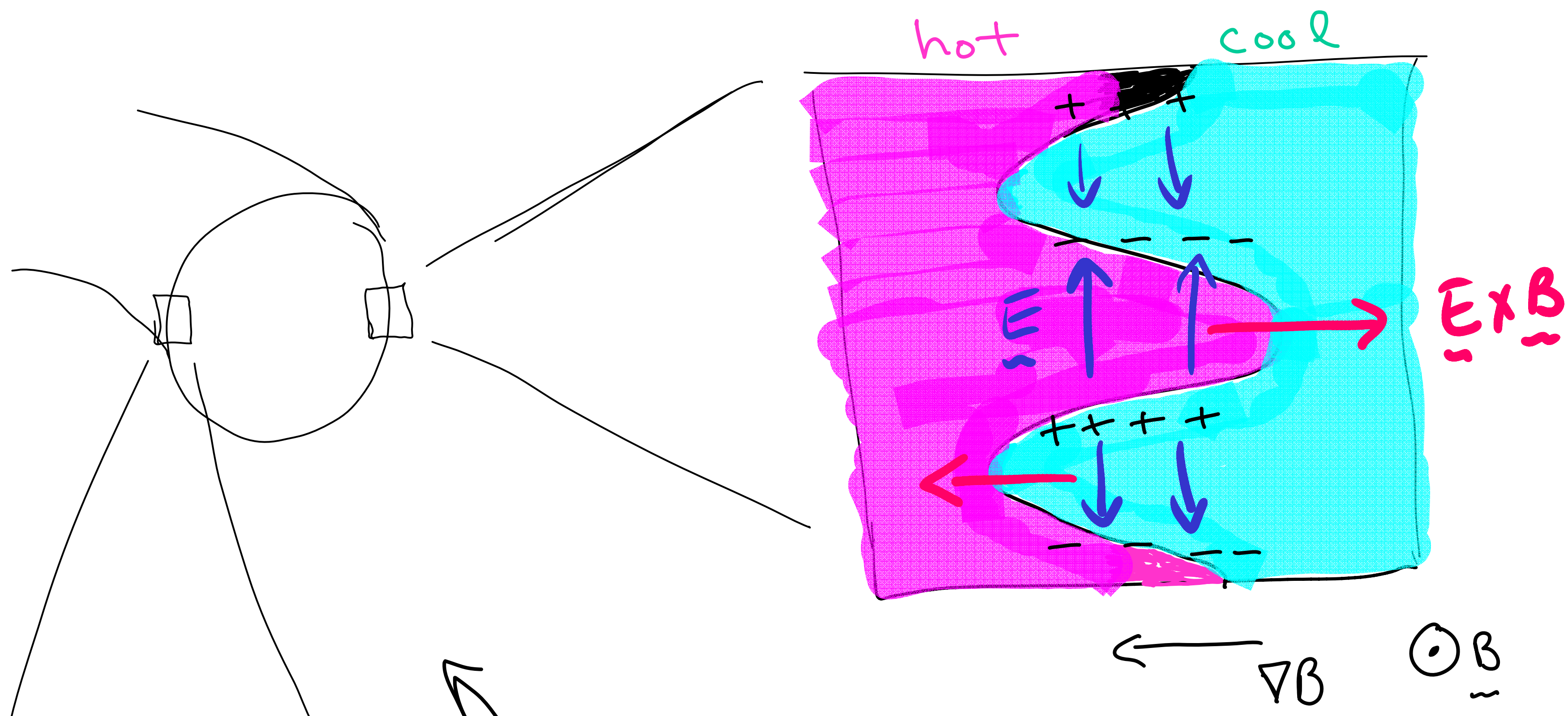
hot
cool
v_d
v_d
∇B
$\odot \underset{\sim}{B}$



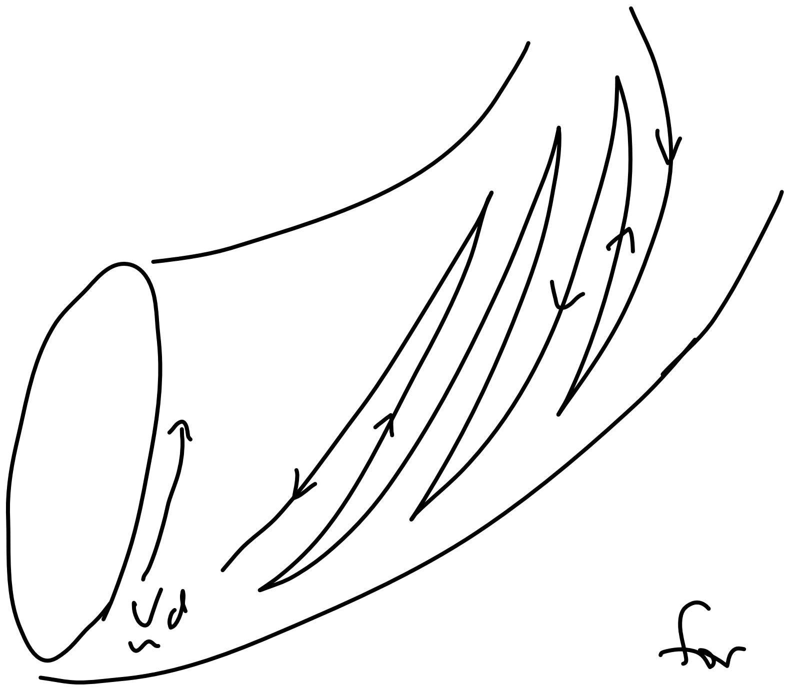
hot
cool
$\underset{\sim}{E}$
∇B
$\odot \underset{\sim}{B}$



Higher energy particles ∇B drift faster, creates charge separation & thus $\underset{\sim}{E}$ field, causes $E \times B$ flow that further accentuates perturbation. Positive feedback $\Rightarrow$ instability.



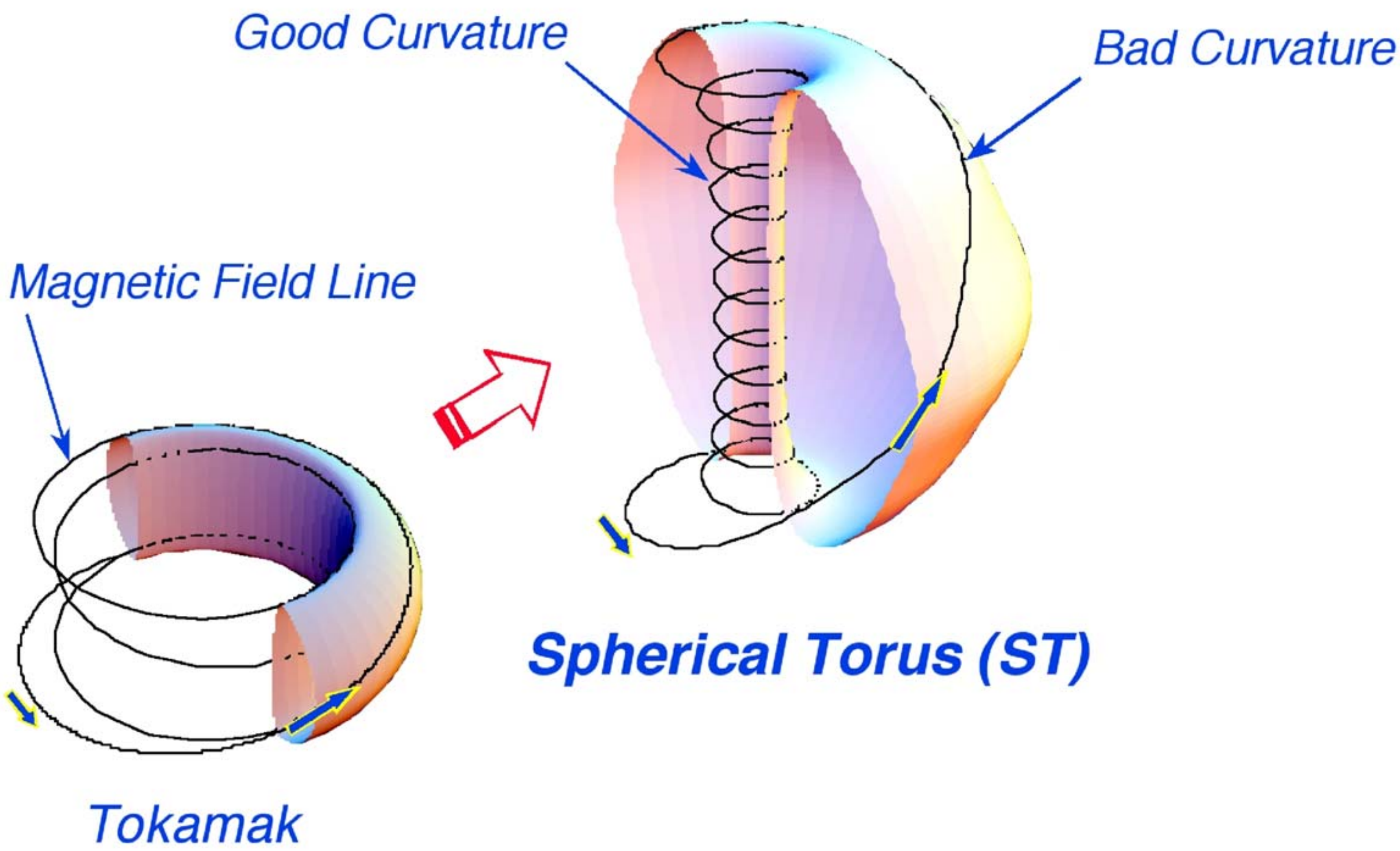
Can repeat this analysis on the good curvature side & find it is stable. (leave as exercise)



