2-FLUID AND FLR OPTIONS FOR NIMROD

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Motivation

- Understand plasma dynamics using fluid models
- What are 2-fluid effects?
- When is MHD valid?
- Can MHD be “extended”?
- What are the computational difficulties?

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Approach

- Assume closure at Π and $\mathbf{q}$
- Understand fluid momentum balance
- Write equations in terms of non-dimensional variables
- Try to understand role of different terms by relative ordering of non-dimensional coefficients

Two-Fluid Equations (m_e=0, n_e=n_i=n)

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{V}_i = -\nabla \cdot n\mathbf{V}_e$$

$$mn\frac{d\mathbf{V}_i}{dt} = -\nabla p_i + ne\left(\mathbf{E} + \mathbf{V}_i \times \mathbf{B}\right) - \nabla \cdot \Pi_i$$

$$0 = -\nabla p_e - ne\left(\mathbf{E} + \mathbf{V}_e \times \mathbf{B}\right) - \nabla \cdot \Pi_e$$

$$\mathbf{J} = ne\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

$$+ \text{Closures, Energy equation, Faraday, and Ampere}$$

Non-dimensional Equations

$$\varepsilon = \frac{\omega}{\Omega} \quad , \quad \xi = \frac{V_0}{V_{thi}} \quad , \quad \delta = \frac{\rho_i}{L} << 1 \quad , \quad \beta = \left(\frac{V_{thi}}{V_A}\right)^2 = \frac{\delta}{\xi}$$

$$E_0 = V_0 B_0 \quad , \quad J_0 = n_0 e V_0 \quad , \quad p_0 = m n_0 V_{thi}^2$$

$$\varepsilon \frac{\partial n}{\partial t} = -\xi\delta\nabla\cdot n\mathbf{V}_i = -\xi\delta\nabla\cdot n\mathbf{V}_e$$

$$\varepsilon\xi\frac{\partial\mathbf{V}_i}{\partial t} + \xi^2\delta\mathbf{V}_i\cdot\nabla\mathbf{V}_i = -\frac{1}{n}\delta\left(\nabla p_i + \frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i\right) + \xi\left(\mathbf{E} + \mathbf{V}_i\times\mathbf{B}\right) \quad ,$$

$$\xi\mathbf{E} = -\xi\mathbf{V}_e\times\mathbf{B} - \frac{1}{n}\delta\left(\nabla p_e + \frac{\Pi_{e0}}{p_0}\nabla\cdot\Pi_e\right)$$

$$\varepsilon\frac{\partial\mathbf{B}}{\partial t} = -\xi\delta\nabla\times\mathbf{E} \quad , \quad \mathbf{J} = \xi\nabla\times\mathbf{B} \quad , \quad \mathbf{J} = n\left(\mathbf{V}_i - \mathbf{V}_e\right)$$

Equation of Motion and Generalized Ohm’s Law

$$\underbrace{\xi \mathbf{J}\times\mathbf{B} - \frac{1}{n}\delta\nabla p}_{\text{"Equilibrium" forces}} = \underbrace{n\left(\varepsilon\xi\frac{\partial \mathbf{V}_i}{\partial t} + \xi^2\delta\mathbf{V}_i\cdot\nabla\mathbf{V}_i\right) - \frac{1}{n}\delta\frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i}_{\text{Dynamical response}}$$

$$\underbrace{\xi\left(\mathbf{E}+\mathbf{V}_i\times\mathbf{B}\right)}_{\text{Ideal MHD}} = \underbrace{\xi\frac{1}{n}\mathbf{J}\times\mathbf{B} - \delta\frac{1}{n}\left(\nabla p_e + \frac{\Pi_{e0}}{p_0}\nabla\cdot\Pi_e\right)}_{\text{2-fluid and FLR effects}}$$

$\mathbf{V}_i\times\mathbf{B}$ and $\mathbf{J}\times\mathbf{B}$ enter formally at the same order

Non-dimensional Stress Tensor

$$\frac{\Pi_{i0}}{p_{i0}}\nabla\cdot\Pi_i = \xi\delta\left[\frac{1}{\nu/\Omega}\nabla\cdot\Pi_{\|} + \nabla\cdot\Pi_{gv} + \frac{\nu}{\Omega}\nabla\cdot\Pi_{\perp}\right]$$

