EXTENDED MHD EQUATIONS[1]

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Theses:[2]

- **Most derivations of plasma MHD equations neglect dissipative effects (e.g., ideal MHD) or use collisional equations, closures (e.g., Braginskii).**

- **Extended MHD (ExMHD) equations developed from two-fluid equations with general closures for $\vec{q}$ and $\overset{\leftrightarrow}{\pi}$ provide a reasonable basis for describing macroscopic plasmas — for arbitrary collisionality regimes along $\vec{B}$.**

- **Closures for $\vec{q}$ and $\overset{\leftrightarrow}{\pi}$ are very anisotropic and must be developed with drives induced by $\vec{\nabla}\rho_m, \vec{\nabla}\vec{V}, \vec{\nabla}P$; different procedures should be used in developing and implementing parallel, cross and perpendicular closures.**

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[2] **Abridged version of poster LP1.00062 at Denver DPP-APS meeting, 24–28 October 2005.**

Outline

- **Goals, assumptions of Extended MHD (ExMHD)**
- **Derivation Of ExMHD equations:**
  - **Complete two-fluid equations**
  - **Extended MHD equations**
  - **Anisotropic nature of closures**
  - **Approximations used in deriving collisional closures**
- **Various types of closures for ExMHD:**
  - **Moment approach — for Spitzer problem, Braginskii closures**
  - **Parallel kinetics — drift-kinetic C-E equation; PIC-based and "continuum" solutions**
  - **Perpendicular closures via fluid moments — diamagnetic/gyroviscosity, perpendicular**
  - **Can microturbulence effects be included?**
- **Complete set of ExMHD equations, specifications**
- **Summary**

Goal — Of Extended MHD (ExMHD) Equations

- **The fundamental goals in developing Extended MHD equations are to:**

  Develop MHD-like equations that accurately model macroscopic plasma behavior in magnetized ($\omega_c >> \nu$, $|\varrho\vec{\nabla}| << 1$) plasmas — for analytics and initial value simulations,

  Allow for arbitrary collisionality along the magnetic field $\vec{B}$ — i.e., $|\lambda_e \nabla_\parallel| \gtrsim 1$ as well as the usual collisional (Braginskii) collisional regime ($|\lambda_e \nabla_\parallel| << 1$),

  Incorporate any needed kinetic effects via closure relations that are obtained from moments of solutions of kinetic descriptions which are consistent with the extended MHD equations — i.e., that are obtained from a Chapman-Enskog-type procedure.

- **Extended MHD equations should include the following MHD models:**

  Ideal MHD — MHD equations with no dissipation $\Longrightarrow$ isentropic equation of state,

  Resistive MHD — MHD equations including dissipation due to plasma resistivity η,

  Reduced MHD – resistive MHD equations with compressional Alfvén waves removed,

  Neclassical MHD — MHD equations including poloidal flow damping, increased perpendicular inertia, and bootstrap current through parallel viscous forces on ions, electrons,

  Electron and ion diamagnetic flow (“two-fluid”) effects — i.e., inclusion of ω_{*e}, ω_{*i}.

Assumptions For Extended MHD Equations, Simulations

- **Some assumptions will be made to develop an Extended MHD model:**

  Plasma has evolved for many collision times before simulation begins so lowest order kinetic distribution is a Maxwellian. Equilibrium flows are assumed to be subsonic.

  In $P_1(\vec{v}/v_T)$ moments, only flow and heat flow will be kept (neglect higher order flows).

  Macroscopic instabilities evolve as plasma is driven slowly through instability threshold.

  Classical, neoclassical and paleoclassical models provide minimum plasma transport.

  Microturbulence is also in steady-state and representable by transport it induces via $\vec{\nabla}\cdot\vec{q}$, $\vec{\nabla}\cdot\overleftrightarrow{\pi}$ — assume no significant flow in $\vec{k}$ space between macro and micro instabilities.

  Sources (e.g., heating) and sinks (e.g., neutrals) are relevant on transport time scale.

- **These assumptions preclude considering the following physical processes:**

  Open field line regions — unless one adds appropriate parallel boundary conditions,

  Velocity-space loss-cones near divertor separatrix — would need "direct-loss" terms,

  Nonaxisymmetric effects of sources and sinks on the $t \lesssim 1/\nu$ time scale,

  Order unity pressure anisotropy — instead, $(p_\parallel - p_\perp)/p \sim \epsilon_\perp << 1$ is being assumed,

  High $\vec{k}, \omega$ microturbulence and any resultant filamentation in velocity space,

  Finite ion gyroradius effects beyond second order (gyroviscosity with diamagnetic flows and $\perp$ viscosity) $\Longrightarrow$ limits poloidal mode numbers to $m \lesssim (r/\varrho_i)^3/S \sim 30$–$100$(ITER)?

  Multi-scale interactions of macro and micro instabilities, except via transport induced by microinstabilities through $\vec{\nabla}\cdot\vec{q}$ and $\vec{\nabla}\cdot\overleftrightarrow{\pi}$.

Extended MHD (ExMHD) Model Has Some New Features

- The collisional friction force density $\vec{R}_e$ is not just the resistivity but includes an electron heat flow $\vec{q}_e$ — i.e., $\vec{R}_e = \dfrac{n_e e}{\sigma_0}\left(\vec{J} + \dfrac{3e}{5T_e}\vec{q}_e\right)$.

