ISLAND module

Glenn Bateman Federico D. Halpern Arnold H. Kritz

21 November 2005

The ISLAND module is used to compute magnetic island widths driven by saturated neoclassical tearing modes in tokamak plasmas using a quasi-linear model published in Phys. Plasmas **11** (2003) 3460–3471.

!

1 Quasi-linear Tearing Mode Equations

In this section, the equations and solution algorithm are derived for computing the widths of saturated tearing mode magnetic islands using the quasi-linear theory approach. The derivation given in Ref. [3, 2] is extended here to include finite plasma pressure and the effect of the neoclassical bootstrap current. In addition, a self-consistent method for implementing this algorithm within an integrated modeling code is developed.

1.1 Summary of the derivation

The derivation for quasi-linear saturated tearing mode equations begins with the three-dimensional static scalar plasma pressure equilibrium force balance equations

$$\mathbf{J} \times \mathbf{B} = \nabla p \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

These equations are perturbed about an axisymmetric equilibrium. The derivation is carried out using magnetic flux coordinates defined in terms of the axisymmetric unperturbed equilibrium since magnetic islands are flux tubes that form around mode-rational surfaces. Specifically, the saturated tearing mode equations are readily derived using Hamada-like coordinates (x, θ, ζ), where x is a surface quantity, θ is an angle-like variable around the poloidal direction (short way around), and ζ is an angle-like variable around the toroidal direction (long way around) [4, 5, 6]. The variable x may represent the volume of the unperturbed magnetic surfaces or any monotonically increasing or decreasing surface quantity.

The static scalar plasma pressure equilibrium force balance equation 1 does not include the effect of differential rotation of the plasma. In reality, magnetic islands on different magnetic surfaces generally rotate with different angular velocities due to sheared plasma rotation and the radial profile of the diamagnetic drift. This effect substantially reduces the mutual coupling between magnetic islands, unless the island chains mode lock with one another. The effects of the differential rotation velocity $\mathbf{v}$ could be implemented by including the convective term $\mathbf{v}\cdot\nabla\mathbf{v}$ in the force balance equation. In addition, non-scalar plasma pressure and viscous stress terms could be included in the force balance equation. The simplest model, which is a static scalar pressure model is implemented in this paper.

In Hamada coordinates, the unperturbed (background) contravariant components of the magnetic field and current density are surface quantities

$$\mathbf{B}^0 = B^{0\theta} \mathcal{J} \nabla\zeta \times \nabla x + B^{0\zeta} \mathcal{J} \nabla x \times \nabla\theta$$

$$\mathbf{J}^0 = J^{0\theta} \mathcal{J} \nabla\zeta \times \nabla x + J^{0\zeta} \mathcal{J} \nabla x \times \nabla\theta.$$

The coordinates are chosen so that the Jacobian ($\mathcal{J}$) is a surface quantity

$$\mathcal{J} \equiv (\nabla x \cdot \nabla\theta \times \nabla\zeta)^{-1} = \mathcal{J}(x)$$

The angle-like variables θ and ζ are assumed to be periodic with period 2π. This choice of Jacobian and periodicity leads to a generalization of the usual Hamada coordinate system, in which the Jacobian and the periods are all unity.

The derivation proceeds by writing the perturbed magnetic field $\mathbf{B}^1$ in terms of both contravariant components $(B^{1x}, B^{1\theta}, B^{1\zeta})$ and covariant components $(B^1_x, B^1_\theta, B^1_\zeta)$:

$$\mathbf{B}^1 = B^{1x}\mathcal{J}\nabla\theta\times\nabla\zeta + B^{1\theta}\mathcal{J}\nabla\zeta\times\nabla x + B^{1\zeta}\mathcal{J}\nabla x\times\nabla\theta \tag{4}$$

$$\mathbf{B}^1 = B^1_x\nabla x + B^1_\theta\nabla\theta + B^1_\zeta\nabla\zeta \tag{5}$$

The perturbed current density $\mathbf{J}^1$ is written in terms of contravariant components

$$\mathbf{J}^1 = J^{1x}\mathcal{J}\nabla\theta\times\nabla\zeta + J^{1\theta}\mathcal{J}\nabla\zeta\times\nabla x + J^{1\zeta}\mathcal{J}\nabla x\times\nabla\theta \tag{6}$$