"Banana" regime: $\quad \nu/\Omega << \varepsilon_A^{3/2}\left(\omega_b/\Omega\right) \sim \varepsilon_A^{3/2}\delta/q \quad \rightarrow \quad \nu/\Omega \sim \delta^2$

Neo - classical parallel viscous force :Ź $\left\langle \mathbf{B}\cdot\nabla\cdot\Pi_i^{nc}\right\rangle = mn\left\langle B^2\right\rangle \mu_i \frac{V_{\theta i}}{B_\theta}\mathbf{e}_\theta$

$$\mu \sim \varepsilon_A^{1/2}\nu \quad \rightarrow \quad \frac{\Pi_0^{nc}}{p_0} = \frac{\xi}{\delta}\varepsilon_A^{1/2}\frac{\nu}{\Omega} \sim \varepsilon_A^{1/2}\xi\delta$$

Artificial numerical viscous force: $\quad \frac{\Pi_0^{visc}}{p_0}\nabla\cdot\Pi^{visc} = -n\mu_A\nabla^2\mathbf{V}$

$$\frac{\Pi_{i0}}{p_0}\nabla\cdot\Pi_i = -n\mu_A\nabla^2\mathbf{V}_i + \xi\delta\left[\nabla\cdot\Pi_i^{gv} + \varepsilon_A^{1/2}\mathbf{bb}\cdot\nabla\cdot\Pi_i^{nc} + \delta^2\nabla\cdot\Pi_{\perp i}\right] \quad .$$

General Force Balance

$$\xi \mathbf{J} \times \mathbf{B} - \delta \nabla p = n\left(\varepsilon \xi \frac{\partial \mathbf{V}_i}{\partial t} + \xi^2 \delta \mathbf{V}_i \cdot \nabla \mathbf{V}_i \right)$$
$$+ \xi \delta^2 \left(\frac{1}{\nu / \Omega} \nabla \cdot \Pi_{\parallel} + \nabla \cdot \Pi_i^{gv} + \frac{\nu}{\Omega} \nabla \cdot \Pi_{\perp} \right)$$
$$+ \xi \frac{\mu}{\Omega} \mathbf{bb} \cdot \nabla \cdot \Pi_i^{nc} - n\mu_A \nabla^2 \mathbf{V}_i \quad .$$

Fast Ordering

$$\varepsilon = \omega / \Omega \sim 1 \quad , \qquad \xi = V / V_{thi} \sim 1/\delta, \quad \Rightarrow \beta \sim \delta^2 \qquad \nu / \Omega \sim \delta \qquad \text{(Classical)}$$

$$\frac{\partial n}{\partial t} = -\nabla \cdot n\mathbf{V}_i$$

$$\mathbf{J} \times \mathbf{B} = n\frac{d\mathbf{V}_i}{dt} + \delta\left(\nabla \cdot \Pi_{\|} + \mathbf{bb} \cdot \nabla \cdot \Pi_i^{nc}\right) + \frac{1}{n}\delta^2\left(\nabla p + \nabla \cdot \Pi_{gv}\right) + O(\delta^3)$$

$$d/dt = \partial / \partial t + \mathbf{V}_i \cdot \nabla$$

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n}\mathbf{J} \qquad \text{(2 fluids)}$$

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \frac{1}{n}\mathbf{J} \times \mathbf{B} + O(\delta^2)$$

- ŹŹVery low β (Poor "confinement")
- $V \sim V_{thi}/\delta$ (Fast flows)ŹŹ,ŹŹŹŹŹ$\omega \sim \Omega$ŹŹŹŹŹ(High frequency)
- Unbalanced forces $\sim O(1)$, $\mathbf{J} \times \mathbf{B} = n\frac{d\mathbf{V}_i}{dt} + \delta\nabla^2\mathbf{V}_i + O(\delta^2)$ "Force - free"
- $\mathbf{J} \times \mathbf{B}$ should be retained in Ohm's law $\Rightarrow$ "Hall MHD"

MHD Ordering

Fast flows, low frequencies, low β: $\varepsilon \sim \delta, \quad \xi \sim 1, \quad \beta \sim \delta$

