Fully Implicit Solution of the full 2-fluid 2D MHD equations using high-order C^1 finite Elements

S.C. Jardin, J. Breslau, J. Chen, N. Ferraro, and the M3D team
Princeton Plasma Physics Laboratory

Presented at

CEMM Meeting
Adams Mark Hotel, Denver CO]
Oct 22, 2005

Summary

- triangular finite element with fifth order accuracy and C^1 continuity
- compact: 3 unknowns per node: spatial derivatives up to 4th order.
- 2-part time-advance for Alfven and Whister: each uses Super-LU.
- Potential, stream function representation
- Non-trivial subsets exist of the full set of 8 equations exist
- GEM reconnection
- Wave Propagation tests
- Gyroviscous force added

7-field model has now been implemented in 2D

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}, \qquad \vec{J} = \nabla \times \vec{B}$$

$$\vec{E} + \vec{V} \times \vec{B} = \frac{1}{ne}\left(\vec{R} + \vec{J} \times \vec{B} - \nabla p_e - \nabla \bullet \overleftrightarrow{\Pi}_e\right)$$

$$nM_i\left(\frac{\partial \vec{V}}{\partial t} + \vec{V} \bullet \nabla \vec{V}\right) + \nabla p_e = \vec{J} \times \vec{B} - \nabla \bullet \overleftrightarrow{\Pi}_i$$

$$\frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \bullet \left(\frac{3}{2} p_e \vec{V}\right) = -p_e \nabla \bullet \vec{V} + \frac{\vec{J}}{ne} \bullet \left[\frac{3}{2}\nabla p_e + \vec{R} - \nabla \bullet \overleftrightarrow{\Pi}_e\right]$$

$$\frac{\partial n}{\partial t} + \nabla \bullet (n\vec{V}) = 0$$

$$\vec{V} = \nabla U \times \hat{z} + \nabla_{\perp} \chi + V_z \hat{z},$$

$$\vec{B} = \nabla \psi \times \hat{z} + I \hat{z}$$

$$p_e, n$$

$$\vec{R} = \eta n e \vec{J}$$

$$\overleftrightarrow{\Pi}_i = -\mu n\left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] + h\mu n \nabla^2 \left[\nabla \vec{V} + \nabla \vec{V}^{\dagger}\right] + GV$$

$$\overleftrightarrow{\Pi}_e = \lambda n e\left[\nabla \vec{J} + \nabla \vec{J}^{\dagger}\right]$$

Projections of the momentum equation:

$-\hat{z}\cdot\nabla\times$

$$n\nabla^2\dot{U}+n\left[\nabla^2 U,U\right]+n\left(\nabla^2 U,\chi\right)+n\nabla^2 U\nabla^2\chi-\mu n\nabla^4 U+\mu nh\nabla^4 w+\left[\psi,\nabla^2\psi\right]=0$$

$\hat{z}\cdot$

$$n\dot{v}_\varphi+n\left[v_\varphi,U\right]+n\left(v_\varphi,\chi\right)-\mu n\nabla^2 v_\varphi+\mu nh\nabla^4 v_\varphi+\left[\psi,I\right]=0$$

$\nabla\bullet$

$$n\nabla^2\dot{\chi}+n\left[\nabla^2\chi,U\right]+\tfrac{1}{2}n\nabla^2\left|\nabla\chi\right|^2+2n\left(U_{xy}^2-U_{xx}U_{yy}+\chi_{xy}[U_{yy}-U_{xx}]+U_{xy}[\chi_{xx}-\chi_{yy}]\right)$$
$$-2\mu n\nabla^4\chi+2\mu nh\nabla^4\Delta+\left(\nabla^2\psi,\psi\right)+\left(\nabla^2\psi\right)^2+\nabla^2\left(\tfrac{1}{2}I^2+p\right)=0$$

$$w\equiv\nabla^2 U,\quad \Delta\equiv\nabla^2\chi$$
$$\left[a,b\right]\equiv\hat{z}\bullet\nabla a\times\nabla b=a_x b_y-a_y b_x$$
$$\left(a,b\right)\equiv\nabla a\bullet\nabla b=a_x b_x+a_y b_y$$