All perturbed variables are written as a series of Fourier harmonics in the toroidal and poloidal angle-like variables, θ and ζ:

$$X^1 = \sum_{m,n} X^1_{mn}(x)e^{im\theta - in\zeta}$$

By linearizing the equilibrium equations, Eqs. (1), (2) and (3), one obtains the equations $\mathbf{J}^1\times\mathbf{B}^0 + \mathbf{J}^0\times\mathbf{B}^1 = \nabla p$, $\mu_0\mathbf{J}^1 = \nabla\times\mathbf{B}^1$, and $\nabla\cdot\mathbf{B}^1 = 0$. After some algebra, the following ordinary differential equations are derived for the mix of contravariant and covariant components of perturbed magnetic field, as well as the perturbed pressure p^1_{mn} [2]

$$\frac{d}{dx}(-i\mathcal{J}B^{1x}_{mn}) = \mathcal{J}(nB^{1\zeta}_{mn} - mB^{1\theta}_{mn}) \tag{7}$$

$$\begin{aligned}(nq-m)B^{0\theta}&\left[\frac{d}{dx}B^1_{\theta mn} - imB^1_{xmn}\right]\\ &= \mu_0(nJ^{0\zeta} - mJ^{0\theta})\mathcal{J}B^{1\zeta}_{mn} - i\mathcal{J}B^{1x}_{mn}B^{0\zeta}\frac{d}{dx}\left(\frac{\mu_0 J^{0\zeta}}{B^{0\zeta}}\right) + \frac{m\mu_0 p^1_{mn}}{B^{0\zeta}}\frac{d}{dx}B^{0\zeta}\end{aligned} \tag{8}$$

$$nB^1_{\zeta mn} = -nB^1_{\theta mn} - i\mathcal{J}B^{1x}_{mn}\frac{\mu_0 J^{0\zeta}}{B^{0\zeta}} - \frac{m\mu_0 p^1_{mn}}{B^{0\zeta}} \tag{9}$$

$$(nq-m)B^{0\theta}p^1_{mn} = -iB^{1x}_{mn}\frac{d}{dx}p^0 \tag{10}$$

In Hamada coordinates, the magnetic q is given by $q = B^{0\zeta}/B^{0\theta}$ [4, 7]. Equations (7) and (8) form a pair of coupled ordinary differential equations for each helical harmonic of the variable $(i\mathcal{J}B^{1x}_{mn}, B^1_{\theta mn})$. Equations (9) and (10) can be used to eliminate $B^1_{\zeta mn}$ and p^1_{mn}. The perturbed covariant and contravariant magnetic field components B^1_x, $B^{1\theta}$, and $B^{1\zeta}$ are determined from the algebraic relations:

$$B^1_x = \frac{B^{1x}}{|\nabla x|^2} - B^1_\theta\frac{\nabla\theta\cdot\nabla x}{|\nabla x|^2} - B^1_\zeta\frac{\nabla\zeta\cdot\nabla x}{|\nabla x|^2} \tag{11}$$

$$B^{1\theta} = B^{1x}\frac{\nabla\zeta\cdot\nabla x}{|\nabla x|^2} + B^1_\theta\left(|\nabla\theta|^2 - \frac{(\nabla\theta\cdot\nabla x)^2}{|\nabla x|^2}\right) + B^1_\zeta\left(\nabla\theta\cdot\nabla\zeta - \frac{(\nabla\theta\cdot\nabla x)(\nabla\zeta\cdot\nabla x)}{|\nabla x|^2}\right) \tag{12}$$

$$B^{1\zeta} = B^{1x}\frac{\nabla\zeta\cdot\nabla x}{|\nabla x|^2} + B^1_\theta\left(\nabla\theta\cdot\nabla\zeta - \frac{(\nabla\theta\cdot\nabla x)(\nabla\zeta\cdot\nabla x)}{|\nabla x|^2}\right) + B^1_\zeta\left(|\nabla\zeta|^2 - \frac{(\nabla\zeta\cdot\nabla x)^2}{|\nabla x|^2}\right) \tag{13}$$

Equations (11), (12), and (13) are derived by equating Eq. (4) and Eq. (5) and taking the dot product with the gradients ∇x, $\nabla\theta$, and $\nabla\zeta$. The harmonic coupling implied by Eqs. (11) through (13) should be regarded as an upper bound estimate of the true harmonic coupling, which can be substantially weakened by differential rotation within the plasma.

