Pressure anisotropy closure.

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Theses

Usually, $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle$ is introduced in the fluid moment equations. Extensions of $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle$ in toroidal geometries have been made.

- $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle \ll \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}$
- We obtained a closure for $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle$ in a dynamic situation.
- The stress tensor can be calculated for $\epsilon \ll 1$.
- The viscous force presents sigularities at field maxima.

Outline

1. Motivation, pressure anisotropy and $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle$.
2. Chapman-Enskog summary.
3. Closure for $\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\rangle$ for dynamic cases.*
4. Closure for $\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}$.
5. Claculation of $\Pi_{\|}$ and progress in the dynamic case.
7. Summary.

*Previous work (“Time-dependent neoclassical viscosity”) is available in the CPTC website (UW-CPTC 04-6) and has been submitted to Phys. Plasmas.

- Consider the momentum balance equation

$$mn\frac{d\mathbf{V}}{dt} = nq(\mathbf{E} + \mathbf{V} \times \mathbf{B}) - \nabla p - \nabla \cdot \mathbf{\Pi}_{\|} + \mathbf{F}_0.$$

- The viscous drive is predominantly in the parallel (to $\mathbf{B}$) direction.

$$\Pi_{\|} \sim \mathcal{O}\left(\rho^0\right) \quad \Pi_{\wedge} \sim \mathcal{O}\left(\rho^1\right) \quad \Pi_{\perp} \sim \mathcal{O}\left(\rho^2\right).$$

- Thus, the relevant dynamics are contained in

$$mn\frac{d}{dt}\left(V_{\|}B\right) = nq\left(E_{\|}^{A}B - \mathbf{B} \cdot \nabla\phi\right) - \mathbf{B} \cdot \nabla p - \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} - nq\frac{J_{\|}B}{\sigma_{\|}}.$$

- When flux-surface-averaged, the total parallel momentum equation gives the evolution of the parallel flow

$$mn\left\langle B^2\right\rangle \frac{\partial U\left(t\right)}{\partial t} = -\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|}\right\rangle.$$

- And from the full electron momentum equation, one can calculate an electrical conductivity

$$m_e\frac{\partial\left\langle BJ_{\|}\right\rangle}{\partial t} = -n_e e^2\left\langle E_{\|}B\right\rangle - e\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|\mathbf{e}}\right\rangle + \nu_e m_e\left\langle BJ_{\|}\right\rangle.$$

The parallel viscous force has averaged and varying parts.

- Consider for simplicity the magnetic field model

$$B(\theta) = B_{\text{min}}\left[1 + 2\epsilon\,\sin^2(\theta/2)\right]$$

- In this simple geometry, the viscous force is:

$$\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|} = \frac{2}{3}\frac{\partial\Pi_{\|}}{\partial\theta} - \frac{1}{B}\Pi_{\|}\frac{\partial B}{\partial\theta}$$

- Note that, for $\Pi_{\|}$ periodic

$$\left\langle\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle \simeq 2\epsilon\left\langle\Pi_{\|} sin\theta\right\rangle \sim \mathcal{O}\left(\sqrt{\epsilon}\right)$$

$$\left\langle\Pi_{\|} sin\theta\right\rangle \sim \mathcal{O}\left(1/\sqrt{\epsilon}\right)$$

- For $\left\langle\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle \neq 0$, $\Pi_{\|}$ is odd and can be written as

$$\Pi_{\|} = \sum_{n=1}^{\infty} a_n sin\,(n\theta)$$

- Since $a_1 = 2\left\langle \mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle/2\epsilon$

$$\frac{1}{B}\frac{\partial B}{\partial\theta}\Pi_{\|} = \left(\sqrt{2\epsilon} sin^2\theta\right) 2.92mnU + 2\epsilon sin\theta\sum_{n=2}^{\infty} a_n sin\left(n\theta\right)$$

  and

$$\frac{\partial\Pi_{\|}}{\partial\theta} = \left(\frac{1}{\sqrt{2\epsilon}}cos\theta\right) 2.92mnU + \sum_{n=2}^{\infty} a_n ncos\left(n\theta\right)$$

- Since $n > 2\epsilon$,

$$\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_{\|} \simeq \frac{2}{3}\frac{\partial\Pi_{\|}}{\partial\theta}$$

- The term that survives the flux-surface-average is smaller than the spatial varying component.