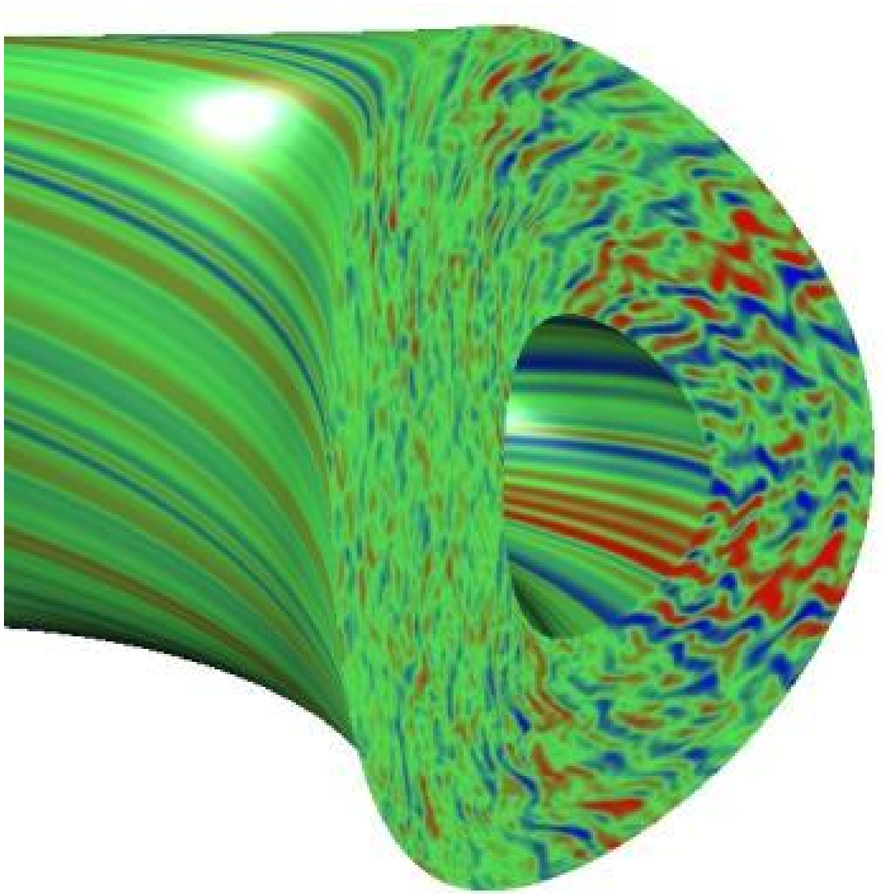
← trapped electrons precess toroidally due to ∇B drift

Similar bad curvature drive for trapped electron modes...

Spherical Torus has improved confinement and pressure limits (but less room in center for coils)



Understanding Turbulence That Affects the Performance of Fusion Device

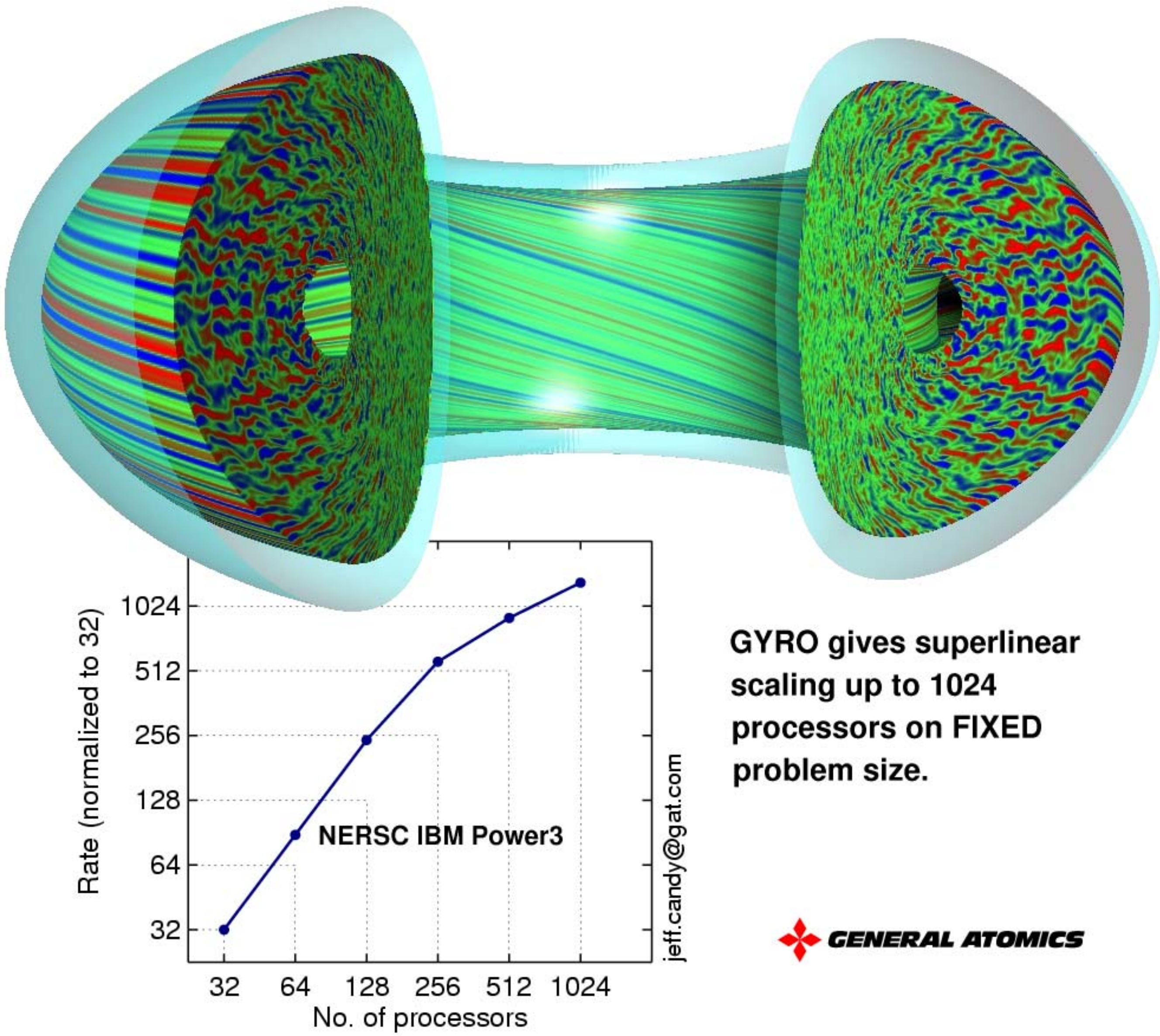


(Candy & Waltz, GA 2003)

Central temp ~ 10 keV ~ 10^8 K

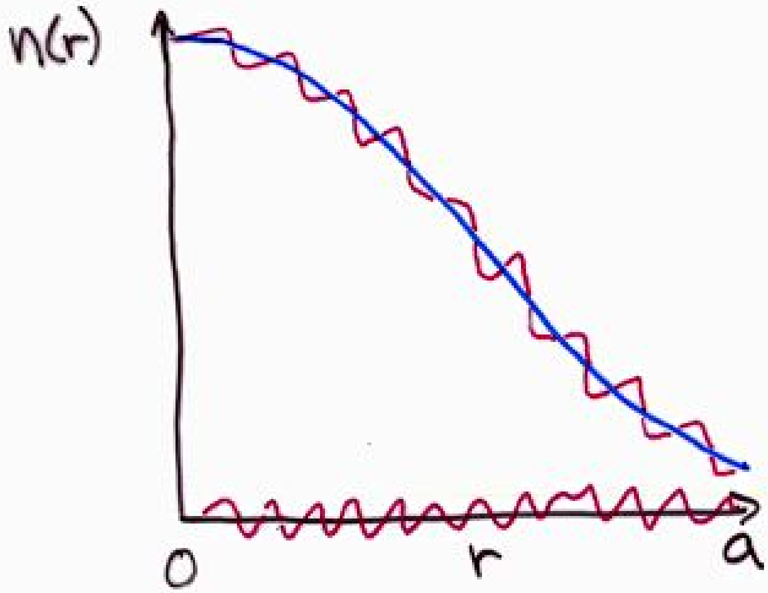
Large temperature gradient
$\rightarrow$ turbulent eddies
$\rightarrow$ cools plasmas
$\rightarrow$ determines plasma size needed for fusion ignition

Major progress in last decade:
detailed nonlinear simulations
(first 3-D fluid approximations,
now 5-D $f(\vec{x},v_{||},v_{\perp},t)$)
& detailed understanding



Comprehensive 5-D computer simulations of core plasma turbulence being developed by Plasma Microturbulence Project. Candy & Waltz (GA) movies shown: d3d.n16.2x_0.6_fly.mpg & supercyclone.mpg, from http://fusion.gat.com/comp/parallel/gyro_gallery.html (also at http://w3.pppl.gov/~hammett/refs/2004).

Microinstabilities are small-amplitude but still nonlinear



$$n = n_0(r) + \tilde{n}(\underset{\sim}{x}, t)$$

$$n_0 \gg \tilde{n}$$

but $\nabla n_0 \sim \nabla \tilde{n}$

Can locally flatten or reverse total gradient that was driving instability.

* Turbulence causes loss of plasma to the wall, but confinement still $\times 10^5$ better than without $\underset{\sim}{B}$.

If no $\underset{\sim}{B}$, loss time $\sim \frac{a}{V_t} \sim 1\ \mu\text{sec}$

with $\underset{\sim}{B}$, expts. measure $\sim 0.1 - 1.0$ sec.