The gradients of (x, θ, ζ) and, consequently, the geometric terms in equations (11), (12), and (13) can be determined from the shape of the equilibrium flux surfaces in the following way: Suppose the shapes of the equilibrium flux surfaces are given by the equations

$$R = R(x, \theta) \tag{14}$$

$$Y = Y(x, \theta) \tag{15}$$

$$\zeta - \phi = Z(x, \theta) \tag{16}$$

where the major radius R, the vertical position Y, and the toroidal angle ϕ are given as functions of the Hamada coordinates (x, θ, ζ). For axisymmetric equilibria, both ϕ and ζ are ignorable coordinates with period 2π. For any value of x, the cross-sectional shape of the corresponding flux surface is traced out by varying θ between 0 and 2π in Eqs. (14) and (15).

Then, from the gradients of Eqs. (14) through (16)

$$\nabla R = \hat{R} = \partial_\theta R \nabla\theta + \partial_x R \nabla x \tag{17}$$

$$\nabla Y = \hat{Y} = \partial_\theta Y \nabla\theta + \partial_x Y \nabla x \tag{18}$$

$$\nabla\phi = \hat{\phi}/R \tag{19}$$

it follows that

$$\nabla x = \frac{\hat{Y}\partial_\theta R - \hat{R}\partial_\theta Y}{\partial_\theta R \partial_x Y - \partial_x R \partial_\theta Y} \tag{20}$$

$$\nabla\theta = \frac{\hat{R}\partial_x Y - \hat{Y}\partial_x R}{\partial_\theta R \partial_x Y - \partial_x R \partial_\theta Y} \tag{21}$$

$$\nabla\zeta = \hat{\phi}/R + \partial_x Z \nabla x + \partial_\theta Z \nabla\theta \tag{22}$$

and the Jacobian is given by $\mathcal{J} \equiv (\nabla x \cdot \nabla\theta \times \nabla\zeta)^{-1} = R(\partial_x R \partial_\theta Y - \partial_x Y \partial_\theta R) = \mathcal{J}(x)$. In these equations, $(\hat{R}, \hat{Y}, \hat{\phi})$ are unit vectors along (R, Y, ϕ). Application of the expressions for the gradients of (x, θ, ζ) above in equations (11), (12), and (13), leads to the following form for the mixing equations:

$$[(\partial_\theta R)^2 + (\partial_\theta Y)^2]B^{1\theta} = -B^{1x}(\partial_x R \partial_\theta R + \partial_x Y \partial_\theta Y) + B^1_\theta + B^1_\zeta \partial_\theta Z \tag{23}$$

$$B^{1\zeta} = B^{1x}\partial_x Z + B^{1\theta}\partial_\theta Z + \frac{B^1_\zeta}{R^2} \tag{24}$$

$$B^1_x = B^{1x}[(\partial_x R)^2 + (\partial_x Y)^2] + B^{1\theta}(\partial_x R \partial_\theta R + \partial_x Y \partial_\theta Y) - B^1_\zeta \partial_x Z \tag{25}$$

Eqs. (7 - 10) for the harmonics of B^{1x}, B^1_θ, B^1_ζ, and p^1 together with Eqs. (23 - 25) for the components B^1_x, $B^{1\theta}$, and $B^{1\zeta}$ form a complete systems of equations.

1.2 Width of a Magnetic Island

At each mode rational surface, $q(x) = m/n$, the expression

$$nB^{0\zeta}(x) - mB^{0\theta}(x) = [nq(x) - m]B^{0\theta}(x)$$

goes to zero, thus giving the appearance that equations (8) and (10) are singular at mode rational surfaces. However, the existence of a magnetic island or stochastic region, which is implied by a nonzero B^{1x}_{mn} at the mode rational surfaces, locally flattens the $p^0(x)$ and $J^{0\zeta}(x)/B^{0\zeta}(x)$ profiles. This removes the singularity in Eqs (8) and (10) [3, 8] as is shown in the subsection below. The perturbation affects the background profiles and permits a saturated island solution, in contrast to a linear tearing mode solution.