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V}_i = 0$$

$$\mathbf{J} \times \mathbf{B} = \delta\left(n\frac{d\mathbf{V}_i}{dt} + \nabla p + \nabla \cdot \Pi_{\|} + \mathbf{bb} \cdot \nabla \cdot \Pi_i^{nc} \right) + \delta^2 \nabla \cdot \Pi_i^{gv} + O(\delta^4)$$

$$\mathbf{V}_{e\|} = \mathbf{V}_{i\|} - \frac{1}{n}\mathbf{J}_{\|} \quad \text{(2 fluids in parallel direction)}$$

$$\mathbf{V}_{\mathrm{e}\perp} = \mathbf{V}_{i\perp} + O(\delta) = \mathbf{V}_E \quad \text{(common } \mathbf{E} \times \mathbf{B} \text{ drift)}$$

$$\mathbf{E} = -\mathbf{V}_i \times \mathbf{B} + \frac{1}{n}\underbrace{\mathbf{J} \times \mathbf{B}}_{O(\delta)} - \delta\frac{1}{n}\nabla p_e = -\mathbf{V}_i \times \mathbf{B} + O(\delta)$$

$$\mathbf{V}_i = \mathbf{V}_{\|i} + \mathbf{V}_E + O(\delta) \quad , \qquad \mathbf{V}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

- $V \sim V_{thi}$ (Fast flows) , $\omega \sim \delta\Omega$ (Low frequency) , $\beta \sim \delta$ (Low β)
- Force balance ("force free equilibrium") to $O(\delta)$
- Hall and electron diamagnetic terms removed by force balance
- Ideal MHD

Drift Ordering

Very low frequencies and slow flows: $\varepsilon \sim \delta^2, \quad \xi \sim \delta, \quad \beta \sim O(1)$

$$-\nabla p + \mathbf{J} \times \mathbf{B} = \delta^2 \left(n \frac{d\mathbf{V}_i}{dt} + \nabla \cdot \Pi_i^{gv} + \mathbf{bb} \cdot \nabla \cdot \Pi_i^{nc} - n\mu_A \nabla^2 \mathbf{V}_i \right) + O(\delta^4)$$

$$\mathbf{E} = -\mathbf{V}_e \times \mathbf{B} - \frac{1}{n} \nabla p_e$$

$$\mathbf{V}_i = \mathbf{V}_E + \mathbf{V}_{*i} + \mathbf{V}_{\|i} + O(\delta^2) \quad , \qquad \mathbf{V}_{*i} = \frac{1}{nB^2} \mathbf{B} \times \nabla p_i$$

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n} \mathbf{J}$$

Force balance to $O(\delta^2)$ (Good confinement)

Lowest order FLR corrections

Two fluids

$\nabla \cdot \Pi^{gv}$ and $\nabla \cdot \Pi^{nc}$ enter at same order as $\mathbf{V} \cdot \nabla \mathbf{V}$

Drift Model

Velocity transformation: $\mathbf{V}_i = \mathbf{V}_{\parallel i} + \mathbf{V}_E + \mathbf{V}_{*i} + O(\delta^2)$

$$\mathbf{V}_e = \mathbf{V}_i - \frac{1}{n}\mathbf{J} = \mathbf{V}_{\parallel i} + \mathbf{V}_E + \mathbf{V}_{*i} - \frac{1}{n}\mathbf{J} + O(\delta^2)$$

$$\mathbf{E} = -\left(\mathbf{V}_E + \mathbf{V}_{*i} - \frac{1}{n}\mathbf{J}_\perp\right)\times\mathbf{B} - \frac{1}{n}\nabla p_e + O(\delta^2) \quad ,$$

$$= -\mathbf{V}_E\times\mathbf{B} - \frac{1}{n}\nabla_\parallel p_e + \frac{1}{n}\underbrace{\left(-\nabla_\perp p + \mathbf{J}\times\mathbf{B}\right)}_{O(\delta^2)} + O(\delta^2) \quad ,$$

$$= -\mathbf{V}_E\times\mathbf{B} - \frac{1}{n}\nabla_\parallel p_e$$

$$\delta^2\left(n\frac{d}{dt}\left(\mathbf{V}_{\parallel i} + \mathbf{V}_E\right) + \underbrace{n\frac{d\mathbf{V}_{*i}}{dt} + \nabla\cdot\Pi_i^{gv}\left(\mathbf{V}_i\right)}_{\text{GVC}}\right) = -\nabla p + \mathbf{J}\times\mathbf{B} - \delta^2\mathbf{b}\cdot\nabla\cdot\Pi_i^{nc}$$

$$+\delta^2 n\mu_A\nabla^2\mathbf{V}_i + O(\delta^4)$$

$$d/dt = \partial/\partial t + \mathbf{V}_i\cdot\nabla$$

Gyro-viscous Cancellation

Gyroviscous force and diamagnetic advective acceleration almost cancel!