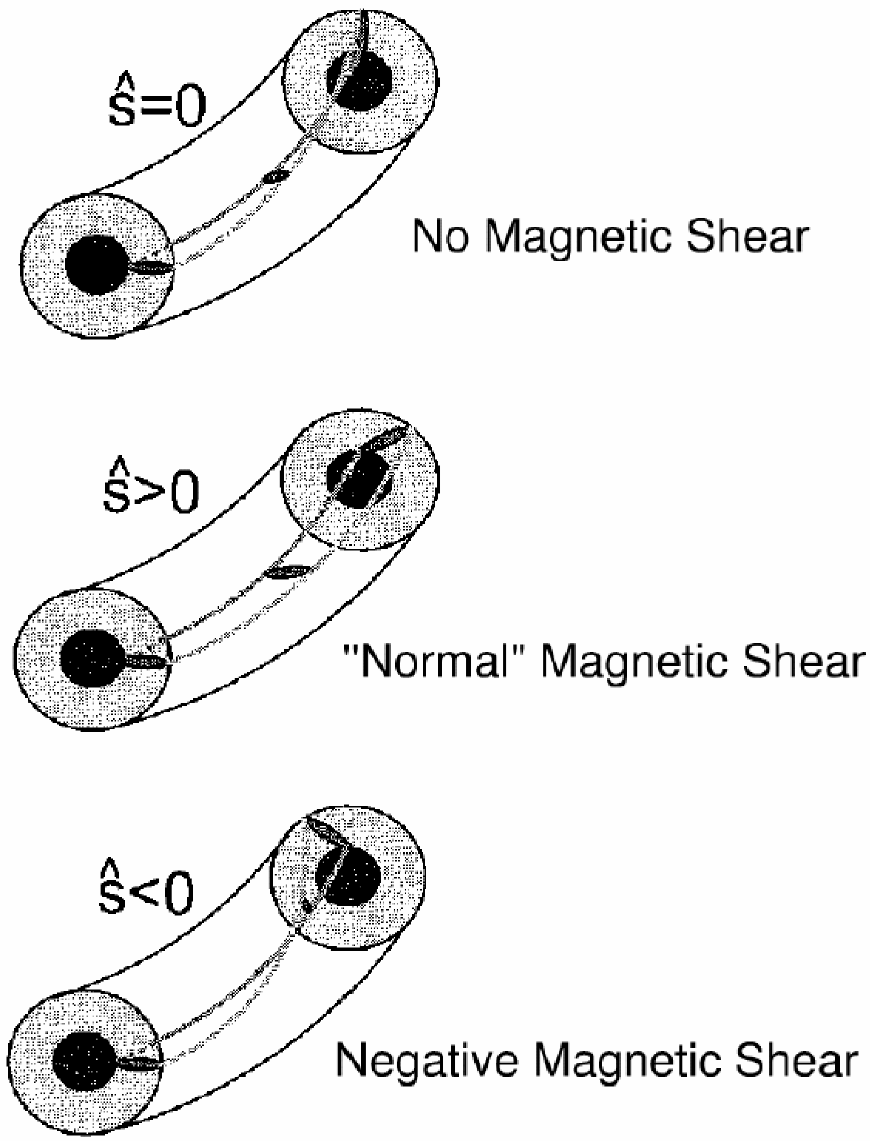
Simple picture of reducing turbulence by negative magnetic shear

Particles that produce an eddy tend to follow field lines.

Reversed magnetic shear twists eddy in a short distance to point in the ``good curvature direction".

Locally reversed magnetic shear naturally produced by squeezing magnetic fields at high plasma pressure: ``Second stability" Advanced Tokamak or Spherical Torus.

Shaping the plasma (elongation and triangularity) can also change local shear

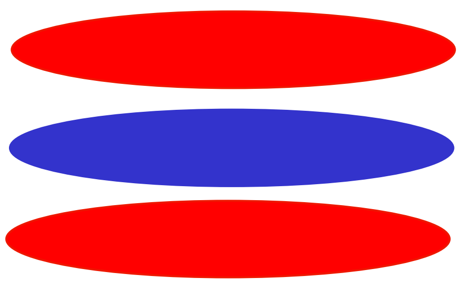


Antonsen, Drake, Guzdar et al. Phys. Plasmas 96

Kessel, Manickam, Rewoldt, Tang Phys. Rev. Lett. 94

Sheared flows can suppress or reduce turbulence

Most Dangerous Eddies:
Transport long distances
In bad curvature direction



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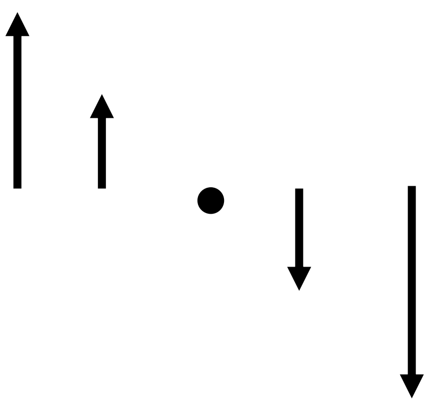
Sheared Flows



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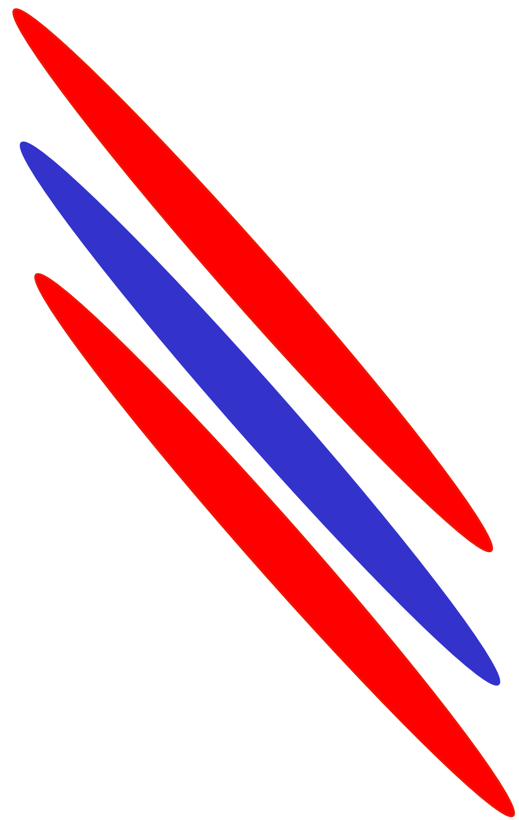
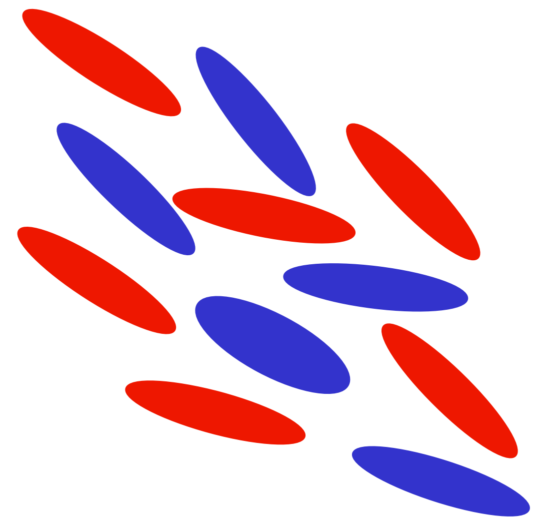
Sheared Eddies
Less effective

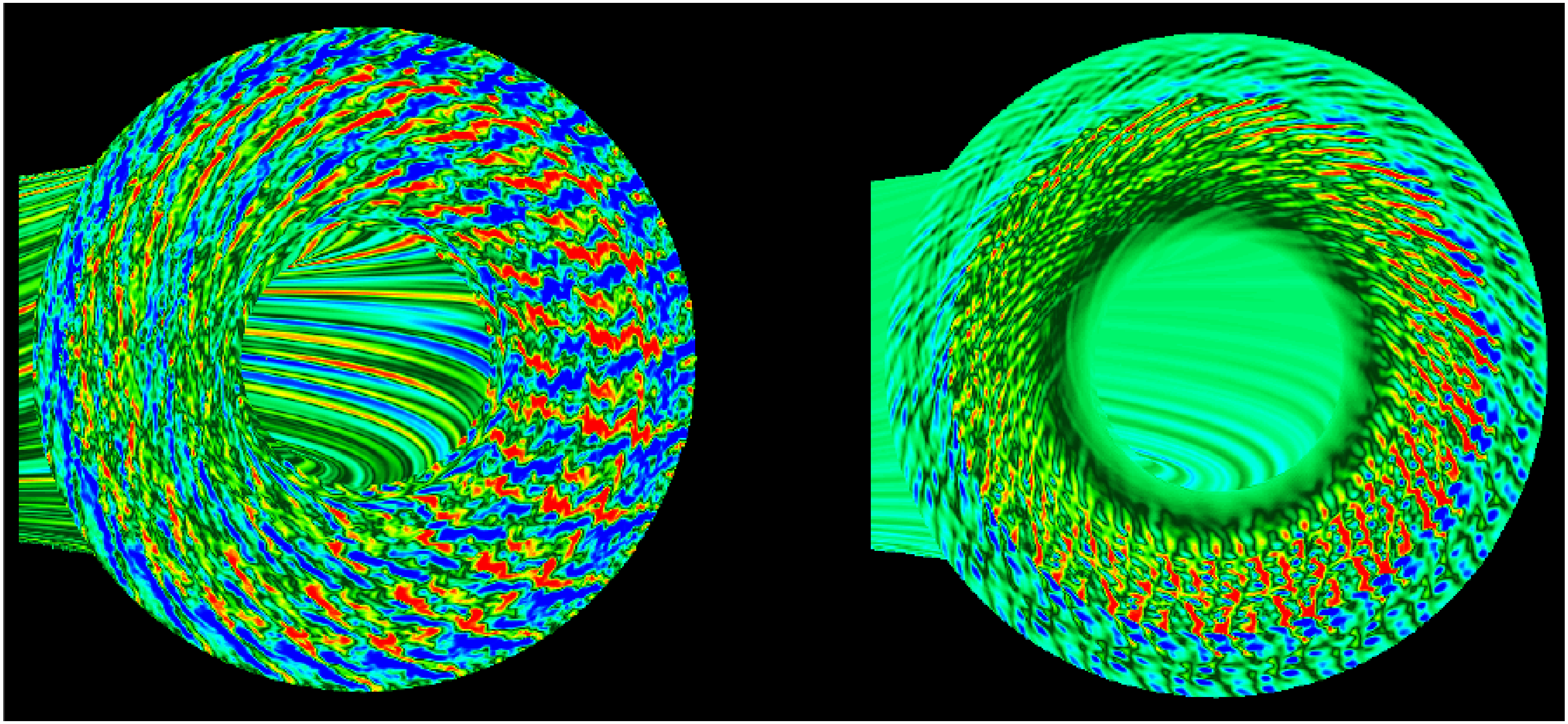


Eventually break up



Biglari, Diamond, Terry (Phys. Fluids1990),
Carreras, Waltz, Hahm, Kolmogorov, et al.