Consider a stream function ψ that is uniform along magnetic field lines:

$$\mathbf{B} \cdot \nabla\psi = 0 \tag{26}$$

If ψ is a smooth function with a local minimum at the mode rational surface, the Taylor series for ψ that satisfies Eq. (26) yields

$$\psi = \psi^0_{mn}(x_{mn}) - \left(\frac{\psi^1_{mn} n B^{0\theta}}{2iB^{1x}_{mn}} \left[\frac{dq}{dx}\right]_{x=x_{mn}} \right) (x - x_{mn})^2 + \cdots + \psi^1_{mn}\cos(m\theta - n\zeta) \tag{27}$$

where x_{mn} is the radius of the mode rational surface and where $\psi^1_{mn}\cos(m\theta - n\zeta)$ is the lowest order harmonic perturbation. With the value of ψ at the edge of the widest part of the island ($m\theta - n\zeta = 0$, $x - x_{mn} = W/2$) set equal to the value of ψ at the x-point ($m\theta - n\zeta = \pi$, $x - x_{mn} = 0$), the following equation for the width of the island is obtained

$$W_{mn} = 4\sqrt{\left[\frac{-iB^{1x}_{mn}}{nB^{0\theta}(dq/dx)}\right]_{x=x_{mn}}} \tag{28}$$

Higher order terms in the Taylor series (27) may be used to determine the asymmetry in the island width and other details of the island's shape. Note that the island width is measured in terms of the flux surface label x. The geometrical width of the island may vary poloidally around a flux surface, being wider where the background flux surfaces are more spread apart and narrower where the background flux surfaces are squeezed together. This effect is automatically taken account when using flux coordinates.

1.3 Effect of Magnetic Islands on the Plasma Profiles

Suppose the pressure profile across the widest part of the island, $m\theta - n\zeta = 0$, is a uniform constant p_0 within the island and is linear in the radius with slope $p_0' = dp_0/dx$, just outside the shoulder of the island,

$$p = \begin{cases} p_0 + 0.5p_0' W_{mn}(u+1) & u < -1 \\ p_0 & |u| < 1 \\ p_0 + 0.5p_0' W_{mn}(u-1) & u > 1 \end{cases} \tag{29}$$

where

$$u \equiv 2(x - x_{mn})/W_{mn}. \tag{30}$$

Since $\mathbf{B} \cdot \nabla p = 0$, the pressure is a function of ψ. At the widest part of the island, Eqs. (27), (28), and (30) can be used to write u as a function of ψ

$$u = \sqrt{\frac{\psi - \psi^0 - \psi^1_{mn}}{2\psi^1_{mn}}} \tag{31}$$

Then, Eq. (29) can be used to express the pressure as a function of ψ in the neighborhood of a magnetic island

$$p(\psi) = \begin{cases} p_0 + 0.5p_0' W_{mn}\left(\left[(\psi^0_{mn} + \psi^1_{mn} - \psi)/2\psi^1_{mn}\right]^{1/2} + 1 \right) & \text{inner edge} \\ p_0 & \text{inside island} \\ p_0 + 0.5p_0' W_{mn}\left(\left[(\psi^0_{mn} + \psi^1_{mn} - \psi)/2\psi^1_{mn}\right]^{1/2} - 1 \right) & \text{outer edge} \end{cases} \tag{32}$$

Consider the poloidal (or, equivalently, the toroidal) average of the pressure

$$\langle p \rangle = \frac{1}{2\pi} \int_0^{2\pi} p(\psi) d\theta$$

at fixed x to determine the background axisymmetric pressure profile to be used in Eq. (10). Substitution of Eq. (27) in Eq. (32) and integration over the poloidal angle leads to the following approximation to the axisymmetric poloidal averaged pressure profile in the neighborhood of a magnetic island

$$\left\langle \frac{dp^0}{dx} \right\rangle \approx p_0' \begin{cases} 0.63u^2 & |u| < 1 \\ 1 - \dfrac{0.25}{u^2} & |u| > 1 \end{cases} \tag{33}$$

It can be seen from Eq. (33) that the presence of a magnetic island has the effect of flattening the axisymmetric background pressure profile at the corresponding mode rational surface. In particular, $\langle dp/dx \rangle$ and $\langle d^2p/dx^2 \rangle$ are both equal to zero at the mode rational surface, $u = 0$.