$$n\left(\frac{\partial \mathbf{V}_{*i}}{\partial t} + \mathbf{V}_i \cdot \nabla \mathbf{V}_{*i}\right) + \nabla \cdot \Pi_i^{gv}(\mathbf{V}_i) \approx \nabla \chi - \mathbf{b} n \mathbf{V}_{*i} \cdot \nabla V_{\parallel i}$$

$$\chi = -p_i \mathbf{b} \cdot (\nabla \times \mathbf{V}_{\perp i})$$

$$\begin{aligned}
\nabla \cdot \Pi_i^{gv} &\sim \nabla \cdot \left[p(\mathbf{b} \times \nabla \mathbf{V}_i)\right] \quad , \\
&\sim \nabla p \cdot (\mathbf{b} \times \nabla)\mathbf{V}_i \quad , \\
&= -(\mathbf{b} \times \nabla p) \cdot \nabla \mathbf{V}_i \quad , \\
&\sim -n\mathbf{V}_{*i} \cdot \nabla \mathbf{V}_i
\end{aligned}$$

Drift Model

$$\frac{\partial n}{\partial t}+\nabla\cdot n\mathbf{V}=-\nabla\cdot n\mathbf{V}_{*i}$$

$$n\delta^2\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)=-\underbrace{\delta^2 n\mathbf{V}_{*i}\cdot\nabla\mathbf{V}_\perp}_{New}-\delta^2\mathbf{bb}\cdot\nabla\cdot\Pi_i^{nc}-\nabla\left(p+\underbrace{\delta^2}_{New}\chi\right)+\mathbf{J}\times\mathbf{B}+\delta^2 n\mu_A\nabla^2\mathbf{V}$$

$$\mathbf{E}=-\mathbf{V}\times\mathbf{B}-\underbrace{\frac{1}{n}\nabla_\parallel p_e}_{New}$$

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}\quad,\qquad\mathbf{J}=\delta\nabla\times\mathbf{B}$$

$$\mathbf{V}=\mathbf{V}_E+\mathbf{V}_{\parallel i}$$

Form very similar to MHD equations

No whistler waves due to force balance

Applies only to well - confined plasmas

Kinetic Alfvén Waves

No whistlers, but.... $\mathbf{E} = -\mathbf{V} \times \mathbf{B} - \underbrace{\frac{1}{n} \nabla_{\|} p_e}_{\text{kinetic Alfvˇn waves}}$

$$\omega^2 = k_{\|}^2 V_A^2 \left(1 + k^2 \frac{V_A^2}{\Omega_i^2} \right) \qquad \text{Whistler}$$

$$\omega^2 = k_{\|}^2 V_A^2 \left(1 + k_{\perp}^2 \frac{V_{th*}^2}{\Omega_i^2} \right) \qquad \text{KAW} \qquad \left(V_{th*}^2 = T_e / m_i \right)$$

FLR modifications to Alfvˇn waves

Whistler removed in drift ordering

KAW dispersive with increasing frequency

Dispersive Wave Operators

$$\text{Whistler:} \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{V_A^2}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla\times\nabla\times\mathbf{B}$$

$$\text{KAW:} \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = \left(\frac{V_A V_{th*}}{\Omega}\right)^2 (\mathbf{b}\cdot\nabla)^2 \nabla\times(\mathbf{bb}\cdot\nabla\times\mathbf{B})$$

Explicit treatment can severly limit time step

4th order operator difficult to invert

Templates for SI operators

Can whistler SI operator stabilize KAW?

Is a separate KAW operator needed?