Sheared ExB Flows can regulate or completely suppress turbulence (analogous to twisting honey on a fork)

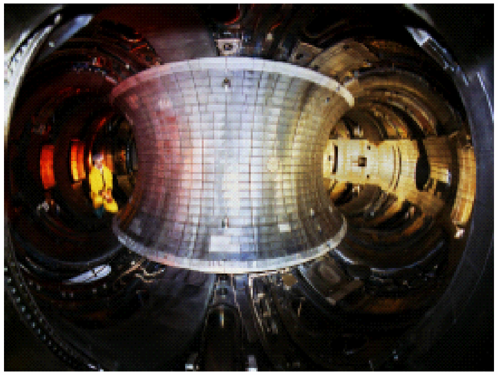


Dominant nonlinear interaction between turbulent eddies and $\pm\theta$-directed zonal flows.

Additional large scale sheared zonal flow (driven by beams, neoclassical) can completely suppress turbulence

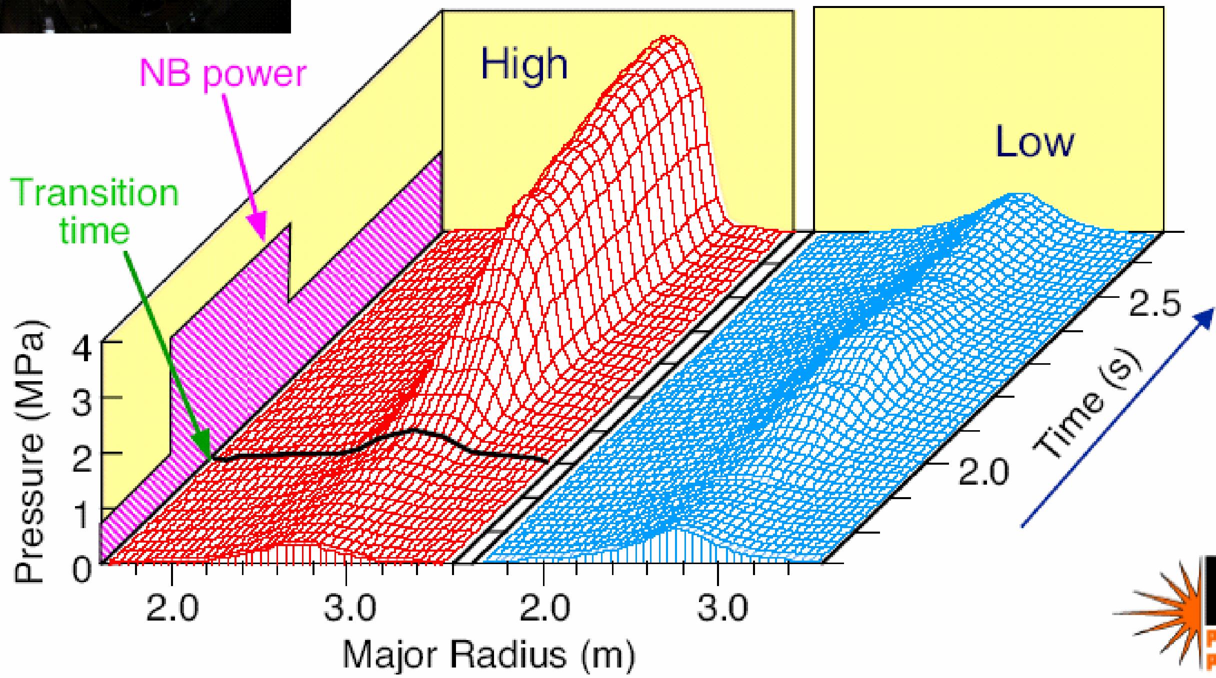
Waltz, Kerbel, Phys. Plasmas 1994 w/ Hammett, Beer, Dorland, Waltz Gyrofluid Eqs., Numerical Tokamak Project, DoE Computational Grand Challenge

Fascinating Diversity of Regimes in Fusion Plasmas. What Triggers Change? What Regulates Confinement?



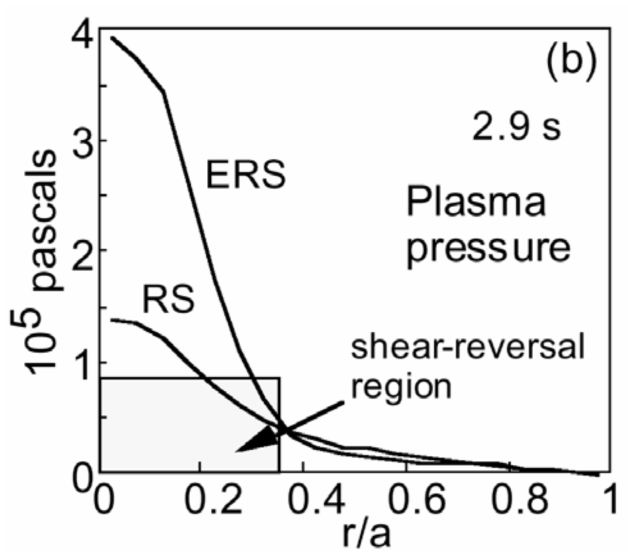
TFTR

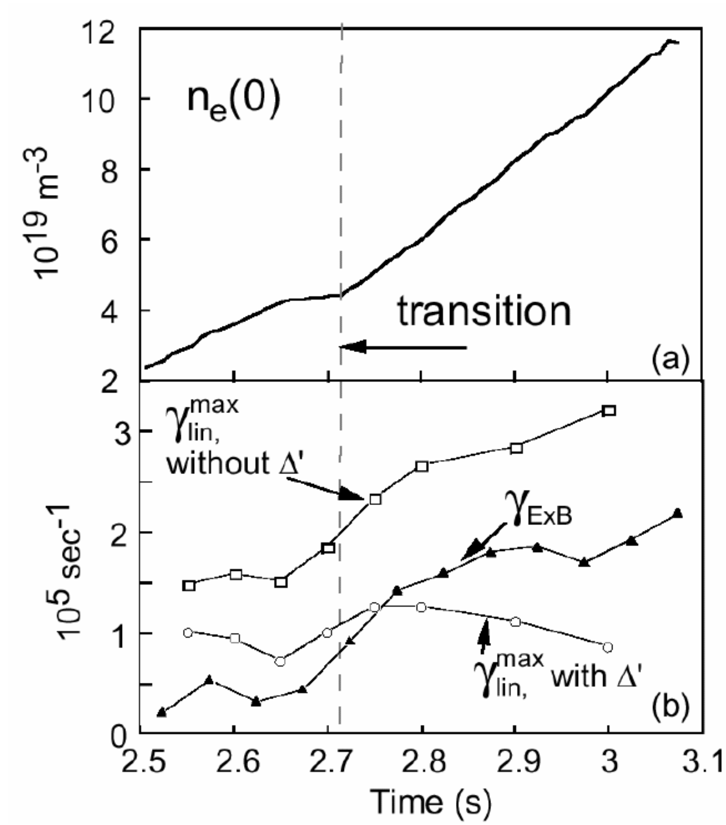
- Two regimes with very different confinement for similar initial conditions and neutral beam heating
- Access depends on plasma heating and reducing current density on axis
- Can we attribute a difference in turbulence to these two different confinement regimes?



R. Nazikian et al.

All major tokamaks show turbulence can be suppressed w/ sheared flows & negative magnetic shear / Shafranov shift





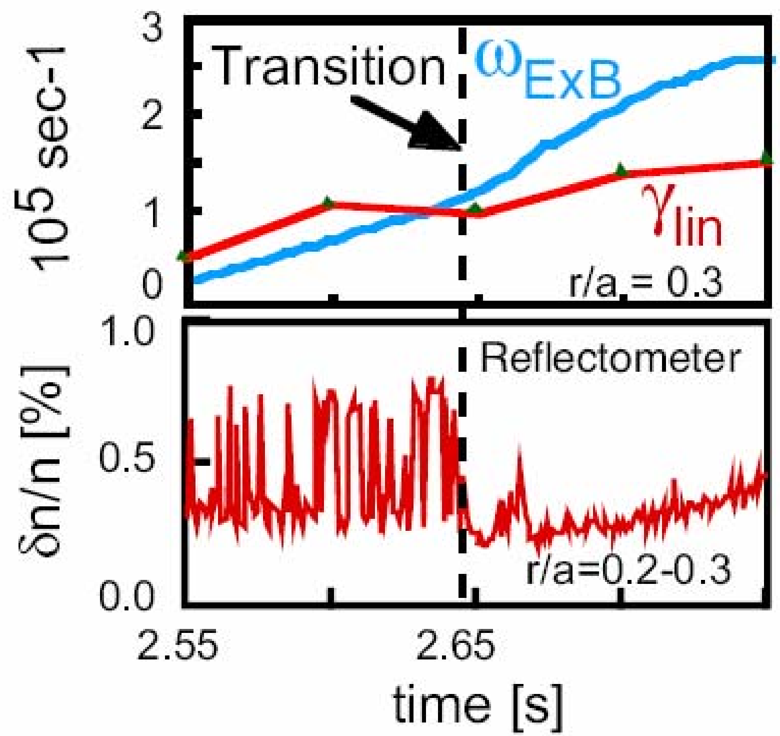
Synakowski, Batha, Beer, et.al. Phys. Plasmas 1997

Internal transport barrier forms when the flow shearing rate $dv_\theta/dr >\sim$ the max linear growth rate γ_{lin}^{max} of the instabilities that usually drive the turbulence.