- The closure relations for $\vec{q}$ and $\overleftrightarrow{\pi}$ (for both electrons and ions) are left in general form — to use different closures for different problems.

- The extreme anisotropy (parallel, cross, perpendicular to $\vec{B}$) in the closure relations is emphasized — to reflect different physics in each direction.

- Different procedures are proposed for obtaining parts of closure relations — kinetics for parallel but fluid for cross and perpendicular directions.

- An attempt is made to include the dissipative, transport effects of microturbulence — via averaging over short scale processes and assuming they can be separated from the macroscopic (ExMHD) processes.

Fluid Moment Equations From Plasma Kinetic Equation

- The rigorous Plasma Kinetic Equation (PKE) to begin from is

$$\boxed{\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{\nabla}_v f = \mathcal{C}\{f\}.}$$

- Exact fluid moment equations for each plasma species result from velocity-space moments ($\int d^3v\,\vec{v}^{\,n},\ n = 0, 1, 2$) of this fundamental kinetic equation:

$$n = 0, |\vec{v}|^{\,0}, \text{density} \qquad \frac{\partial n}{\partial t} + \vec{\nabla}\cdot n\vec{V} = 0, \qquad \{\vec{\nabla}\vec{V}\} \equiv \frac{1}{2}\left[\vec{\nabla}\vec{V} + (\vec{\nabla}\vec{V})^T\right] - \frac{1}{3}\overleftrightarrow{\mathrm{I}}(\vec{\nabla}\cdot\vec{V}),$$

$$n = 1, \vec{v}, \text{momentum} \qquad mn\frac{d\vec{V}}{dt} = nq(\vec{E} + \vec{V}\times\vec{B}) - \vec{\nabla}p - \vec{\nabla}\cdot\pi + \vec{R}, \qquad \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}\cdot\vec{\nabla},$$

$$n = 2, v^2, \text{energy} \qquad \frac{3}{2}n\frac{dT}{dt} + nT\,\vec{\nabla}\cdot\vec{V} = -\vec{\nabla}\cdot\vec{q} - \overleftrightarrow{\pi} : \{\vec{\nabla}\vec{V}\} + Q, \qquad p \equiv nT,$$

$$\text{or entropy} \quad \frac{\partial(ns)}{\partial t} + \vec{\nabla}\cdot\left(ns\vec{V} + \frac{\vec{q}}{T}\right) = \frac{1}{T}(-\vec{q}\cdot\vec{\nabla}\ln T - \overleftrightarrow{\pi} : \{\vec{\nabla}\vec{V}\} + Q), \quad s \equiv \ln(T^{3/2}/n).$$

- These moment equations need closure moments for $\vec{q}$ and $\overleftrightarrow{\pi}$ ($\vec{v}_r \equiv \vec{v} - \vec{V}$):

$$\text{heat flux } \vec{q} \equiv \int d^3v\,\vec{v}_r\left(\frac{mv_r^2}{2} - \frac{5}{2}\right)f, \qquad \text{stress tensor } \overleftrightarrow{\pi} \equiv \int d^3v\,m\left(\vec{v}_r\vec{v}_r - \frac{v_r^2}{3}\overleftrightarrow{\mathrm{I}}\right)f.$$

Magnetized Plasmas Are Very Anisotropic ($\|$, $\wedge$ $\perp$, to $\vec{B}$)

- **Braginskii [1] used a Chapman-Enskog procedure and an ordering scheme for magnetized ($\omega_c \equiv qB/m >> \nu$), collisional ($\nu >> \omega, k_\| v_T$) plasmas:**

  $\perp$ **to $\vec{B}$: small gyroradius,** $\varrho \equiv v_T/\omega_c \implies \epsilon_\perp \sim |\varrho \vec{\nabla}_\perp| << 1.$

  $\|$ **to $\vec{B}$: short collision length,** $\lambda \equiv v_t/\nu \implies \epsilon_\| \sim |\lambda \nabla_\|| << 1.$

- **Conductive heat flux closure moment is found to have parallel ($\|$), cross ($\wedge$, in flux surface) and perpendicular ($\perp$, across flux surfaces) components:**

  $$\vec{q} = -\,n\,\chi_\| \nabla_\| T - n\,\chi_\wedge (\vec{B}/B)\times\vec{\nabla}T - n\,\chi_\perp \vec{\nabla}_\perp T, \quad \text{in which } \vec{\nabla}_\perp \equiv -\,(1/B^2)\vec{B}\times(\vec{B}\times\vec{\nabla}),$$

  parallel heat conduction: $\chi_\| \sim \nu\lambda^2 \sim \epsilon_\|^2 \epsilon_\perp^0 \implies$ **fast ($t \sim 1/\nu$), $\|$ T_e equilibration,**

  cross (diamagnetic heat flow): $\chi_\wedge \sim v_T \varrho \sim \epsilon_\perp \implies$ **slower, diamag. flows in surface,**

  perpendicular heat conduction: $\chi_\perp \sim \nu\varrho^2 \sim \epsilon_\perp^2 \implies$ **slowest, radial heat transport.**

- **Stress tensor has similar form:** $\overleftrightarrow{\pi} = \overleftrightarrow{\pi}_\| + \overleftrightarrow{\pi}_\wedge + \overleftrightarrow{\pi}_\perp$ **with similar scalings** $\overleftrightarrow{\pi}_\| \sim \epsilon_\|^2 \epsilon_\perp^0$ **(parallel stress),** $\overleftrightarrow{\pi}_\wedge \sim \epsilon_\perp$ **(gyroviscosity) and** $\overleftrightarrow{\pi}_\perp \sim \epsilon_\perp^2$ **($\perp$ visc.).**