- **The part that one misses when considering flux-surface-averaged quantities has at least one term $\mathcal{O}\left(1/\epsilon\right)$ larger than the average!**

Chapman-Enskog (CE).

- Simplified DKE for the distortion ($f \equiv f_M + F$)

$$\frac{dF}{dt} + v_\parallel \mathbf{b}\cdot\nabla\left(F + \frac{m}{T} v_\parallel B U f_M\right) - \frac{\nu_\perp}{2}\mathcal{L}\left(F\right) = \frac{v_\parallel}{p}\left(\mathbf{b}\cdot\nabla\cdot\mathbf{\Pi}_\parallel\right) f_M .$$

- The kinetic distortion is expanded as follows

$$F = F_0 + \nu_* F_1 + \ldots$$

where $\nu_* = \nu_\perp / \epsilon^{3/2}\omega_b \ll 1$ in the banana regime.

- Usually, to solve for F_0 a bounce average is calculated.

- The lowest order solution is in terms of averaged quantities and from it one obtains (for $\partial/\partial t \ll \omega_b$)

$$\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel \right\rangle \simeq mn\mu \left\langle B^2 \right\rangle U$$

where $U\left(\psi\right) = V_\parallel\left(\theta\right)/B\left(\theta\right)$.

- The solution in the dynamic case is in terms of Cordey eigenfunctions which satisfy

$$\langle C_R(\Lambda_n)\rangle \propto \kappa_n \Lambda_n,$$

- The closure obtained is

$$\left\langle \mathbf{B}\cdot\widehat{\nabla\cdot\mathbf{\Pi}_{\|}}\right\rangle = nm\upsilon(\omega)\left\langle B^2\right\rangle \widehat{U}$$

- Quantities are in terms of the eigenfunctions ($\bar{\nu}\equiv\nu_\perp/2$)

$$\upsilon(\omega) = \int d^3v\,\frac{\nu_\perp}{v_{th}^2}\frac{v^2}{3}\frac{f_M}{n}\frac{\widehat{f}_t(v,\omega)}{\widehat{f}_c(v,\omega)}, \qquad \frac{\widehat{f}_t(v,\omega)}{\widehat{f}_c(v,\omega)} \sim \left(1-\frac{i\omega}{\bar{\nu}}\right)\sum \frac{\eta_n\int_0^{\lambda_c}\Lambda_n d\lambda}{\kappa_n - i\omega/\bar{\nu}},$$

$$\eta_n = \frac{2}{\varsigma v}\frac{\int_0^{\lambda_c}\Lambda_n d\lambda}{\int_0^{\lambda_c}\Lambda_n^2\left\langle B/v_{\|}\right\rangle d\lambda}.$$

- The Laplace tranform can be inverted analitically in a small ϵ expansion

$$\begin{aligned}\left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_{\|}\right\rangle &= \left\langle B^2\right\rangle mn\int d^3v\,\frac{v^2}{3}\frac{f_M}{n}\frac{\nu_\perp}{v_{th}^2}\left\{U(t)\,f_t + \frac{2}{\nu_\perp}\frac{\partial U(t)}{\partial t}\left(1-\sum\gamma_n\right)\right. \\ &\left. + \sum\frac{\gamma_n}{\kappa_n}(\kappa_n-1)^2\int_0^t \frac{dU}{d\tau}e^{-\bar{\nu}\kappa_n(t-\tau)}d\tau\right\}.\end{aligned}$$