Shafranov shift Δ' effects (self-induced negative magnetic shear at high plasma pressure) also help reduce the linear growth rate.

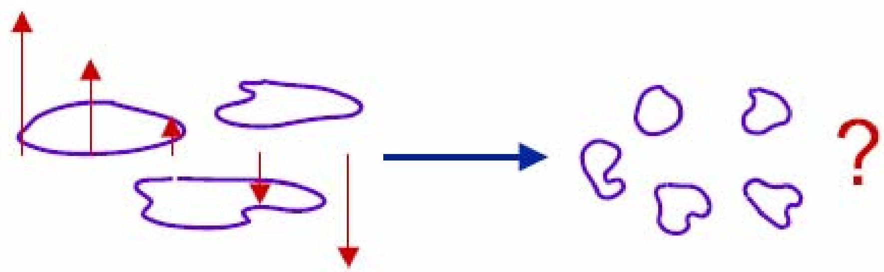
Advanced Tokamak goal: Plasma pressure ~ x 2, $P_{fusion} \propto$ pressure^2 ~ x 4

Transition to Enhanced Confinement Regime is Correlated with Suppression of Core Fluctuations in TFTR



- Theory predicts fluctuation suppression when rate of shearing (ω_{ExB}) exceeds rate of growth (γ_{lin})

- Outstanding issue:
  Is suppression accompanied by radial decorrelation?

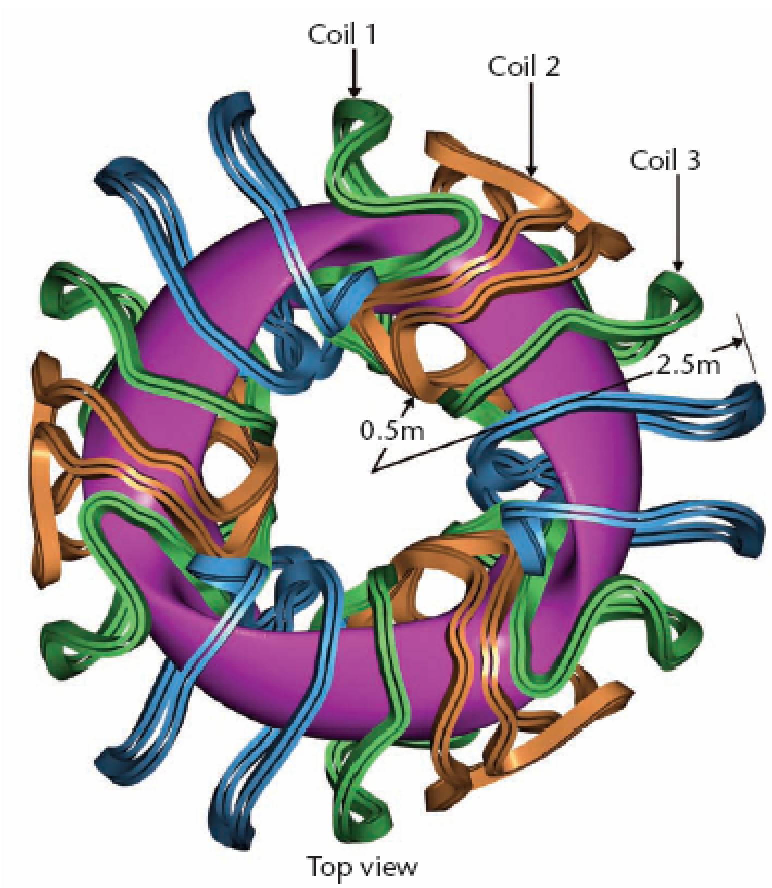


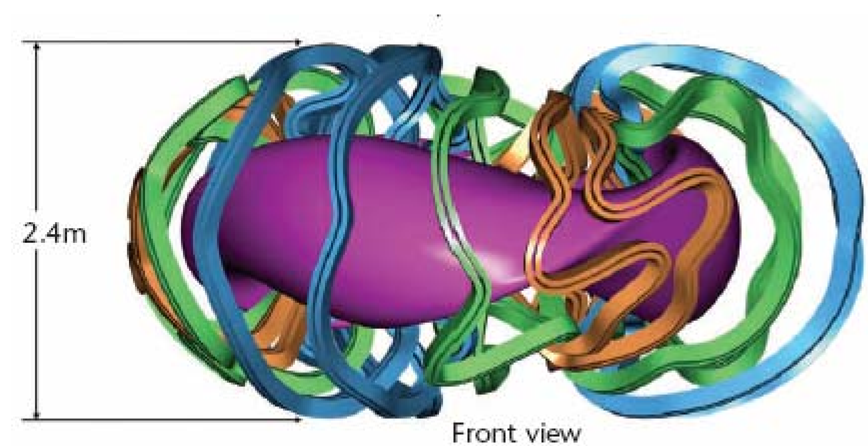
- Similar suppression observed on JET (X-mode reflectometer) and DIII-D (FIR Scattering)

Hahm, Burrell, Phys. Plas. 1995, E. Mazzucato et al., PRL 1996.

Improved Stellarators Being Built

- Magnetic field twist and shear provided by external coils, not plasma currents. More stable?
- Computer optimized designs much better than 1950-60 slide rules?
- Quasi-toroidal symmetry allows plasma to spin toroidally: shear flow stabilization?





Part 2: Rigorous derivation of ITG growth rate & threshold (in a simple limit) starting from the Gyrokinetic Eq.

Our starting point will be the electrostatic Gyrokinetic Eq. written in a Drift-Kinetic-like form for the full, gyro-averaged, guiding center density $\bar{f}(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d) \cdot \nabla \bar{f} + \left(\frac{q}{m} E_\parallel - \mu \nabla_\parallel B + v_\parallel (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) \cdot \mathbf{v}_E\right) \frac{\partial \bar{f}}{\partial v_\parallel} = 0}$$

$$\underset{\sim}{V}_E \equiv -\frac{c}{B} \nabla \langle \Phi \rangle \times \hat{b} \qquad E_\parallel = -\hat{b} \cdot \nabla \langle \Phi \rangle$$

$$\mathbf{v}_d = \frac{v_\parallel^2}{\Omega} \hat{\mathbf{b}} \times (\hat{\mathbf{b}} \cdot \nabla \hat{\mathbf{b}}) + \frac{\mu}{\Omega} \hat{\mathbf{b}} \times \nabla B \simeq \frac{v_\parallel^2 + v_\perp^2/2}{\Omega B^2} \underset{\sim}{B} \times \nabla B$$

$$\mu = \frac{1}{2} \frac{v_\perp^2}{B}$$

Gyro-averaged

$$\bar{f}(\underset{\sim}{R}, v_\parallel, \mu, t) = \langle f(\underset{\sim}{R} + \underset{\sim}{\rho}(\theta), v_\parallel, \mu, \theta, t) \rangle_\theta$$

details:

* this is not the original Drift-Kinetic Eq. of Chew, Goldberger, & Low (1956), which was for the strong E-field "MHD ordering" (see Kulsrud, Handbook of Plasma Physics, 1983)

$$V_E \sim V_t \gg V_d \sim \frac{v_\perp^2}{\Omega R} \sim v_t \frac{\rho}{R}$$

* closer to the form of the Drift-Kinetic Eq. used in neoclassical theory, where $\underset{\sim}{v}_E \sim \underset{\sim}{v}_d$ ("weak E-field")

even though $\frac{v_E}{v_t} \sim \frac{\rho}{R} \sim \varepsilon$, $\frac{v_E \cdot \nabla}{v_\parallel \hat{b} \cdot \nabla} \sim \frac{v_t \frac{\rho}{R} k_\perp}{v_t k_\parallel} \sim \frac{k_\perp \rho}{k_\parallel R}$

$$\sim 1$$

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_{\parallel}, \mu, t)$:

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\parallel}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_{\parallel} - \mu\nabla_{\parallel}B + v_{\parallel}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial\bar{f}}{\partial v_{\parallel}} = 0}$$

In the uniform $\underset{\sim}{B}$ slab limit, this is = to Krommes GK Eq. 4 (~ p. 11-13)

Homework: show that expanding the Boltzmann factor in Cowley's Eq. 37, & gyroaveraging to get $\boxed{\bar{f} = F_0 - \frac{q\langle\Phi\rangle}{T_0}F_0 + h}$ & subst. into above GK Eq. gives exactly Cowley's (Frieman-Chen) form of the GK Eq. (Cowley Eq. 40) for $\frac{\partial h}{\partial t}$ (Use uniform B slab limit for simplicity).