Present subject of research by Sovinec, Tian, and Barnes

Sugiyama-Park Drift Model

$$\mathbf{V}_i = \mathbf{V}_E + \mathbf{V}_{di} + \mathbf{V}_{\parallel i} = \mathbf{V} + \mathbf{V}_{di} \qquad \text{Exact!}$$

$\mathbf{V}_{di}$ contains all the ion drifts

$$\mathbf{V}_e = \mathbf{V}_E + \mathbf{V}_{*e} + \mathbf{V}_{\parallel e} = \mathbf{V}_i - \frac{1}{n}\mathbf{J} \quad , \qquad \left(\mathbf{V}_{*e} = -\frac{1}{nB^2}\mathbf{B}\times\nabla p_e \right)$$

$$= \mathbf{V}_{\parallel i} - \frac{1}{n}\mathbf{J}_{\parallel} + \mathbf{V}_E + \mathbf{V}_{di} - \frac{1}{n}\mathbf{J}_{\perp}$$

$$\rightarrow \quad \mathbf{V}_{\parallel e} = \mathbf{V}_{\parallel i} - \frac{1}{n}\mathbf{J}_{\parallel} \quad , \qquad \mathbf{V}_{di} = \frac{1}{n}\mathbf{J}_{\perp} + \mathbf{V}_{*e} \sim \mathbf{V}_{*i} + O(\delta^2)$$

$$\mathbf{E} = -\mathbf{V}\times\mathbf{B} - \frac{1}{n}\nabla_{\parallel} p_e \qquad \text{Ohm's law is exact!}$$

$$n\delta^2\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V}\cdot\nabla\mathbf{V}\right) = -\delta^2 n \underbrace{\mathbf{V}_{di}}_{\mathbf{V}_{di}=\mathbf{V}_{*i}+O(\delta^2)} \cdot\nabla\mathbf{V}_{\perp} - \delta^2\mathbf{b}\cdot\nabla\cdot\Pi_i^{nc} - \nabla\left(p+\delta^2\chi\right) + \mathbf{J}\times\mathbf{B} + \delta^2 n\mu_A\nabla^2\mathbf{V}$$

Summary of Fluid Models

Model	V_i	ω	β	$\mathbf{J} \times \mathbf{B}$		
Hall MHD	V_{thi} / δ	Ω_{ci}	$O(\delta^2)$	$mn \frac{d\mathbf{V}_i}{dt} + O(\delta)$		
Ideal MHD	V_{thi}	$\delta\Omega_{ci}$	$O(\delta)$	$O(\delta)$		
Drift	δV_{thi}	$\delta^2\Omega_{ci}$	$O(1)$	$\nabla p + O(\delta^2)$		

Whistler waves are high frequency phenomena that disappear as the frequency is ordered successively lower.

Kinetic Alfvˇn waves are finite pressure phenomena that appear as β becomes successively larger.

- Models of increasingly better confined plasmas
- Drift model applicable to very well confined plasmas (e.g., tokamaks)

Two-fluid Options

Use SP drift model for ions: $\mathbf{V}_{di} = \frac{1}{n}\mathbf{J}_{\perp} + \mathbf{V}_{*e}$

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{V} = -\nabla \cdot n\mathbf{V}_{di}$$

$$mn\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V}\right) = -mn\mathbf{V}_{di} \cdot \nabla \mathbf{V}_{\perp} - \nabla\left(p + \chi\right) + \mathbf{J} \times \mathbf{B} + mn\mu_A \nabla^2 \mathbf{V}$$

Options for OhmÕs law:

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} - \frac{1}{ne}\nabla_{\|} p_e \hat{\mathbf{Z}}$$

Stablilze KAW with new SI operator

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} + \frac{1}{ne}\left(-\nabla p_e + \mathbf{J} \times \mathbf{B}\right)$$

Stablilze KAW and whistlers with whistler SI operator and use V_* in EOM

$$\mathbf{E} = -\mathbf{V} \times \mathbf{B} - \frac{1}{ne}\nabla_{\|} p_e \hat{\mathbf{Z}}$$

Stablilze KAW with whistler SI operator

Discussion

- S&P drift model for ions is relatively small modification to present NIMROD formulation
  - Diamagnetic drift terms are additive
  - No need to explicitly calculate the full gyro-viscous force
  - Applicable primarily to tokamaks
- One option for Ohm's law:
  - Use S&P Ohm's law to eliminate whistlers and modified SI operator to stabilize KAW
  - Under study
- Other options:
  - Use full Ohm's law and whistler SI operator for both KAW and whistlers
  - Use S&P Ohm's law and stabilize KAW with whistler operator
  - Use full Ohm's law and gyroviscous stress (for non-tokamak applications)