Scalar Field Equations:

$$\dot{p}_e + [p_e, U] + (p_e, \chi) + \gamma p_e \nabla^2 \chi = \frac{1}{ne}[p_e, I] + S_e$$

$$\dot{\psi} + [\psi, U] + (\psi, \chi) = \eta \nabla^2 \psi - \lambda \nabla^4 \psi + \tfrac{1}{ne}[\psi, I]$$

$$\dot{I} + [I, U] + (I, \chi) + I \nabla^2 \chi + [\psi, v_\varphi] = \eta \nabla^2 I - \lambda \nabla^4 I + \tfrac{1}{ne}[\nabla^2 \psi, \psi]$$

$$S_e = \tfrac{2}{3}\left[\frac{1}{ne}\vec{J} \cdot (\vec{R} - \nabla \bullet \overleftrightarrow{\Pi}_e)\right]$$

Derivation of Implicit Equations

Taylor expand in time to get derivatives at advanced time. Use field equations to eliminate field time derivatives from momentum equation. For example:

$$
\begin{aligned}
& n\nabla^2\dot{U} + n\left[\nabla^2 U + \theta\delta t\nabla^2\dot{U}, U + \theta\delta t\dot{U}\right] + n\left(\nabla^2 U + \theta\delta t\nabla^2\dot{U}, \chi + \theta\delta t\dot{\chi}\right) \\
&\quad + n\left(\nabla^2 U + \theta\delta t\nabla^2\dot{U}\right)\left(\nabla^2\chi + \theta\delta t\nabla^2\dot{\chi}\right) + \left[\psi + \theta\delta t\dot{\psi}, \nabla^2\psi + \theta\delta t\nabla^2\dot{\psi}\right] \\
&\quad - \mu n(\nabla^4 U + \theta\delta t\nabla^4\dot{U}) = 0
\end{aligned}
$$

$$
\dot{\psi} + \left[\psi, U + \theta\delta t\dot{U}\right] + \left(\psi, \chi + \theta\delta t\dot{\chi}\right) = S_\psi
$$

Multiply by the time step, δt, and center the time derivatives about time *n+1/2,* so that , $\delta t\dot{U}_j = \left[U_j^{n+1} - U_j^n\right]$ etc.

Expand everything in *C*[1] finite elements: $U(x, y, t^n) = \sum_{j=1}^{18} \nu_j(x, y) U_j^n$

Multiply by each test function, integrate over domain, shift derivatives as needed, collect terms

M3D-C^1 code has full Extended MHD (2-fluid) equations with implicit differencing that allows time step to be determined by accuracy only:

$$\vec{V} = \nabla U \times \hat{z} + \nabla_\perp \chi + V_z$$
$$\vec{B} = \nabla \psi \times \hat{z} + I\hat{z}$$

$$\begin{bmatrix} S_{11}^v & S_{12}^v & S_{13}^v \\ S_{21}^v & S_{22}^v & S_{23}^v \\ S_{31}^v & S_{32}^v & S_{33}^v \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^v & D_{12}^v & D_{13}^v \\ D_{21}^v & D_{22}^v & D_{23}^v \\ D_{31}^v & D_{32}^v & D_{33}^v \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^v & R_{12}^v & R_{13}^v \\ R_{21}^v & R_{22}^v & R_{23}^v \\ R_{31}^v & R_{32}^v & R_{33}^v \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n}$$

Equations expressed in a form that allows non-trivial subsets of lower rank equations:

Alfven Wave physics

$$\begin{bmatrix} S_{11}^p & S_{12}^p & S_{13}^p \\ S_{21}^p & S_{22}^p & S_{23}^p \\ S_{31}^p & S_{32}^p & S_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n+1} = \begin{bmatrix} D_{11}^p & D_{12}^p & D_{13}^p \\ D_{21}^p & D_{22}^p & D_{23}^p \\ D_{31}^p & D_{32}^p & D_{33}^p \end{bmatrix} \bullet \begin{bmatrix} \psi \\ I \\ T_e \end{bmatrix}^{n} + \begin{bmatrix} R_{11}^p & R_{12}^p & R_{13}^p \\ R_{21}^p & R_{22}^p & R_{23}^p \\ R_{31}^p & R_{32}^p & R_{33}^p \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n+1} + \begin{bmatrix} Q_{11}^p & Q_{12}^p & Q_{13}^p \\ Q_{21}^p & Q_{22}^p & Q_{23}^p \\ Q_{31}^p & Q_{32}^p & Q_{33}^p \end{bmatrix} \bullet \begin{bmatrix} U \\ V_z \\ \chi \end{bmatrix}^{n}$$