[& expand in consistent assumptions: $\frac{q\langle\Phi\rangle}{T} \ll 1$ but

$$F_0 \nabla_{\perp}\frac{q\langle\Phi\rangle}{T_0} \sim \nabla_{\perp}F_0 \qquad F_0\nabla_{\perp}\frac{q\langle\Phi\rangle}{T_0} \sim \nabla_{\perp}F_0$$]

Gyrokinetic Eq. for full guiding-center density $f(\underset{\sim}{R}, v_\parallel, \mu, t)$:

$$\boxed{\frac{\partial \bar f}{\partial t} + (v_\parallel \hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla \bar f + \left(\frac{q}{m}E_\parallel - \mu\nabla_\parallel B + v_\parallel(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial \bar f}{\partial v_\parallel} = 0}$$

Homework: show that substituting the gyro-average of Cowley's Eq. 37:

$$\bar f = F_0 - \frac{q\langle\Phi\rangle}{T_0}F_0 + h$$

(Straight & uniform $\underset{\sim}{B}$ limit for simplicity)

$$\frac{\partial h}{\partial t} - \frac{q}{T_0}\frac{\partial\langle\Phi\rangle}{\partial t}F_0 + v_\parallel\hat b\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla h + \underset{\sim}{v}_E\cdot\nabla\left(F_0\underbrace{\left(1 - \frac{q\langle\Phi\rangle}{T_0}\right)}_{\text{drop}}\right)$$

$$- v_\parallel\hat b\cdot\nabla\langle\Phi\rangle\frac{q}{T_0}F_0 \qquad \underbrace{-\frac{q}{m}\hat b\cdot\nabla\langle\Phi\rangle\frac{\partial h}{\partial v_\parallel}}_{\text{drop}}$$

$$-\frac{q}{m}\hat b\cdot\nabla\langle\Phi\rangle\frac{\partial F_0}{\partial v_\parallel}\underbrace{\left(1 - \frac{q\langle\Phi\rangle}{T_0}\right)}_{\text{drop}} = 0$$

These 2 terms cancel

Use $\frac{\partial F_0}{\partial v_\parallel} = -\frac{m v_\parallel}{T_0}F_0$

Homework Solution outline

$$\boxed{\frac{\partial \overline{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\overline{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial\overline{f}}{\partial v_{\|}} = 0}$$

Linearize: $\overline{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

$$\frac{\partial}{\partial t} = 0 \qquad \underset{\sim}{E} = 0$$

$$(v_{\|}\hat{b} + \underset{\sim}{v_d})\cdot\nabla F_0 - \mu\nabla_{\|}B\,\frac{\partial F_0}{\partial v_{\|}} = 0$$

Basically says F_0 = const. along trajectories of banana orbits or passing orbits in a tokamak.

General Equilibrium solution could be an arbitrary function of the constants of the motion (E, μ, P_ϕ) where

$$E = \tfrac{1}{2}mv_{\|}^2 + \mu B.$$

& P_ϕ = canonical angular momentum

But if we neglect $\frac{|\underset{\sim}{v_d}|}{v_{\|}} \sim \frac{\rho}{R}$ get simpler Eq:

$$v_{\parallel} \hat{b} \cdot \nabla F_0 - \mu \left(\hat{b} \cdot \nabla B \right) \frac{\partial F_0}{\partial v_{\parallel}} = 0$$

Will consider Equilibrium of the form:

$$F_0 (\underset{\sim}{R}, v_{\parallel}, \mu) \propto \frac{n_0(\psi)}{T_0^{3/2}(\psi)} e^{-\frac{m\left(\frac{1}{2}v_{\parallel}^2 + \mu B(\underset{\sim}{x})\right)}{T(\psi)}} \propto e^{-\frac{E}{T}}$$

Exercise: Plug this in to the previous Eq. & show it is a solution.

$$\boxed{\frac{\partial \bar{f}}{\partial t} + (v_{\|}\hat{\mathbf{b}} + \mathbf{v}_E + \mathbf{v}_d)\cdot\nabla\bar{f} + \left(\frac{q}{m}E_{\|} - \mu\nabla_{\|}B + v_{\|}(\hat{\mathbf{b}}\cdot\nabla\hat{\mathbf{b}})\cdot\mathbf{v}_E\right)\frac{\partial\bar{f}}{\partial v_{\|}} = 0}$$

Linearize: $\bar{f} = F_0 + \tilde{f}$, where F_0 satisfies Equilibrium Eq.

Next order Eq:

$$\frac{\partial \tilde{f}}{\partial t} + \left(v_{\|}\hat{b} + \underset{\sim}{v}_d\right)\cdot\nabla\tilde{f} - \mu\nabla_{\|}B\frac{\partial\tilde{f}}{\partial v_{\|}} = -\underset{\sim}{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

$$\left(-i\omega + i v_{\|}k_{\|} + i\underset{\sim}{v}_d\cdot\underset{\sim}{k}_{\perp}\right)\tilde{f} = -\underset{\sim}{v}_E\cdot\nabla F_0 - \left(\frac{q}{m}E_{\|} + v_{\|}(\hat{b}\cdot\nabla\hat{b})\cdot\underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_{\|}}$$

Important Subtlety: $\bar{F}(\underset{\sim}{R}, v_\parallel, \mu, t)$ so

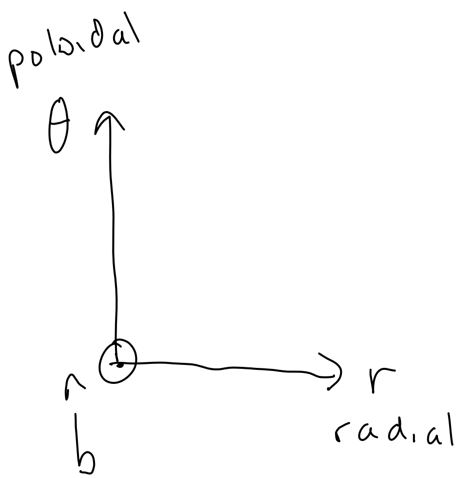
$$-\underset{\sim}{v}_E \cdot \nabla F_0 = -\underset{\sim}{v}_E \cdot \nabla\Big|_{v_\parallel, \mu, t} F_0$$

using $\quad F_0 \propto \dfrac{n_0(r)}{T_0^{3/2}(r)} e^{-\frac{\left(\frac{1}{2} m v_\parallel^2 + m\mu B(\underset{\sim}{x})\right)}{T_0(r)}}$

will give terms proportional to ∇n_0, ∇T_0, & $\mu \nabla B$

∇n_0 terms: $\quad -\underset{\sim}{v}_E \cdot \nabla F_0 \Rightarrow + \dfrac{c}{B}\left(\nabla \Phi \times \hat{b} \cdot \dfrac{\nabla n_0}{n_0}\right) F_0 \qquad \dfrac{\nabla n_0}{n_0} = -\dfrac{\hat{r}}{L_n}$

$$= -\frac{c}{B} \nabla \Phi \times \hat{b} \cdot \hat{r} \frac{1}{L_n} F_0$$

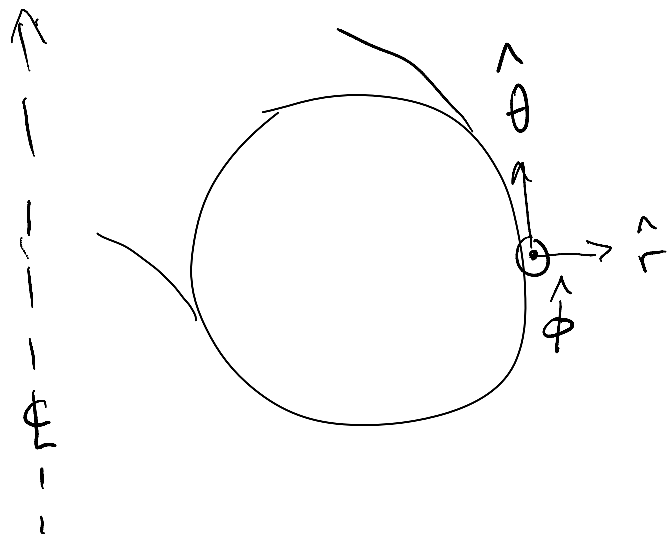


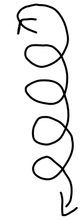
$$= -\frac{c}{B} i k_\theta \Phi \frac{1}{L_n} F_0$$

$$= + i \omega_* \frac{e\Phi}{T_0} F_0$$

$$\boxed{\omega_{*i} \equiv -\frac{cT}{eB}\frac{k_\theta}{L_n} \equiv -k_\theta \rho_s \frac{c_s}{L_n}}$$

Note on sign conventions:





With $\underset{\sim}{B}$ field out of the page, the ∇B drift for ions is downward

$$\underset{\sim}{v}_d \approx -\hat{\theta}\, v_t \frac{\rho}{R} \quad (\text{at } \theta = 0)$$

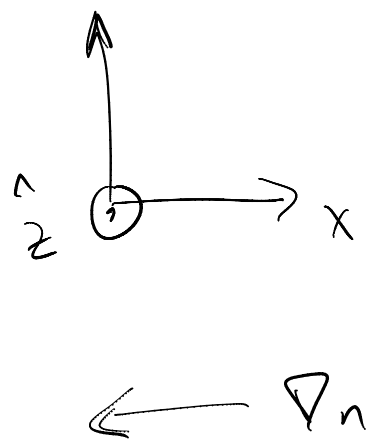
defining $\omega_{dv} = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$

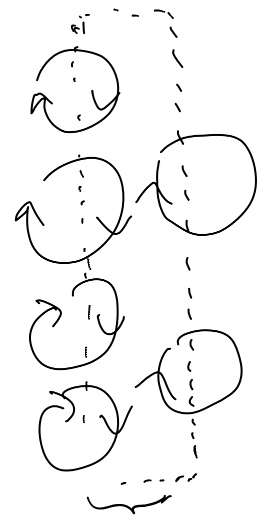
gives convention used in Beer's thesis!