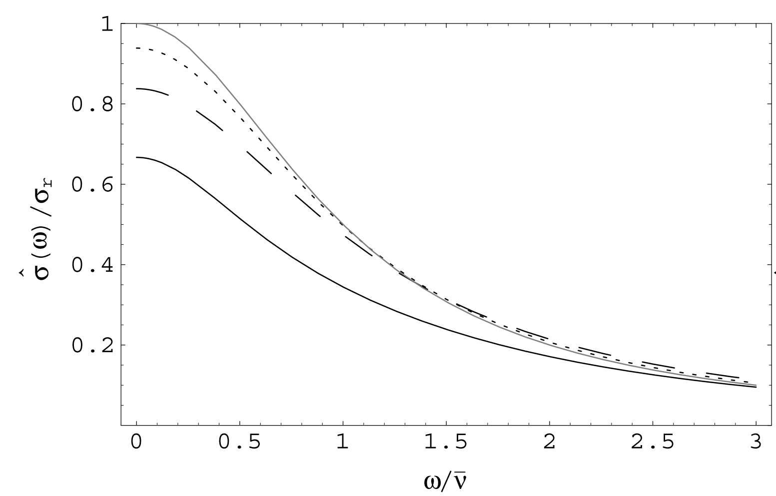
- With the closure obtained, the parallel flow evolution yields an integral equation

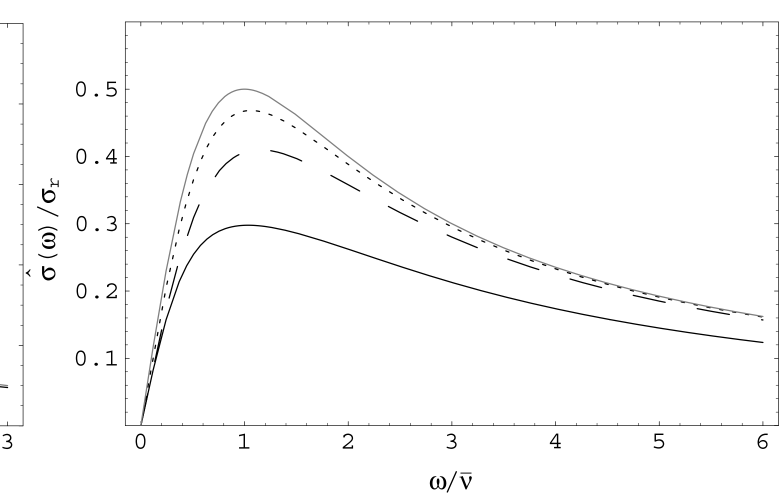
$$U(t) = h(t) + \int_0^t K(t;\tau)\, U(\tau)\, d\tau.$$

- Also, a frequency-dependent electrical conductivity can be calculated from electron parallel momentum balance:

$$\widehat{\sigma}(\omega) = \sigma_r \left\{ 1 + \frac{1}{\nu_e} \left[\upsilon(\omega) - i\omega \right] \right\}^{-1}, \qquad \sigma_r = \frac{n_e e^2}{m_e \nu_e}.$$

- The real (left) and imaginary (right) parts of $\widehat{\sigma}/\sigma_r$ for $\epsilon = 0$ (gray), $\epsilon = 10^{-3}$ (dotted), $\epsilon = 10^{-2}$ (dashed) and $\epsilon = 0.1$ (solid).





Spatial variation of the viscous force

- The first term that has non-zero P_2 moment is $F_1(\lambda, v, \theta)$ which can be obtained by integrating the 1st order DKE

$$F_1 \simeq \int_0^\theta \frac{C(F_0)}{v_\parallel} d\theta + \frac{2}{3B}\frac{f_M}{p}\Pi_\parallel + h\left(\psi, \lambda, v\right) + \mathcal{O}\left(\sqrt{\epsilon}\right).$$

- The parallel viscous force can then be written as

$$\mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel = \left\langle \mathbf{B}\cdot\nabla\cdot\mathbf{\Pi}_\parallel \right\rangle \left[f_1\left(\theta\right) + f_2\left(\theta\right)\right].$$

- The geometric factors are given by in terms of elliptic functions (E and K).