[1] **S.I. Braginskii, in** *Reviews of Plasma Physics* **(Consultants Bureau, NY, 1965), Vol I, p 205.**

Comments On Collisional Magnetized Plasma Equations

- **Braginskii collisional closures and equations are derived using the following major approximations, which determine their range of validity:**

  short collision length, $\epsilon_{\|} \sim \lambda \nabla_{\|} << 1$ — not valid for most tokamak plasma regimes,

  small gyroradius, $\epsilon_{\perp} \sim \varrho \vec{\nabla}_{\perp} << 1$ — equil. ok, but need $k_{\perp} \varrho << 1$ for perturbations,

  slow processes, $\partial / \partial t << \nu$ — equilibrium ok, but need $\omega / \nu << 1$ for perturbations,

  negligible anomalous transport — add transport coefficients from microturbulence?

- **Critiques of the Braginskii equations:**

  They neglect effects due to collisions with neutrals or energetic (e.g., fast ion) particles — but these transport-time-scale (slow) effects can mostly just be added as "sources."

  They do not include direct loss processes (e.g., near separatrix, on open field lines).

  The stress tensor $\overleftrightarrow{\pi}$ is driven not just by $\{\vec{\nabla}\vec{V}\}$ but also by a comparable $\{\vec{\nabla}\vec{q}\}$ [2].

[2] **A.B. Mikhailovskii,** *Theory of Plasma Instabilities* **(Atomdat, Moscow, 1977), Vol 2, p 307-325 (in Russian); A.B. Mikhailovskii and V.S. Tsypin, Plasma Physics 13, 785 (1971);** *ibid.*, **Beitr. Plasmaphys. 24, 335 (1984).**

Extended MHD Model Derived From Two-Fluid Equations

- Assume for the moment that anisotropic closures for $\vec{q}$ and $\overleftrightarrow{\pi}$ can be obtained for both electrons and ions for relevant situations.

- Then, adding, subtracting electron and ion density and momentum equations one obtains general "Extended MHD" equations:

$$
\begin{array}{lll}
\text{density} & \dfrac{\partial \rho_m}{\partial t} + \vec{\nabla}\cdot\rho_m\vec{V} = 0, & \rho_m \equiv \dfrac{\sum_s n_s m_s}{\sum_s m_s} \simeq n_i, \;\; \vec{V} \equiv \dfrac{\sum_s n_s m_s \vec{V}_s}{\sum_s n_s m_s} \simeq \vec{V}_i, \\
\text{charge density} & \vec{\nabla}\cdot\vec{J} = 0, & \vec{J} \equiv e(n_i Z_i \vec{V}_i - n_e \vec{V}_e), \\
\text{momentum} & \rho_m \dfrac{d\vec{V}}{dt} = \vec{J}\times\vec{B} - \vec{\nabla}P - \vec{\nabla}\cdot\overleftrightarrow{\Pi}, & P \equiv p_e + p_i, \;\; \overleftrightarrow{\Pi} \simeq \overleftrightarrow{\pi}_i + \overleftrightarrow{\pi}_e \simeq \overleftrightarrow{\pi}_i, \\
\text{Ohm's law} & \vec{E} + \vec{V}\times\vec{B} = \underbrace{\dfrac{\vec{R}_e}{n_e e}}_{\sim\,\eta\vec{J}} + \underbrace{\dfrac{\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overleftrightarrow{\pi}_e}{n_e e}}_{\text{Hall terms}} + \underbrace{\dfrac{m_e}{e^2}\dfrac{d}{dt}\left(\dfrac{\vec{J}}{n_e}\right)}_{\text{electron inertia}}. &
\end{array}
$$

- Main effects of closures come in parallel Ohm's law and equation of state for the total plasma pressure P obtained from plasma entropy evolution:

$$
\frac{d}{dt}\left(\ln\frac{P}{\rho_m^{\Gamma}}\right) = \frac{\Gamma-1}{P}\left(p_e\frac{ds_e}{dt} + p_i\frac{ds_i}{dt}\right) \simeq \frac{\Gamma-1}{P}\Big(-\underbrace{\vec{\nabla}\cdot\vec{q}_e}_{\sim\nu_e\epsilon_\parallel^2} - \underbrace{\{\vec{\nabla}\vec{V}_i\}:\overleftrightarrow{\pi}_i}_{\sim\nu_i\epsilon_\perp^2} + \underbrace{\eta J^2}_{\sim 1/\tau_E}\Big), \quad \Gamma \equiv \frac{5}{3}.
$$

Comments On Closures For Extended MHD Equations

- **The main limitation in using Braginskii closures is the high collisionality requirement for the parallel kinetics: $\epsilon_\parallel \sim (v_T/\nu)\nabla_\parallel = \lambda\nabla_\parallel << 1$.**

- **The closures should be determined from a Chapman-Enskog-type procedure so the kinetics used to obtain them does not produce "extra" $\partial\delta n/\partial t$, $\partial\delta\vec{V}/\partial t$, and/or $\partial\delta T/\partial t$ contributions to the equations:**

  The usual drift-kinetic and gyro-kinetic equations are not developed using a Chapman-Enskog-like procedure and hence usually produce δn, $\delta\vec{V}$, and/or δT terms.