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_d = -k_\theta \rho v_t / R$$

More on Sign Conventions





with $\underset{\sim}{B}$ out of page, the diamagnetic flow v_{*i} is downward if ∇n is inward. Thus

$$\omega_{*i} \equiv \underset{\sim}{k} \cdot v_{*i} = - k_\theta v_t \frac{\rho}{L_n}$$

$$= - \frac{cT}{eB} \frac{k_\theta}{L_n}$$

(Back to RHS of linearized GK Eq., 4 slides back)

$$\mathrm{RHS} = \underbrace{-\underset{\sim}{v}_E \cdot \nabla F_0}_{\text{part of this}} - \underbrace{\left(\frac{q}{m} E_\parallel + v_\parallel (\hat{b}\cdot\nabla\hat{b}) \cdot \underset{\sim}{v}_E\right) \frac{\partial F_0}{\partial v_\parallel}}$$

part of this

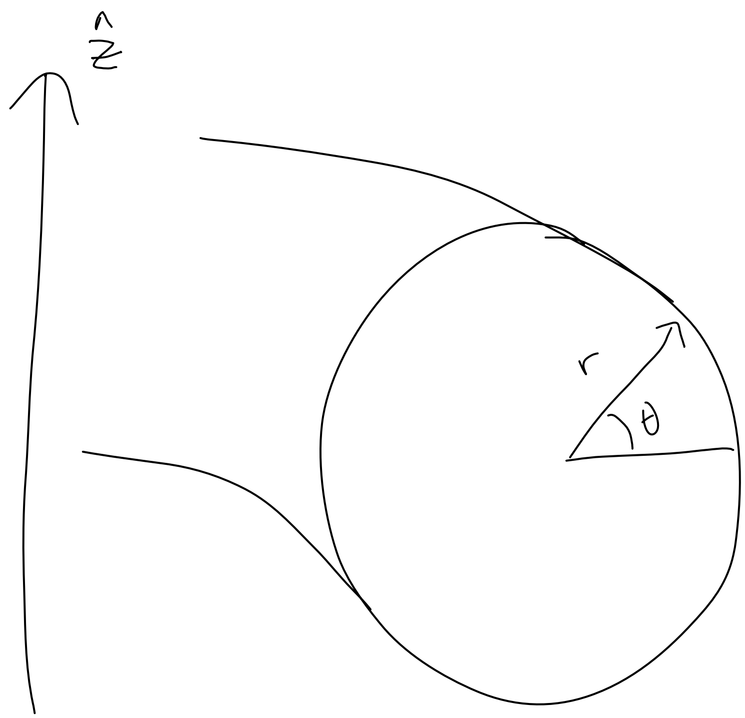
$$\propto -\frac{c}{B} \nabla\Phi \times \hat{b} \cdot \mu \nabla B \qquad\qquad \propto + v_\parallel^2 (\hat{b}\cdot\nabla\hat{b}) \cdot \left(\frac{\hat{b}\times\nabla\Phi}{B}\right)$$

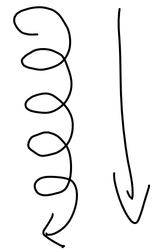
$$\propto -\nabla\Phi \cdot \Big[\underbrace{\mu\, \hat{b}\times\nabla B}_{\nabla B} + \underbrace{v_\parallel^2\, \hat{b}\times(\hat{b}\cdot\nabla\hat{b})}_{+\text{ curvature drift}} \Big]$$

$$\mathrm{RHS} = +i\left(\omega_{*v}^T - \omega_{dv} - k_\parallel v_\parallel\right) \frac{e\Phi}{T_0} F_0$$

$$\omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)] \qquad\qquad \omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2$$

$$\omega_* = k_\theta \rho \frac{v_t}{L_n} \qquad \eta = \frac{L_n}{L_T} \qquad \omega_d = -\frac{v_t}{R} \rho \left(k_\theta \cos\theta + k_r \sin\theta\right)$$





downward

$\underset{\sim}{v}_d$ from ∇B + curvature drift

$$\omega_d = \underset{\sim}{k} \cdot \underset{\sim}{v}_d$$

$$= -\frac{v_t \rho}{R}\left(k_\theta \cos\theta + k_r \sin\theta\right)$$

will focus on $\theta \approx 0$ here

(where bad-curvature drive is the strongest)

$$\left(-i\omega + i v_{\parallel} k_{\parallel} + i \underset{\sim}{v}_d \cdot \underset{\sim}{k}_{\perp}\right)\tilde{f} = -\underset{\sim}{v}_E \cdot \nabla F_0 - \left(\frac{q}{m} E_{\parallel} + v_{\parallel}\left(\hat{b}\cdot\nabla\hat{b}\right)\cdot \underset{\sim}{v}_E\right)\frac{\partial F_0}{\partial v_{\parallel}}$$

subst. for RHS

$$\left(-i\omega + i v_{\parallel} k_{\parallel} + i\omega_{dv}\right)\tilde{f} = -i\left(-\omega_{*v}^{T} + \omega_{dv} + k_{\parallel} v_{\parallel}\right)\frac{e\Phi}{T_0} F_0$$

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^{T} + \left(k_{\parallel} v_{\parallel} + \omega_{dv}\right)}{\omega - \left(k_{\parallel} v_{\parallel} + \omega_{dv}\right)} \frac{e\Phi}{T_0} F_0}$$

Note: recover Boltzmann response when $k_{\parallel} v_{\parallel}$ & or ω_{dv} large

$$\boxed{\tilde{f} = \frac{-\omega_{*v}^{T} + (k_\parallel v_\parallel + \omega_{dv})}{\omega - (k_\parallel v_\parallel + \omega_{dv})} \frac{e\Phi}{T_0} F_0}$$

Look for modes with

$$k_\parallel v_{ti} \ll \omega, \omega_{*v}^{T}, \omega_{dv} \ll k_\parallel v_{te}$$

(slab "η_i" version of ITG requires finite $k_\parallel v_{ti}$, but not toroidal version).

assume Boltzmann electrons

(additional polarization contribution to density gives $k_\perp^2 \rho_i^2$ corrections but not critical for basic ITG.)

Quasineutrality: $\tilde{n}_e = \tilde{n}_i$

$$\boxed{n_{e0} \frac{e\Phi}{T_e} = \int d^3v \frac{-\omega_{*v}^{T} + \omega_{dv}}{\omega - \omega_{dv}} F_0 \frac{e\Phi}{T_{i0}}}$$

$$n_0 \frac{e\Phi}{T_{e0}} = n_0 \frac{e\Phi}{T_{i0}} \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega - \omega_{dv}}$$

"Cold plasma" or "fast wave" approx. $\omega \gg \omega_{dv}$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \mu B/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n$$

$$v_\perp^2 = v_x^2 + v_y^2$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv} = \int d^3v \, \frac{F_0}{n_0} \omega_d \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right) / v_t^2$$

$$= 2\omega_d$$

Using useful I.D. for Maxwellian F_0 :

$$\langle v_x^{2n} \rangle = \int d^3v \, \frac{F_0}{n_0} v_x^{2n} = v_t^{2n} \underbrace{(2n-1)!!}_{(2n-1)(2n-3)(2n-5)\cdots 5\cdot 3\cdot 1}$$

$$\frac{T_{i0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$\omega_{dv} = \omega_d(v_\parallel^2 + \mu B)/v_t^2$ $\quad\omega_*^T = \omega_*[1 + \eta(v_\parallel^2/2v_t^2 + \underbrace{\mu B}/v_t^2 - 3/2)]$

$\omega_d = -k_\theta \rho v_t / R$ $\quad\omega_* = -k_\theta \rho v_t / L_n$ $\quad= \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)$

$$\int d^3v \, \frac{F_0}{n_0} \omega_*^T = \omega_* \left(1 + \eta \left(\frac{1}{2} + 1 - \frac{3}{2} \right) \right) = \omega_*$$

$$\int d^3v \, \frac{F_0}{n_0} \omega_{dv}^2 = \int d^3v \, \frac{F_0}{n_0} \omega_d^2 \left[v_\parallel^4 + 2 v_\parallel^2 \frac{1}{2} v_\perp^2 + \frac{1}{4} (v_x^2 + v_y^2)^2 \right] \frac{1}{v_t^4}$$

$$= \omega_d^2 \left[3 + 2 \cdot \frac{1}{2}(1+1) + \frac{1}{4} \left(\frac{\langle v_x^4 + 2 v_x^2 v_y^2 + v_y^4 \rangle}{v_t^4} \right) \right]$$

$$= \omega_d^2 \left[5 + \frac{1}{4}(8) \right] = 7 \omega_d^2$$

$$\frac{T_{\wedge 0}}{T_{e0}} = \int d^3v \, \frac{F_0}{n_0} \, \frac{\omega_{dv} - \omega_{*T}}{\omega} \left(1 + \frac{\omega_{dv}}{\omega} + \cdots \right)$$

$$\omega_{dv} = \omega_d (v_\parallel^2 + \mu B)/v_t^2 \qquad \omega_*^T = \omega_* [1 + \eta (v_\parallel^2 / 2 v_t^2 + \underbrace{\mu B}/v_t^2 - 3/2)]$$

$$\omega_d = -k_\theta \rho v_t / R \qquad \omega_* = -k_\theta \rho v_t / L_n \qquad = \frac{1}{2} v_\perp^2 = \frac{1}{2}(v_x^2 + v_y^2)$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T = \omega_d \omega_* \Big\{ 2$$

$$+ \eta \int d^3v \, \frac{F_0}{n_0} \, \frac{\left(v_\parallel^2 + \frac{1}{2} v_\perp^2\right)}{v_t^2} \, \frac{\left(\frac{1}{2} v_\parallel^2 + \frac{1}{2} v_\perp^2 - \frac{3}{2} v_t^2\right)}{v_t^2} \Big\}$$

$$= \omega_d \omega_* \Big\{ 2 + \eta \Big[\frac{1}{2} 3 + \frac{1}{2} 2 - \frac{3}{2} + \frac{1}{2} \cdot 2 \cdot \frac{1}{2} + \frac{1}{4} 8$$

$$- \frac{1}{2} \cdot 2 \cdot \frac{3}{2} \Big] \Big\}$$

$$\int d^3v \, \frac{F_0}{n_0} \, \omega_{dv} \, \omega_*^T$$

$$= \omega_d \omega_* \left\{ 2 + \eta \left[\cancel{\tfrac{1}{2} 3} + \cancel{\tfrac{1}{2} 2} - \cancel{\tfrac{3}{2}} + \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{1}{2}} + \tfrac{1}{4} 8 - \cancel{\tfrac{1}{2} \cdot 2 \cdot \tfrac{3}{2}} \right] \right\}$$

$$= \omega_d \omega_* 2 (1 + \eta)$$

Combine results from last 2 pages:

$$\boxed{\frac{T_{10}}{T_{e0}} = 2 \frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7 \frac{\omega_d^2}{\omega^2} - 2 \frac{\omega_d \omega_*}{\omega^2} (1 + \eta)}$$