$$f_1\left(\theta\right) \propto \frac{1}{\sqrt{2\epsilon}}\frac{1}{f_t}\left\{\int_0^1 \frac{ds}{\left|1 - s^2 sin^2\theta\right| E\left(s\right)}\left[1 + \frac{3}{2}\frac{s^2 sin\left(2\theta\right) E\left(\theta, s\right)}{\left(1 - s^2 sin^2\theta\right)^{3/2}}\right] - 1 - \frac{sin\left(2\theta\right) E\left(\theta, 1\right)}{2\left(1 - sin^2\theta\right)^{3/2}}\right\} \sim \mathcal{O}\left(\right.$$

$$f_2\left(\theta\right) \propto \sqrt{2\epsilon}\frac{sin\left(2\theta\right)}{f_t}\left\{\frac{E\left(\theta, 1\right)}{\sqrt{1 - sin^2\theta}} - \int_0^1 \frac{ds}{E\left(s\right)}\frac{E\left(\theta, s\right)}{\left(1 - s^2 sin^2\theta\right)^{3/2}}\right\} \sim \mathcal{O}\left(1\right).$$

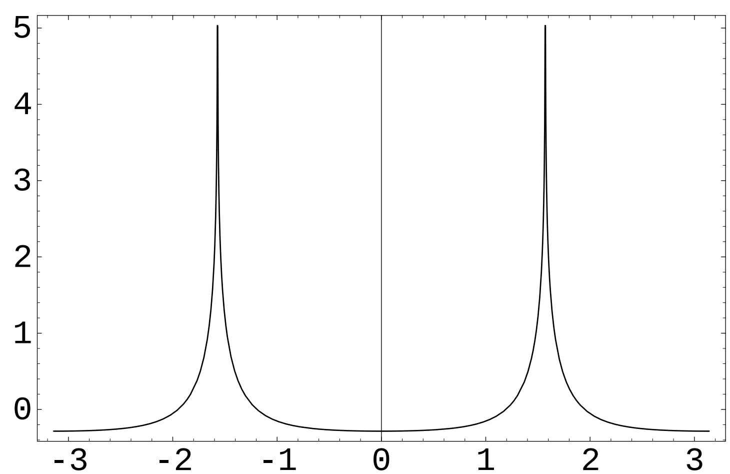
$$\mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} = \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle [f_1(\theta) + f_2(\theta)] \, .$$

- Note that

$$f_1(\theta) \sim 1/cos^2\theta\epsilon, \qquad f_2(\theta) \sim sin^2\theta,$$

- It can be verified that $\langle f_1(\theta) \rangle = 0$ and $\langle f_2(\theta) \rangle = 1$.

- The result has $1/cos^2$ singularities arising from integrating up to the boundary at $s = 1$.



$f_1(\theta)$ vs. θ.

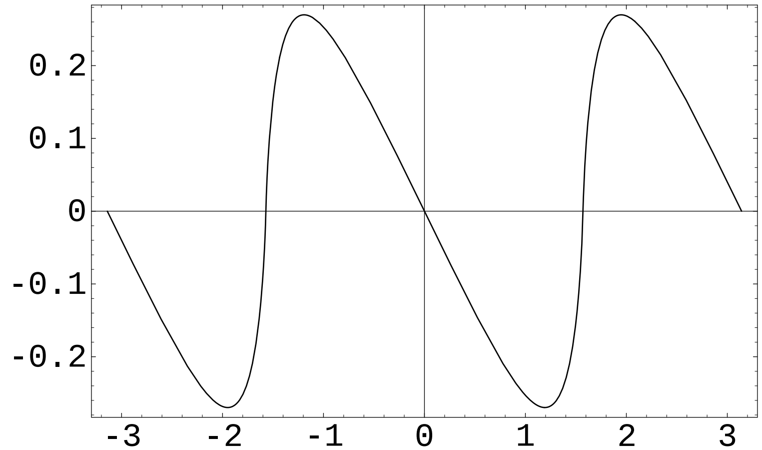
- Considering $B = B_{min}\left[1+2\epsilon\tau(\theta)\right]$ with $\sqrt{\epsilon} \ll 1$ [defining $\zeta(A) = \int_0^\theta (A/B)\,d\theta$]