  Formal Chapman-Enskog-type procedures and resultant drift-kinetic equations have been developed for arbitrary ∥ collisionality [3-5], but they are rather complicated.

- **The anisotropic components of the closures can be handled differently:**

parallel:	**in general a kinetic analysis must be used, including collisional effects,**
cross:	**fluid-type analysis, gyroviscosity for these diamagnetic flow type effects,**
perpendicular:	**fluidlike radial transport due to collisional effects on diamagnetic flows.**

[3] K.C. Shaing and D.A. Spong, Phys. Fluids B 2, 1190 (1990) — first Chapman-Enskog-like formalism.
[4] J.P. Wang and J.D. Callen, Phys. Fluids B 4, 1139 (1992) — axisymmetric $\vec{B}$, neoclassical formalism.
[5] Z. Chang and J.D. Callen, Phys. Fluids B 4 1167 (1992) — sheared slab model, with Landau damping.

Comments On Closures For Extended MHD (continued)

- **Friction forces $\vec{R}$ and stress tensors $\overset{\leftrightarrow}{\pi}$ are most fundamentally, generally written in terms of $\vec{V}$ and $\vec{q}$ — rather than $\vec{V}$ and $\vec{\nabla} T$ Braginskii uses.**

- **The parallel Ohm's law is governed experimentally by the neoclassical Ohm's law and apparently not affected [6,7] by microturbulence — because $k_\parallel << k_\perp$ and hence their parallel momentum transfer is small**

- **Temporal regimes — it seems there are two MHD regimes of interest:**

  "fast MHD" ($\omega >> \nu$) — little entropy production, closures not very important?

  "slow MHD" ($\omega << \nu$) — collision-dominated closures and dissipation critical.

- **Spatial regimes — very anisotropic and different physics each direction:**

  parallel: need more general kinetic-based formalism, closures for $k_\parallel \lambda \sim 1$

  cross (in flux surface): need separation from drift-wave-type microturbulence
  $\Longrightarrow k_\theta \varrho_S < 0.3? \Longrightarrow$ poloidal mode numbers $m \lesssim 0.3\, r/\varrho_S \sim$ 30–100 (ITER)?

  perpendicular (across flux surface): avoid FLR effects on resistive layer widths
  $\Longrightarrow k_x \varrho_i < 1$ with $\delta_\eta \sim r/(mS)^{1/3} \Longrightarrow mS \lesssim (r/\varrho_i)^3 \sim 10^6$–$3 \times 10^7$(ITER)?

[6] **K.C. Shaing, Phys. Fluids 31, 8 (1988) — for electrostatic microturbulence.**

[7] **F.L. Hinton, R.E. Waltz, and J. Candy, Phys. Plasmas 11, 2433 (2004) — including $\delta \vec{B}_\perp$ effects.**

Moment Expansion Solution Of "Kinetic" Spitzer Problem

- **Electron flow, current induced by electric field is called Spitzer problem:**

$$\frac{q_e}{m_e}\vec{E}\cdot\frac{\partial f_M}{\partial\vec{v}} = \mathcal{C}\{\delta f\} \quad\Longrightarrow\quad \delta f = -\,\mathcal{C}^{-1}\left\{\frac{q_e\vec{v}\cdot\vec{E}}{T}f_M\right\} \quad\Longrightarrow\quad \vec{J} = q_e\int d^3v\,\vec{v}\,\delta f \equiv \sigma_{\rm Sp}\vec{E}.$$

- **In moment approach one takes $\int d^3v\,\vec{v}\,L_i^{(3/2)}$ moments of kinetic equation and obtains a matrix equation to be solved for $\vec{V}$, $\vec{q}$, etc. induced by $\vec{E}$:**

$$n_e e\begin{pmatrix}\vec{E}\\ 0\\ \vdots\end{pmatrix} = -\,\frac{m_e n_e \nu_e}{Z}\underbrace{\begin{pmatrix}\ell_{00} & \ell_{01} & \cdots\\ \ell_{10} & \ell_{11} & \cdots\\ \vdots & \vdots & \ddots\end{pmatrix}}_{L_{ij}}\begin{pmatrix}\vec{V}_e-\vec{V}_i\\ -\frac{2}{5n_eT_e}\vec{q}_e\\ \vdots\end{pmatrix} \quad\Longrightarrow\quad \left\{\begin{aligned}\vec{J} &\equiv -n_e e(\vec{V}_e-\vec{V}_i)\\ &= \frac{n_e e^2}{m_e\nu_e}Z[L_{ij}^{-1}]_{00}\vec{E}.\end{aligned}\right.$$

- **One can show [8] that inverting the friction matrix L_{ij} yields variational solution of Spitzer problem and hence plasma electrical conductivity σ:**

1×1 matrix inversion yields $\sigma_0 \equiv \dfrac{n_e e^2}{m_e\nu_e}$, which is reference ($\perp$) conductivity,

2×2 matrix inversion yields $\sigma_{\rm Sp} = \dfrac{1}{\alpha_e}\sigma_0$, $\alpha_e = \frac{\sqrt{2}+Z}{\sqrt{2}+13Z/4} \lesssim 5\%$ accuracy ($< \frac{1}{\ln\Lambda} \sim 0.07$),

3×3 matrix inversion yields $\sigma_{\rm Sp} = \dfrac{1}{\alpha_e}\sigma_0$, with $\lesssim 1\%$ accuracy in α_e ($\simeq 0.51$ for $Z=1$).