The same derivation can be applied to compute the flat spot in the current density profile caused by the magnetic island at each mode rational surface. Suppose the current density profile across the widest part of the island is

$$K \equiv \frac{\mu_0 J^\zeta}{B^\zeta} \approx \begin{cases} K_0 + K_1(u+1) & u < -1 \\ K_0 - \gamma K_1(1-u^2) - K_2 & |u| < 1 \\ K_0 + K_1(u-1) & u > 1 \end{cases} \tag{34}$$

with $K_0 = mu_0 J^\zeta / B^\zeta$ at the edge of the island, $K_1 = -d\left(\mu_0 J^\zeta / B^\zeta\right)/du$, at the edge of the island. The coefficient, γ, is a current peaking factor (if $\gamma > 0$, assuming $K_1 < 0$) or current suppression factor (if $\gamma < 0$) within the island, and $K_2 \equiv \mu_0 J^\zeta_{\text{bootstrap}} / B^\zeta$ is the bootstrap current density just outside the edge of the island. Finally, $u \equiv 2(x - x_{mn})/W_{mn}$. Since the pressure profile is assumed to be flat within the magnetic island, and the bootstrap current density is roughly proportional to the pressure gradient [11], there is no bootstrap current density within the island. Hence, the abrupt transition from the presence of bootstrap current outside of the island to the complete absence of the bootstrap current within the island produces a square wave contribution to the current suppression within the island. Close to the mode rational surface, where $|u| << 1$, the axisymmetric average of the radial derivative of the toroidal current density can be approximated with the following expression

$$\begin{aligned} \left\langle \frac{d}{dx}\left(\frac{\mu_0 J^{0\zeta}}{B^{0\zeta}}\right)\right\rangle \approx & \frac{2K_1 u}{W_{mn}\sqrt{u^2+0.5}}\left[1 + \frac{3}{32(u^2+0.5)^2}\right] \\ & + \frac{2\gamma K_1 u \cos^{-1}(2u^2-1)}{W_{mn}\pi} - \frac{2.52 K_2 u}{W_{mn}}. \end{aligned} \tag{35}$$

Thus, near the mode rational surface, the apparently singular terms on the right side of equations (8) and (10) reduce to the following non-singular forms

$$\frac{nJ^{0\zeta} - mJ^{0\theta}}{(nq-m)B^{0\theta}} \rightarrow \frac{J^{0\zeta}}{B^{0\zeta}}$$

$$\frac{p^1_{mn}}{(nq-m)B^{0\theta}} \rightarrow \frac{-2.52 i B^{1x}_{mn} p_0'}{[nB^{0\theta} W_{mn}(dq/dx)]^2}$$

$$\frac{\dfrac{d}{dx}\left(\mu_0 J^{0\zeta}/B^{0\zeta}\right)}{(nq-m)B^{0\theta}} \rightarrow \frac{6.4K_1 + 4\gamma K_1 - 5.04 K_2}{W^2_{mn} B^{0\theta} n (dq/dx)}$$

Consequently the Eqs. (8) and (10) are not singular, and they can be integrated from the magnetic axis to the edge of the plasma with a numerical ordinary differential equation solver.

It should be noted that the interaction between different harmonics can produce stochastic regions in the magnetic field near the X-points of magnetic islands. The effect of the stochastic regions will be to flatten the profiles more completely near the mode rational surface than is estimated here. In addition, multiple Fourier harmonics with the same ratio of m/n will produce magnetic islands with a cross-sectional shape that is different from the single harmonic given in Eq. (27). The distortion in the island shape will also modify the estimates given above for the profile flattening.