Whistler, KAW, field diffusion physics

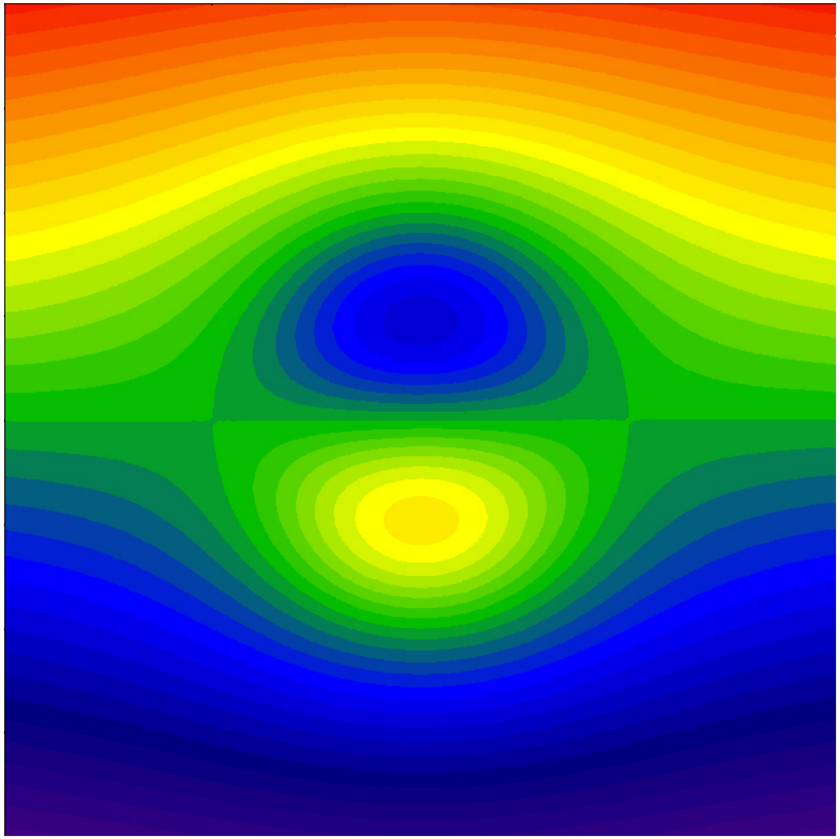
Phase-I: Reduced 2-field MHD:

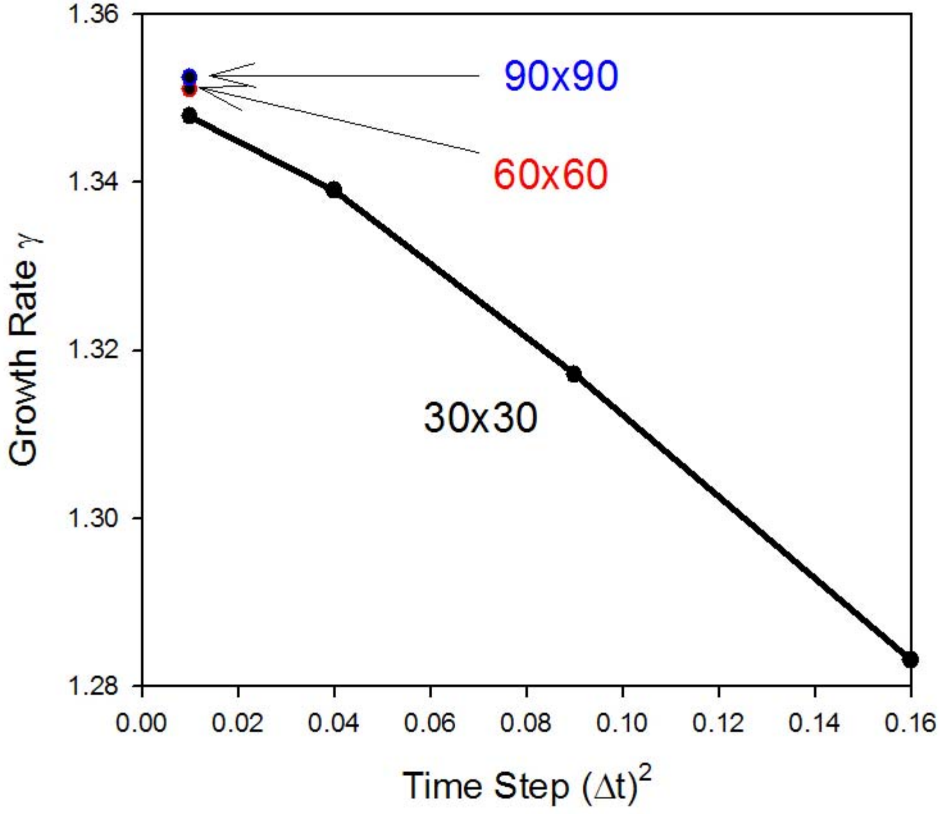
$$\frac{\partial}{\partial t}\nabla^2 U + \left[\nabla^2 U, U\right] - \left[\nabla^2 \psi, \psi\right] = \mu \nabla^4 U$$
$$\frac{\partial \psi}{\partial t} + \left[\psi, U\right] = \eta \nabla^2 \psi$$

Phase-II: Fitzpatrick-Porcelli 4-field model:

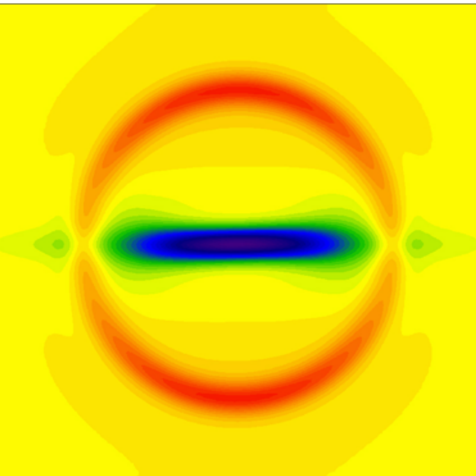
$$\frac{\partial}{\partial t}\nabla^2 U = \left[U, \nabla^2 U\right] + \left[\nabla^2 \psi, \psi\right] + \mu \nabla^4 U$$
$$\frac{\partial V_z}{\partial t} = \left[U, V_z\right] + c_\beta \left[I, \psi\right] + \mu \nabla^2 V_z$$
$$\frac{\partial \psi}{\partial t} = \left[U, \psi\right] + d_\beta \left[\psi, I\right] + \eta \nabla^2 \psi$$
$$\frac{\partial I}{\partial t} = \left[U, I\right] + d_\beta \left[\nabla^2 \psi, \psi\right] + c_\beta \left[V_z, \psi\right] + c_\beta^2 \eta \nabla^2 I$$