[8] S.P. Hirshman, 21, 1295 (1978) — variational solution of Spitzer problem via moments.

Comments On Moment Approach Solutions Of Kinetics

- **Moment approach matrix solution of Spitzer problem also produces electron heat flux induced by the electric field, $\frac{2}{5n_eT_e}\vec{q}_e = \frac{Ze}{m_e\nu_e}[L_{ij}^{-1}]_{01}\vec{E}$.**

This is a key contribution to Spitzer conductivity (with $\geq 2\times 2$ matrix inversion) since it converts $\|$ friction force $R_{e\|}$ from reference ($\perp$) to $\|$ Spitzer electrical conductivity:

$$\vec{R}_e = -m_en_e\nu_e\Big[(\vec{V}_e-\vec{V}_i)-\frac{3}{5n_eT_e}\vec{q}\Big] = \frac{n_ee}{\sigma_0}\Big(\vec{J}-\frac{9Z/4}{\sqrt{2}+Z}\sigma_0\vec{E}\Big) \Longrightarrow \vec{J} = \sigma_0\Big(\frac{\sqrt{2}+13Z/4}{\sqrt{2}+Z}\Big)\vec{E} = \sigma_{\mathrm{Sp}}\vec{E}.$$

- **Braginskii collisional closures were obtained using moment approach:**

Effects of all "forces" ($\vec{E}$, $\vec{\nabla}\ln p$, $\vec{\nabla}\ln T$, $\{\vec{\nabla}\vec{V}\}$) were determined simultaneously $\Longrightarrow$ Onsager symmetry, thermal force effect ($0.71\nabla_{\|}T_e$), Ettinghausen effect, etc.;

4×4 matrix inversion was used for accurate numerical coefficients;

However, really only need 2×2 approach for order $1/\ln\Lambda \sim 5\%$ accuracy in resistivity but factor of 2 accuracy in thermal diffusivity χ — need 3×3 for similarly accurate χ.

Complete Set Of Extended MHD (ExMHD) Equations

- **Combining plasma fluid and Maxwell's equations, one obtains the complete set of "Extended MHD" equations:**

density	$\frac{\partial \rho_m}{\partial t} + \vec{\nabla}\cdot\rho_m \vec{V} = 0,$	
momentum	$\rho_m \frac{d\vec{V}}{dt} = \vec{J}\times\vec{B} - \vec{\nabla}P - \vec{\nabla}\cdot \overset{\leftrightarrow}{\Pi},$	$P \equiv p_e + p_i, \;\; \overset{\leftrightarrow}{\Pi} \simeq \overset{\leftrightarrow}{\pi}_i + \overset{\leftrightarrow}{\pi}_e \simeq \overset{\leftrightarrow}{\pi}_i,$
magnetic field	$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E},$	$\vec{J} = \vec{\nabla}\times\vec{B}/\mu_0, \;\; \vec{\nabla}\cdot\vec{B} = 0,$
Ohm's law	$\vec{E} = -\vec{V}\times\vec{B} + \frac{\vec{R}_e}{n_e e} + \frac{\vec{J}\times\vec{B} - \vec{\nabla}p_e - \vec{\nabla}\cdot\overset{\leftrightarrow}{\pi}_e}{n_e e} + \frac{m_e}{e^2}\frac{d}{dt}\left(\frac{\vec{J}}{n_e}\right).$	
Eq. of state	$\frac{d}{dt}\left(\ln\frac{P}{\rho_m^{\Gamma}}\right) = \frac{\Gamma - 1}{P}\left(p_e \frac{ds_e}{dt} + p_i \frac{ds_i}{dt}\right), \quad \Gamma = \frac{5}{3}.$	

Additional Specifications Needed For ExMHD Equations

- **Electron temperature T_e, pressure $p_e = n_e T_e$, flow $\vec{V}_e \equiv -\vec{J}/n_e e + \vec{V}_i$:**

$$\frac{dT_e}{dt} \equiv \frac{\partial T_e}{\partial t} + \vec{V}_e \cdot \vec{\nabla} T_e = \frac{2}{3} T_e \left(-\vec{\nabla} \cdot \vec{V}_e + \frac{ds_e}{dt} \right) \Longrightarrow \frac{3}{2} \frac{\partial p_e}{\partial t} = -\vec{V}_e \cdot \vec{\nabla} p_e - \vec{\nabla} \cdot \left(\frac{5}{2} p_e \vec{V}_e \right) + p_e \frac{ds_e}{dt}$$

- **Electron entropy $s_e \equiv \ln (T_e^{3/2}/n_e)$:**

$$\frac{ds_e}{dt} \equiv \frac{\partial s_e}{\partial t} + \vec{V}_e \cdot \vec{\nabla} s_e = -\,(\vec{\nabla} \cdot \vec{q}_e + \overleftrightarrow{\pi}_e : \vec{\nabla} \vec{V}_e - Q_e)/n_e T_e$$

- **Ion temperature T_i, pressure $p_i = n_i T_i$, flow $\vec{V}_i \simeq \vec{V}$:**

$$\frac{dT_i}{dt} \equiv \frac{\partial T_i}{\partial t} + \vec{V}_i \cdot \vec{\nabla} T_i = \frac{2}{3} T_i \left(-\vec{\nabla} \cdot \vec{V}_i + \frac{ds_i}{dt} \right) \Longrightarrow \frac{3}{2} \frac{\partial p_i}{\partial t} = -\vec{V}_i \cdot \vec{\nabla} p_i - \vec{\nabla} \cdot \left(\frac{5}{2} p_i \vec{V}_i \right) + p_i \frac{ds_i}{dt}$$