1.4 Iterative Procedure for Determining Saturated Tearing Mode Island Widths

The quasi-linear three-dimensional scalar pressure plasma equilibrium formulation developed above can be solved using an iterative procedure to determine the widths of saturated magnetic islands in tokamak plasmas. The basic idea is to vary the island widths until the solutions for the perturbed fields match the boundary conditions at the magnetic axis and at a conducting wall at (or outside) the edge of the plasma. In effect, the island widths can be thought of as eigenvalues of the equations.

The procedure works in the following way: Start with an initial guess for magnetic island widths for the resonant helical harmonics being considered. Determine the effect of each magnetic island on the background current density and pressure profile near each mode rational surface. The background axisymmetric current density and pressure profile and flux surface shapes (such as elongation and triangularity) can be taken, for example, from the evolving solution in an integrated modeling code. A solution is found for each harmonic of the perturbation $(-i\mathcal{J}B^{1x}_{mn}(x), B^{1}_{\theta mn})$ from the set of coupled differential equations (7) and (8) together with ancillary equations (9) and (10) for $B^1_{\zeta mn}$ and p^1_{mn}. The Hamada coordinate metric elements (the geometric terms in Eqs. (11-13)) are used to compute the harmonic mixing of the perturbed magnetic field components $(B^1_{xmn}, B^{1\theta}_{mn}, B^{1\zeta}_{mn})$. An adaptive ordinary differential equation solver [9] can be used to integrate the equations for B^{1x}_{mn} and $B^1_{\theta mn}$ from the magnetic axis to the plasma boundary or perfectly conducting wall. The island widths are adjusted iteratively until all the solutions match the boundary conditions at the wall. The width of each magnetic island acts like a nonlinear eigenvalue for the two point boundary value problem in which the perturbed magnetic field radial component vanishes at the magnetic axis and at a perfectly conducting wall at the plasma edge. For a self-consistent treatment, this algorithm is used in an integrated modeling code and the transport is enhanced across each island, as described in the next section.

At each stage in the iteration, the amplitudes of the perturbed magnetic field harmonics are adjusted to be consistent with the widths of the corresponding magnetic islands, using Eq. (28). In most other treatments of tearing modes, the differential equation for the perturbation is integrated from the magnetic axis outward to the island, and then from the wall inward to the island, and the solution is matched across the island. When there are multiple harmonics that are mixing with one another (through Eqs. (11-13)), it is simpler to integrate all the harmonics together from the magnetic axis to the outer boundary condition at the wall. With finite island widths and the resulting modifications of the profiles described in section 1.3 above, an adaptive ODE solver [9] can integrate all the harmonics from the magnetic axis through the islands to the outer boundary condition.

The algorithm described above applies to an arbitrary number of helical harmonics in any background axisymmetric geometry with arbitrary cross-section, aspect ratio, and plasma β. The algorithm is simple enough to be used in an integrated predictive modeling code. Island widths change as the plasma profiles evolve. The algorithm does not apply to tearing modes that do not saturate in time, such as the $m/n = 1/1$ tearing mode that drives sawtooth crashes.

1.5 Transport Enhanced by Saturated Magnetic Islands

Magnetic islands enhance transport by locally short-circuiting the plasma. Since parallel diffusivity is much larger than perpendicular diffusivity, the temperature and density profiles are flattened within each island [10]. This assumption breaks down for small magnetic islands where the pressure flattening is incomplete. The critical island width, below which pressure flattening is incomplete, is given as $W_0/r_s \approx 5.1(\chi_\perp/\chi_\parallel)^{1/4}/\sqrt{\epsilon s n}$, where ϵ is the localized inverse aspect ratio and s is the localized magnetic shear [10]. For islands that are wider than W_0, however, the perturbation on the background pressure and temperature profiles near the magnetic island is given by Eq. (33), which can be written in the form

$$S(u) \equiv \frac{p'_{\text{perturbed}}}{p'_{\text{unperturbed}}} \approx \begin{cases} 0.63u^2 + \cdots & |u| < 1 \\ 1 - \dfrac{0.25}{u^2} + \cdots & |u| > 1 \end{cases} \tag{36}$$