Tilting cylinder with 6-field 2-fluid model

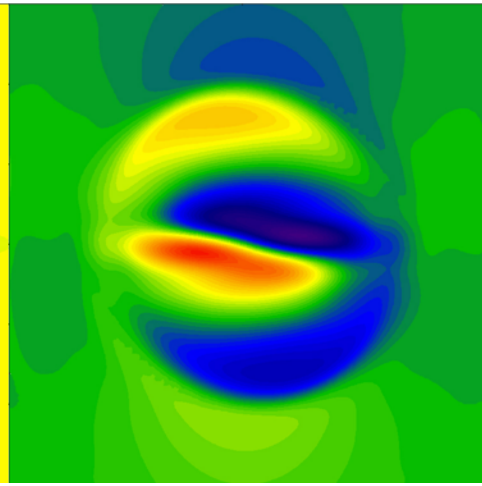




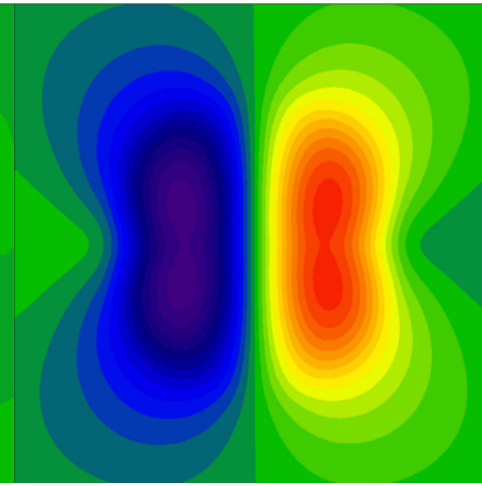
Linear eigenmode of tilting cylinder in 6-field 2-fluid model