- **Ion entropy $s_i \equiv \ln (T_i^{3/2}/n_i)$:**

$$\frac{ds_i}{dt} \equiv \frac{\partial s_i}{\partial t} + \vec{V}_i \cdot \vec{\nabla} s_i = -\,(\vec{\nabla} \cdot \vec{q}_i + \overleftrightarrow{\pi}_i : \vec{\nabla} \vec{V}_i - Q_i)/n_i T_i$$

- **Collisional friction force $\vec{R}_e$:**

$$\vec{R}_e = \frac{n_e e}{\sigma_0} \left(\vec{J} + \frac{3e}{5T_e} \vec{q}_e \right), \qquad \sigma_0 \equiv \frac{n_e e^2}{m_e \nu_e}.$$

- **Collisional energy exchange:**

$$Q_e = -\,Q_i + \frac{1}{\sigma_0} \left(|\vec{J}\,|^2 + \frac{3e}{5T_e} \vec{J} \cdot \vec{q}_e \right), \qquad Q_i = \frac{3}{2} n_e \nu_e (T_e - T_i).$$

Extended MHD Equations Require Various Closures

- **Closures for $\vec{q}$ and $\overleftrightarrow{\pi}$ need to have their parallel ($\|$), cross ($\wedge$) and perpendicular ($\perp$) components specified:**

$$\vec{q} = \vec{q}_{\|} + \vec{q}_{\wedge} + \vec{q}_{\perp}, \qquad \text{and} \qquad \overleftrightarrow{\pi} = \overleftrightarrow{\pi}_{\|} + \overleftrightarrow{\pi}_{\wedge} + \overleftrightarrow{\pi}_{\perp}.$$

- **Parallel heat flow $\vec{q}_{\|} = q_{\|}\vec{b}$, $\vec{b} \equiv \vec{B}/B$, $q_{\|} \equiv -\int d^3v \, v_{\|} L_1^{3/2} F$ determined using F obtained solving a Chapman-Enskog-type drift kinetic equation [4,5]:**

$$\frac{\partial F}{\partial t} + v_{\|}\,\vec{b}\cdot\vec{\nabla}F = \mathcal{C}_R\{F\} + v_{\|}L_1^{(3/2)} f_M\,\vec{b}\cdot\vec{\nabla}T - \frac{m}{T}\left(v_{\|}^2 - \frac{v_{\perp}^2}{2}\right) f_M\,(\vec{b}\cdot\{\vec{\nabla}\vec{V}\}\cdot\vec{b}) + \cdots.$$

- **Various approaches used to obtain $q_{\|}$ from this parallel kinetic equation:**

Collisional regime (Braginskii) — neglect $\partial F/\partial t$, $v_{\|}\,\vec{b}\cdot\vec{\nabla}F$; invert collision operator;
Collisionless — linearize and obtain Hammett-Perkins [9] Landau-type closures [5,10];
PIC-type δf code (Barnes) — but higher order moments are "noisier?"
"Continuum" type solutions [11] — expand F in pitch-angle eigenfunctions of $\mathcal{C}_R\{F\}$.

- **Stress $\overleftrightarrow{\pi}_{\|} \equiv \pi_{\|}(\vec{b}\vec{b} - \overleftrightarrow{\mathrm{I}}/3)$, $\vec{b} \equiv \vec{B}/B$, $\pi_{\|} \equiv p_{\|} - p_{\perp} = \int d^3v \, m(v_{\|}^2 - v_{\perp}^2/2)F$ is also determined from the solution of the parallel kinetic equation [12] $\Longrightarrow$ neoclassical closures for $\pi_{\|}$ and $\langle \vec{B}\cdot\vec{\nabla}\cdot\overleftrightarrow{\pi}_{\|}\rangle$ for $k_{\|}v_T << \nu$.**

[9] G.W. Hammett and F.W. Perkins, Phys. Rev.Lett. 64, 3019 (1990) — simplest Landau closure for $q_{\|}$.
[10] A.I. Smolyakov, M. Yagi, J.D. Callen, Fields Inst. Comm. 46, 243 (2005) — neutral fluid nonlocal clos.
[11] E.D. Held *et al.*, Phys. Plasmas 11, 2419 (2004) — and references cited therein.
[12] E.D. Held, "Unified form for parallel ion viscous stress in magnetized plasmas," PoP 10, 4708 (2003).

Some Complications In Obtaining Parallel Closures

- **The "usual" Chapman-Enskog-like drift-kinetic equation (DKE) [4,5] has many ($\gtrsim$ 5) "drives" on its right side in terms of the form**

  $v_\parallel f_M \vec{B} \cdot \vec{\nabla} \cdot \overleftrightarrow{\pi} \Longrightarrow$ causes $\sim \sqrt{\epsilon}$ correction to parallel viscous forces,

  $v_\parallel R_{e\parallel} \Longrightarrow$ additional corrections to parallel flow?, part of Spitzer problem?

  dissipative $L_1^{(1/2)}$ terms due to $\overleftrightarrow{\pi} : \{\vec{\nabla}\vec{V}\}$, $\vec{\nabla} \cdot \vec{q}$ and $Q \Longrightarrow$ temperature change δT?