This defines a dispersion relation ω vs. $\underset{\sim}{k}$

$$\boxed{\frac{T_{i0}}{T_{e0}} = 2\frac{\omega_d}{\omega} - \frac{\omega_*}{\omega} + 7\frac{\omega_d^2}{\omega^2} - 2\frac{\omega_d\omega_*}{\omega^2}(1+\eta)}$$

Consider the flat density limit: $\nabla n \to 0$, but $\nabla T \neq 0$

$$\omega_* = -k_\theta \rho \frac{v_t}{L_n} \longrightarrow 0 \qquad \eta = \frac{\frac{1}{T}\nabla T}{\frac{1}{n}\nabla n} = \frac{L_n}{L_T} \to \infty$$

$$\omega_*\eta = -k_\theta \rho \frac{v_t}{L_n}\frac{L_n}{L_T} \equiv \bar{\omega}_{*T}$$

$$\omega^2 \frac{T_{i0}}{T_{e0}} - 2\omega_d\omega + 2\omega_d\bar{\omega}_{*T} - 7\omega_d^2 = 0$$

$$\boxed{\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d\bar{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}}$$

From last page:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{io}}{T_{eo}}\left(2\omega_d \overline{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{io}/T_{eo})}$$

Consider large temperature gradient limit: $\omega_{*T} \propto \nabla T$ $\uparrow$

Growth rate:

$$\gamma = \frac{\sqrt{2\omega_d \overline{\omega}_{*T}}}{\sqrt{T_{io}/T_{eo}}} = \frac{\sqrt{2}\, k_\theta \rho_i}{\sqrt{T_{io}/T_{eo}}} \frac{v_{ti}}{\sqrt{R L_T}}$$

Fundamental scaling of bad-curvature driven instabilities.

Go back to general D.R.:

$$\omega = \frac{2\omega_d \pm \sqrt{4\omega_d^2 - 4\frac{T_{i0}}{T_{e0}}\left(2\omega_d\overline{\omega}_{*T} - 7\omega_d^2\right)}}{2(T_{i0}/T_{e0})}$$

$$= \frac{2\omega_d \pm \sqrt{\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)\omega_d^2 - 8\frac{T_{i0}}{T_{e0}}\omega_d\overline{\omega}_{*T}}}{2(T_{i0}/T_{e0})}$$

Instability exists if

$$8\frac{T_{i0}}{T_{e0}}\omega_d\overline{\omega}_{*T} > \omega_d^2\left(4 + 28\frac{T_{i0}}{T_{e0}}\right)$$

$$\frac{1}{R}\frac{1}{L_T} > \frac{1}{R^2}\left(\frac{1}{2}\frac{T_{e0}}{T_{i0}} + \frac{1}{2}7\right)$$

$$\boxed{\frac{R}{L_T} > \frac{1}{2}\left(7 + \frac{T_{e0}}{T_{i0}}\right)}$$

$\bigstar$

Compare w/ Romanelli 1990 (Eq. 12):

$$\eta_i = (\tfrac{5}{3} + \tau/4)2\epsilon_n$$

or

$$\frac{L_n}{L_T} = \left(\frac{5}{3} + \frac{1}{4}\frac{T_e}{T_i}\right) 2 \frac{L_n}{R}$$

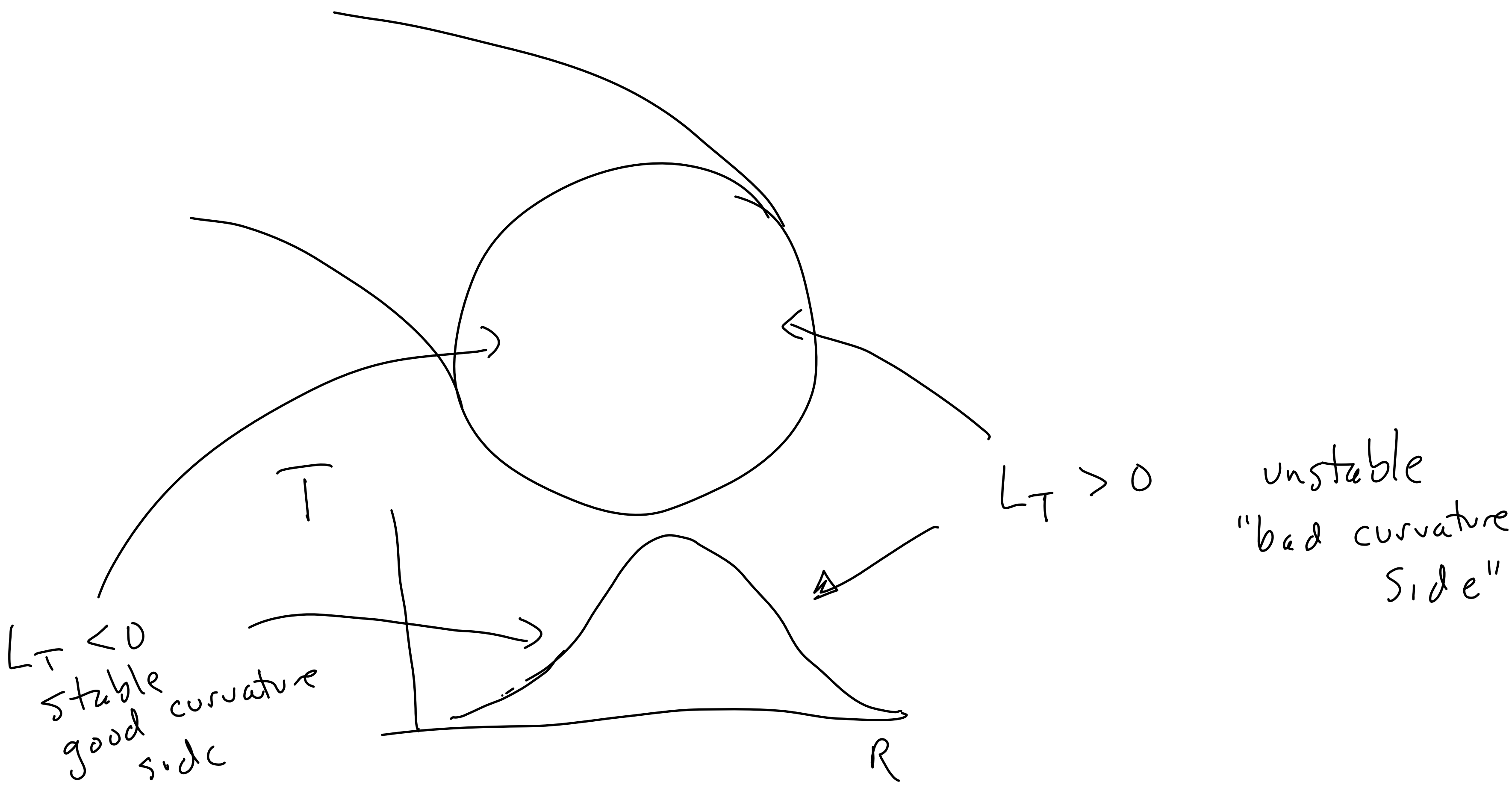
$$\boxed{\frac{R}{L_{Tcrit}} = \frac{10}{3} + \frac{1}{2}\frac{T_{e0}}{T_{i0}}}$$

$$= 3.33 + 0.5\frac{T_{e0}}{T_{i0}}$$

vs. my $\frac{R}{L_{Tcrit}} = 3.5 + 0.5\frac{T_{e0}}{T_{i0}}$ } Very close. Diff. is presumably because Romanelli simplifies ω_D (See after his Eq. 6)

Note there is an instability only if $\omega_d \bar{\omega}_{*T} > 0$

$$\omega_d \bar{\omega}_{*T} = (k_\theta \rho)^2 \frac{v_t^2}{R L_T} \qquad \frac{1}{L_T} \equiv -\frac{1}{T}\frac{\partial T}{\partial R}$$



Why does this get the $\frac{T_{i0}}{T_{e0}}$ dependence of

$\frac{R}{L_{T crit}}$ wrong?

More accurate: $\boxed{\frac{R}{L_T} > \frac{R}{L_{T crit}} = \frac{4}{3}\left(1 + \frac{T_{i0}}{T_{e0}}\right)}$

Because near marginal stability, the expansion of the resonant denominator

$$\frac{1}{\omega - \omega_{dv}} \simeq \frac{1}{\omega}\left(1 + \frac{\omega_{dv}}{\omega} + \dots\right)$$

breaks down, since $\omega \sim \omega_d$ near marginal stability...

More general result for threshold for instability:

$$\frac{R_0}{L_{T\,crit}} = \mathrm{Max}\left[\left(1+\frac{T_i}{T_e}\right)\left(1.33+1.91\,\frac{\hat{s}}{q}\right)\left(1-1.5\,\frac{r}{R_0}\right)\left(1+0.3\,\frac{r\,d\kappa}{dr}\right),\ 0.8\,\frac{R_0}{L_n}\right]$$

Found by fits to lots of GS2 Gyrokinetic stability calculations (Jenko, Dorland Hammett, PoP 2001), guided by previous analytic results (Romanelli, Hahm + Tang) in some limits.

ITG References

- Mike Beer's Thesis 1995
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- Romanelli & Briguglio, Phys. Fluids B 1990
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- Dorland et al, PRL …
- Dimits et al.…
- …
- Earlier history:
  - slab eta_i mode: Rudakov and Sagdeev, 1961
  - Sheared-slab eta_i mode: Coppi, Rosenbluth, and Sagdeev, Phys. Fluids 1967
  - Toroidal ITG mode: Coppi and Pegoraro 1977, Horton, Choi, Tang 1981, Terry et al. 1982, Guzdar et al. 1983… (See Beer's thesis)