$$\Pi_\parallel \simeq \frac{1}{\sqrt{2\epsilon}}\frac{f_c}{f_t} mn\mu U \left\langle B^2 \right\rangle \int \frac{1}{\sqrt{1+s^2\tau(\theta)}}\frac{s^3 ds}{(s^2+2\epsilon)^{3/2}} \times$$

$$\frac{1}{\left\langle \sqrt{1+s^2\tau(\theta)} \right\rangle}\left[\frac{\zeta\left(\sqrt{1+s^2\tau(\theta)}\right)}{\left\langle \sqrt{1+s^2\tau(\theta)} \right\rangle}\left\langle \frac{B}{\sqrt{1+s^2\tau(\theta)}} \right\rangle - \zeta\left(\frac{B}{\sqrt{1+s^2\tau(\theta)}}\right)\right]$$

- For the bumpy cylinder magnetic field $\tau(\theta) = \sin^2\theta$ and the solution is

$$\Pi_\parallel = \frac{1}{\sqrt{2\epsilon}}\frac{f_c}{f_t} mn\mu U \left\langle B^2 \right\rangle \left\{ \int_0^1 \frac{ds}{E(s)} \frac{E(\theta,s)}{(1-s^2 sin^2\theta)^{3/2}} - \frac{E(\theta,1)}{\sqrt{1-sin^2\theta}} \right\}.$$



Dynamic pressure anisotropy

- Retaining the time derivative term in the 1st order unaveraged DKE

$$F_1 = \int_0^\theta d\theta \frac{C(\widehat{F_0})}{v_{||}} + i\omega \int_0^\theta d\theta \frac{1}{v_{||}} \widehat{F_0}.$$

- Taking the P_2 moment we get, for small ϵ,

$$\Pi_{||} \simeq \frac{1}{2\epsilon} nm\widehat{U} \sum_1^\infty \eta_n I_n(\theta) \int d^3v \frac{\bar{\nu}}{v_{th}^2} v^2 \frac{f_M}{n} \frac{1}{\widehat{f_c}} \frac{1}{\kappa_n - i\omega/\bar{\nu}}$$

$$I_n(\theta) = \frac{1}{4}\left\{ \int \frac{s^2 E(\theta, s)}{(1 - s^2 sin^2\theta)^{3/2}} \frac{\partial \Lambda_n}{\partial s} ds - \frac{E(\theta, 1)}{\sqrt{1 - sin^2\theta}} \left(\frac{\partial \Lambda_n}{\partial s}\right)_1 \right\}$$

- Note that, since the eigenfunction equation for Λ_n is for $\langle C_R(\Lambda_n)\rangle$, there is no trivial way of introducing the eigenvalue or using the orthogonality condition.

Summary

- We previously obtained a time-dependent closure for $\left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle$ and explored the dynamics of parallel flow damping and the electrical conductivity.

- The variation of the pressure anisotropy and the viscous force within a flux surface were calculated in a small ϵ approximation.

- For $t \gg \nu$ (steady state) the parallel viscous force can be written as

$$\begin{array}{ccccccc} \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} & = & \left\langle \mathbf{B} \cdot \nabla \cdot \mathbf{\Pi}_{\|} \right\rangle & \{ f_1(\theta) & + & f_2(\theta) \} \\ 1/\sqrt{\epsilon} & & \sqrt{\epsilon} & \sim 1/\epsilon cos^2\theta & & \sim sin^2\theta \\ & & & \langle f_1(\theta) \rangle = 0 & & \langle f_2(\theta) \rangle = 1 \end{array}$$

Issues / Future work

- In the dynamic case, F_0 is given in terms of Cordey eigenfunctions. Orthogonality conditions only apply for flux-surface-averaged quantities.

- The θ-dependence of the $\Pi_{\|}$ results in a coefficient $I_n(\theta)$. Can it be numerically evaluated?