- **Also, $\epsilon_\perp^2$ additions to DKE — Catto & Simakov [13], paleoclassical [14].**

- **Unfortunately, to obtain parallel closures correct to $\mathcal{O}(\epsilon_\perp^2)$ one needs to keep many (most?) of the $\mathcal{O}(\epsilon_\perp)$ terms, particularly for 3D geometry.**

- **Shaing and Spong [3] have exhibited some of the 3D complications that arise in long collision length plasmas by obtaining a "local" closure relation for $\pi_\parallel \equiv p_\parallel - p_\perp$ in the plateau collisionality regime.**

- **Also, Shaing emphasizes that in general there are not enough free parameters in kinetic analysis to satisfy all Chapman-Enskog constraints $\Longrightarrow$ residual "extra" (but usually higher order) δn, δT, $\delta \vec{V}$ terms.**

[13] A.N. Simakov, P.J. Catto, Phys. Plasmas 12, 012105 (2005) — additional $\mathcal{O}(\epsilon_\perp^2)$ terms in DKE (for $\overleftrightarrow{\pi}_\wedge$).
[14] J.D. Callen, Phys. Rev. Lett. 94, 055002 (2005); Nucl. Fus. 45, 1120 (2005); Phys. Pl. 12, 092512 (2005).

Extended MHD Equations Require Various Closures (cont'd)

- **Components of $\vec{q}$ perpendicular to $\vec{B}$ can be obtained from the fluid moment equation for $\partial\vec{q}/\partial t$ [4], $\vec{R}_q \sim mn\nu[l_{10}(\vec{V}_e - \vec{V}_i) + l_{11}(-2\,\vec{q}_e/5p_e)]$:**

$$\frac{d\vec{q}}{dt} = \frac{\omega_c}{B}\vec{q}\times\vec{B} - \frac{5}{2}\frac{nT}{m}\vec{\nabla}T + \frac{T}{m}(\vec{\nabla}\cdot\overleftrightarrow{\Theta} + \vec{R}_q) - \cdots, \text{ which upon taking } \vec{B}\times \text{ yields}$$

$$\vec{q}_\wedge = \underbrace{\frac{5}{2}\frac{\vec{B}\times\vec{\nabla}T}{\omega_c}}_{\text{diamagnetic}} \sim \epsilon_\perp, \qquad \text{and} \qquad \vec{q}_\perp = \frac{1}{\omega_c B}\vec{B}\times\left[\frac{T}{m}\underbrace{(\vec{R}_q + \vec{\nabla}\cdot\overleftrightarrow{\Theta})}_{\text{classical + neo}} - \frac{d\vec{q}}{dt} + \cdots\right] \sim \epsilon_\perp^2.$$

- **A similar analysis of the $d\overleftrightarrow{\pi}/dt$ equation can be performed to yield**

$$\overleftrightarrow{\pi}_\wedge = \underbrace{\frac{2p}{\omega_c}\overleftrightarrow{\mathcal{K}}^{-1}\{\{\vec{\nabla}\vec{V}\} + \frac{4}{5nT}\{\vec{\nabla}\vec{q}\}\}}_{\text{gyroviscous stress}} \sim \epsilon_\perp, \qquad \text{and} \qquad \overleftrightarrow{\pi}_\perp = \underbrace{\frac{\nu_{\text{eff}}}{\omega_c}\overleftrightarrow{\mathcal{K}}^{-1}\{\overleftrightarrow{\pi}_\wedge\} + \cdots}_{\text{perpendicular stress}} \sim \epsilon_\perp^2,$$

in which the inverse tensor operator [15] $\overleftrightarrow{\mathcal{K}}^{-1}\{\overleftrightarrow{\mathrm{S}}\} = \frac{1}{4}\Big([\vec{b}\times\overleftrightarrow{\mathrm{S}}\cdot(\overleftrightarrow{\mathrm{I}} + 3\vec{b}\vec{b})] + \text{transpose}\Big)$.

- **Recently, Ramos [16] used fluid moments to obtain a compact form for the gyroviscous stress tensor $\overleftrightarrow{\pi}_\wedge$ for arbitrary magnetic geometry.**

[15] C.T. Hsu, R.D. Hazeltine and P.J. Morrison, Phys. Fluids 29, 1480 (1986) — see Appendix A.

[16] J.J. Ramos, "Fluid formalism for collisionless magnetized plasmas," Phys. Plasmas 12, 052102 (2005).

Can One Include Microturbulence Effects In ExMHD?

- **Effects of turbulence are usually at small scales:**

  Neutral fluid turbulence closure models seek Reynolds stress closures that represent non-dissipative transfer of energy to higher $\vec{k}$ in inertial range, then dissipation.

  Drift-wave-type turbulence has non-inertial unstable $\vec{k}$-space region ($k_\perp \varrho_i \sim 0.2$–$1$) but some mode coupling to other $\vec{k}$ space regions (e.g., to $k_\theta = k_\zeta = 0$ zonal flows) — can these reactive and dissipative effects be approximated by the transport they induce?

- **Some micoturbulence effects are included via the closure moments:**

  Zonal flow damping via neoclassical viscous damping of poloidal flow

  Nonambipolar radial particle flux and toroidal momentum damping via "anomalous" toroidal ion viscous force

  Radial electron heat transport via paleoclassical plus anomalous χ_e

  Radial ion heat transport via anomalous χ_i

- **By averaging over microscopic scales? — still being worked on:**

  Separate shorter wavelengths via $n = \bar{n} + \tilde{n}$ with $\bar{n}$ including all k up to say $k\varrho \lesssim 0.2$ and $\tilde{n}$ representing all higher k processes (i.e., microturbulence).