The diffusive transport is modified to produce this local flattening of the profiles. If the transport equations have the following basic form

$$\chi \frac{\partial p}{\partial x} = F \tag{37}$$

where p is the plasma pressure, χ is the diffusivity and F is the diffusive heat flux, it follows that the diffusivity is inversely proportional to the radial pressure gradient and

$$\frac{\chi_{\rm perturbed}}{\chi_{\rm unperturbed}} \approx 1/S(u). \tag{38}$$

Consider a finite difference scheme in which the pressure $p_{j+1/2}$ is computed on zone centers $x_{j+1/2}$, while the thermal diffusivity χ_j and heat flux F_j are computed on zone boundaries x_j. Then, a simple finite difference approximation for Eq. (37) is

$$\chi_j \frac{p_{j+1/2} - p_{j-1/2}}{x_{j+1/2} - x_{j-1/2}} = F_j$$

at the jth zone boundary. To determine the finite difference form for χ, integrate Eq. (37) from one zone center to the next

$$p_{j+1/2} - p_{j-1/2} = \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{F(x)}{\chi(x)} dx \approx F_j \int_{x_{j-1/2}}^{x_{j+1/2}} \frac{dx}{\chi}. \tag{39}$$

Hence, an approximation for the thermal diffusivity at zone boundary j is

$$\chi_j \approx \frac{x_{j+1/2} - x_{j-1/2}}{\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} \frac{dx}{\chi}} \tag{40}$$

¿From Eq. (38), it is concluded that the diffusivity is enhanced near a magnetic islands with widths W_{mn} located at mode rational surfaces $x = x_{mn}$ by the following factor

$$\frac{\chi_{j\rm perturbed}}{\chi_{j\rm unperturbed}} \approx \sum_{mn} \frac{x_{j+1/2} - x_{j-1/2}}{\displaystyle\int_{x_{j-1/2}}^{x_{j+1/2}} S[2(x - x_{mn})/W_{mn}] dx}. \tag{41}$$

Eq. (41) provides a reasonable approximation to the enhancement of the diffusivities caused by each magnetic island even if the width of the magnetic island is much smaller than the grid spacing or even if part of an island lies in one grid zone while the other parts of the same island lie in one or more adjacent zones. Even though the underlying physics that results in this flattening of the profiles makes use of the fact that the diffusivity along magnetic field lines is much larger than the diffusivity across magnetic field lines, the magnitudes of the parallel and perpendicular diffusivities are not needed in this derivation. When the enhanced diffusivity given by Eq. (41) is used in the transport equations in an integrated modeling code, the effect is to flatten the pressure profile to a form that approximates Eq. (36). As the electron temperature profile is flattened by each island, the resulting resistivity profile will be flattened and the magnetic diffusion equation will produce a corresponding flattening in the current density profile. Hence, the use of the enhanced diffusivities in an integrated modeling code produces an approximately self-consistent treatment of the axisymmetric effects of the magnetic islands.

References

[1] Canh N. Nguyen, Glenn Bateman and Arnold H. Kritz, "Simulation of saturated tearing modes in tokamaks," Phys. Plasmas **11** (2003) 3460–3471.

[2] Glenn Bateman and R.N. Morris, "Saturated tearing modes in toroidal geometry," Phys. Fluids **29** (1986) 753–761.

[3] R. B. White, D. A. Monticello, M. N. Rosenbluth, and B. V. Waddell, Phys. Fluids **20 (1977) 800.**

[4] G. Bateman, "MHD Instabilities," The MIT Press, Cambridge MA, (1978).

[5] S. Hamada, Nuclear Fusion 2 (1962) 23.

[6] J. M. Greene and J. L. Johnson, Phys. Fluids 5 (1962) 510.

[7] W. D. Dhaeseler, W. N. G. Hitchon, J. D. Callen, and J. L. Shohet, "Flux Coordinates and Magnetic Field Structure," Springer-Verlag, Germany, (1991).

[8] R. N. Morris, "Saturated Tearing Modes in Low Aspect Ratio Tokamaks," Ph.D. Thesis, Georgia Institute of Technology (1984).

[9] L. F. Shampine and M. K. Gordon, "Computer Simulation of Ordinary Differential Equations: The Initial Value Problem," W.H. Freeman and Company, San Francisco, (1991).

[10] R. Fitzpatrick, Phys. Plasmas, 2 (1995) 825.

[11] J. Wesson, "Tokamaks," Clarendon Press, Oxford, (1997).