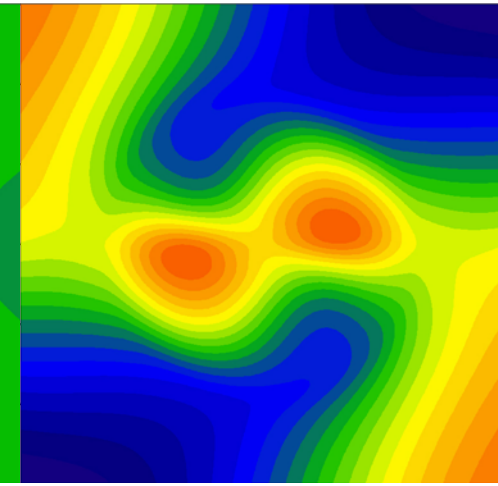
$\nabla^2 U$
Vorticity



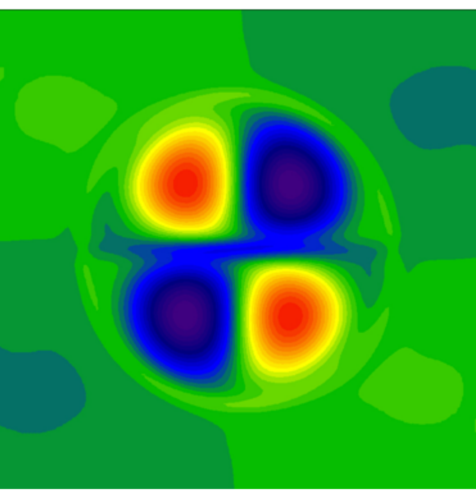
V
Toroidal Velocity



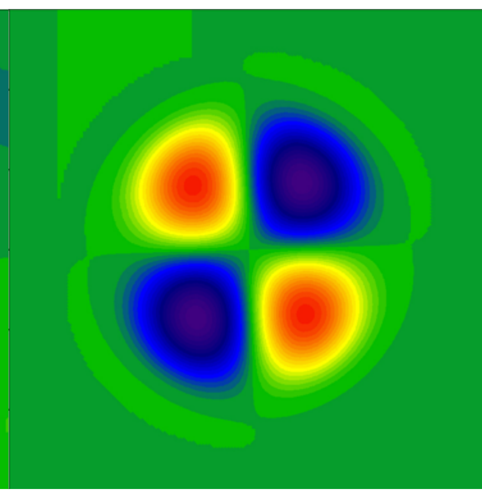
Ψ
Poloidal Flux



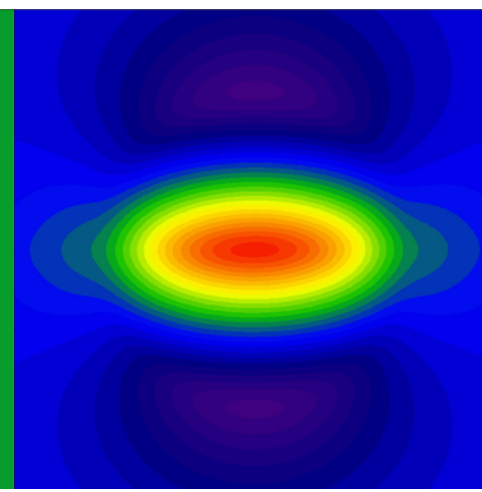
χ
Velocity Potential



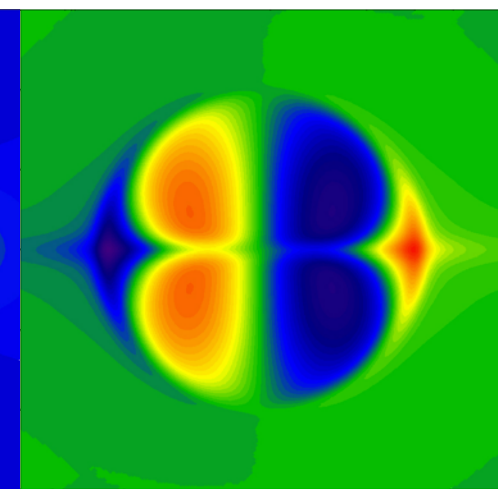
I
Toroidal field