  Then, average over the short scale stuff to obtain effects of microturbulence in the macroscopic description: $\vec{\Gamma} = \langle \tilde{n} \tilde{\vec{V}} \rangle \simeq -D\vec{\nabla}\bar{n}$ which leads, for example, to a mass density equation $\quad \partial\bar{\rho}_m/\partial t + \vec{\nabla}\cdot(\bar{\rho}_m \bar{\vec{V}} + \vec{\Gamma}_\perp) = 0, \quad \vec{\Gamma}_\perp = \langle \tilde{\rho}_m \tilde{\vec{V}}_\perp \rangle \sim -D_{\mu\mathrm{turb}} \vec{\nabla}_\perp \bar{\rho}_m.$

There Are Two Generic Types Of Extended MHD Problems

- **"Fast MHD" ($\omega > \nu_i \sim 10^3$ s^{-1}) phenomena occur on Alfvénic timescale:**

  Examples: sawtooth crashes, disruption precursors (DIII-D #87009), ELMs.

  Physically, need ideal MHD plus diamagnetic flow & gyroviscosity (two-fluid) effects — ω_{*i} stabilization for 1/1 sawtooth crashes, plus ω_{*e} for stabilizing high mode numbers.

  Dissipative closures operate on longer time scales ($t > 1/\nu$) and hence are negligible — except for parallel T_e equilibration in irregular magnetic fields, destabilizing resistivity effects, and possibly stabilizing diffusive effects on high mode numbers [17].

- **"Slow MHD" ($\omega < \nu_i \sim 10^3$ s^{-1}) phenomena occur on the resistive time scale and involve many physical processes:**

  Examples: $\Delta' > 0$ tearing modes, NTMs, RWMs.

  Neoclassical MHD effects important — poloidal flow damping $\Longrightarrow$ only toroidal flow, enhanced inertia (by $B^2/B_\theta^2 >> 1$), neoclassical parallel restivity, bootstrap current.

  Since nonlinear evolution (tearing modes $\Longrightarrow$ magnetic islands, RWMs $\Longrightarrow$ kink in plasma growing on wall time scale $\sim$ 10 ms) is on transport time scale, all transport effects are important — need complete ($\|$, $\wedge$, $\perp$) dissipative closures for $\vec{q}$, $\overleftrightarrow{\pi}$.

  Diamagnetic flow (ω_*) effects are ultimately not so critical — vanish on separatrix where $\vec{\nabla} P \to 0$, or just lead to slight changes in toroidal flow velocity.

  The second order (in gyroradius) effects are needed for perturbations, but they may not be needed for equilibrium since they represent negligible classical diffusion effects.

[17] B.A. Carreras, L. Garcia and P.H. Diamond, Phys. Fl. 30, 1388 (1987) — $\chi_\perp$, $\mu_\perp$ effects on res-g modes.

Discussion: Develop Closures For Classes Of Problems?

- **No general closures can be derived for long collision length λ regimes — because $\|$ kinetics depends on geometry over the collision length.**

- **Also, needed (for $\wedge$, $\perp$ closures) first and second order terms in finite gyroradius expansion are complicated and depend on gradients of $\vec{B}$.**

- **Thus, one is led to consider key closures needed for classes of problems:**

  <u>Fast MHD</u> ($\omega > \nu_i$)

  mainly just diamagnetic flows & gyroviscosity, but maybe with some diffusivities to stabilize high mode numbers.

  <u>Slow MHD</u> ($\omega < \nu_i$)

  tearing modes, NTMs — mainly just neo $\|$ viscous force (but local with dynamics),

  RWMs — mainly just equilibrium neoclassical parallel viscous force, plus toroidal flow damping induced as mode kinks the plasma and magnetic field.

Summary[3]

- **Extended MHD equations developed from two-fluid equations provide a reasonable basis for simulating macroscopic plasma behavior — if suitable closures for $\vec{q}$ and $\overleftrightarrow{\pi}$ are available and/or numerically implementable.**

- **Rigorous analysis in collisional regime (Braginskii) shows that in a magnetized plasma closures are very anisotropic — $\vec{q}_\parallel \sim \epsilon_\perp^0 \epsilon_\parallel^2$ (parallel), $\vec{q}_\wedge \sim \epsilon_\perp$ (diamagnetic), $\vec{q}_\perp \sim \epsilon_\perp^2$(perpendicular) and similarly for $\overleftrightarrow{\pi}$.**

- **Determinations of closure components depend on direction:**
  - **$\parallel$ (kinetic, $\epsilon_\parallel \sim |\lambda \nabla_\parallel| \gtrsim 1$) — parallel Chapman-Enskog-type drift-kinetic equation,**
  - **$\wedge$ (diamagnetic, $\epsilon_\perp \sim |\varrho \vec{\nabla}_\perp| << 1$) — can use $\vec{B}\times$ fluid moment equations for $\vec{q}$, $\overleftrightarrow{\pi}$,**
  - **$\perp$ (perp, $\epsilon_\perp^2 << 1$) — gyroradius smaller parts of $\vec{B}\times$ fluid moment equations.**

- **A comprehensive set of Extended MHD (ExMHD) equations and specifications are being developed — but closures or procedures for determining them are needed for them to be complete.**

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[3] **These viewgraphs will be available from** `http://w3.pppl.gov/CEMM/workshops.html`.