P_e
Electron Pressure

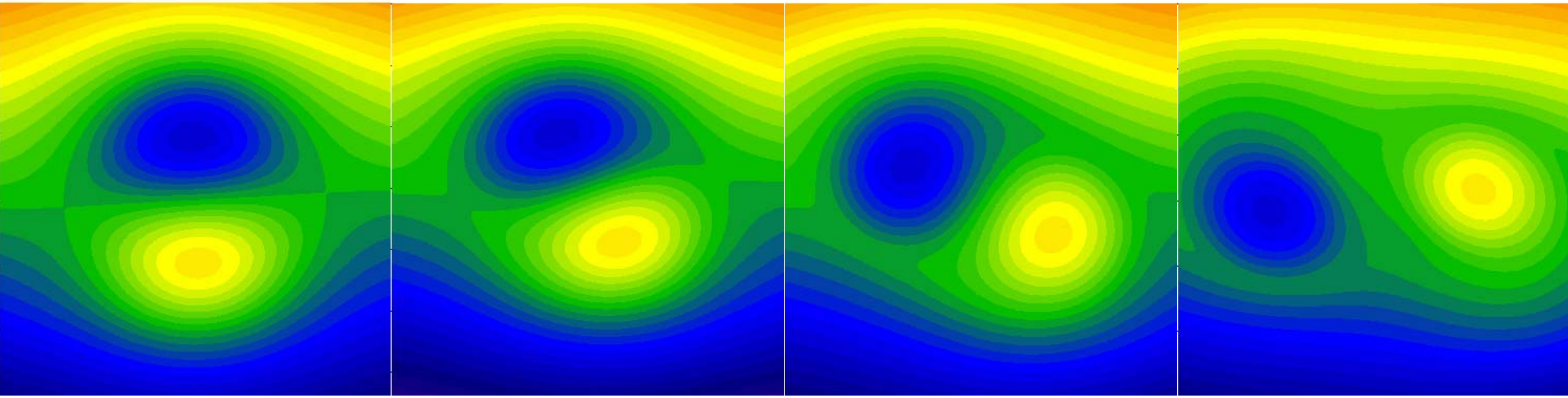


U
Stream Function

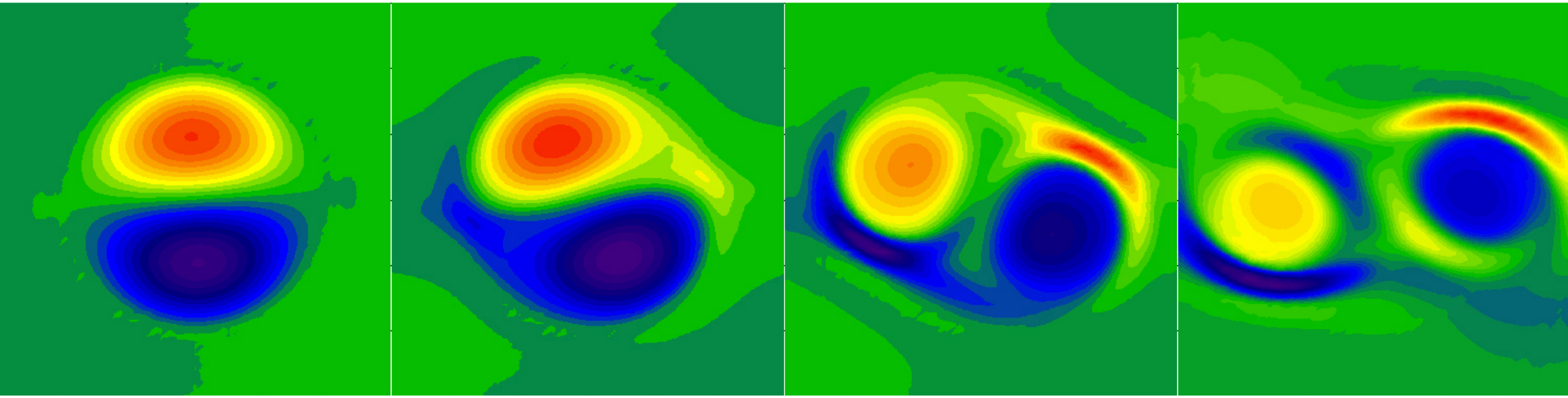


J
Toroidal Current

Non-linear evolution of tilting cylinder in full 6-field 2-fluid model

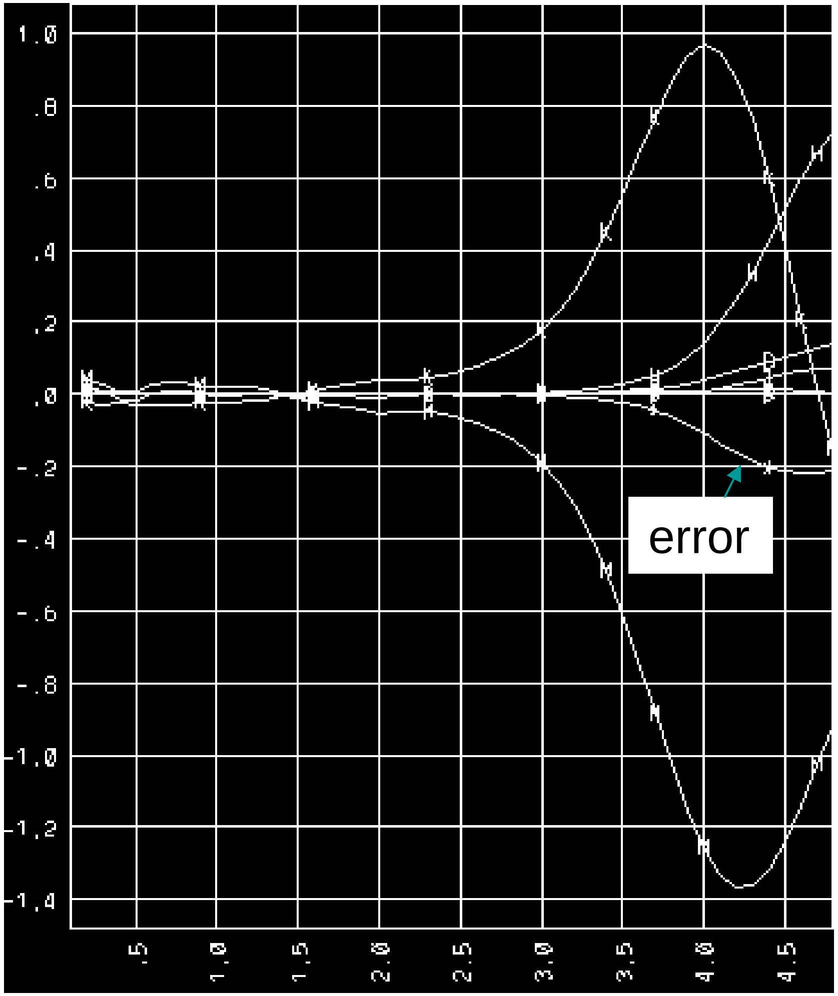


Ψ: t=0.8 | Ψ: t=3.8 | Ψ: t=4.0 | Ψ: t=4.8

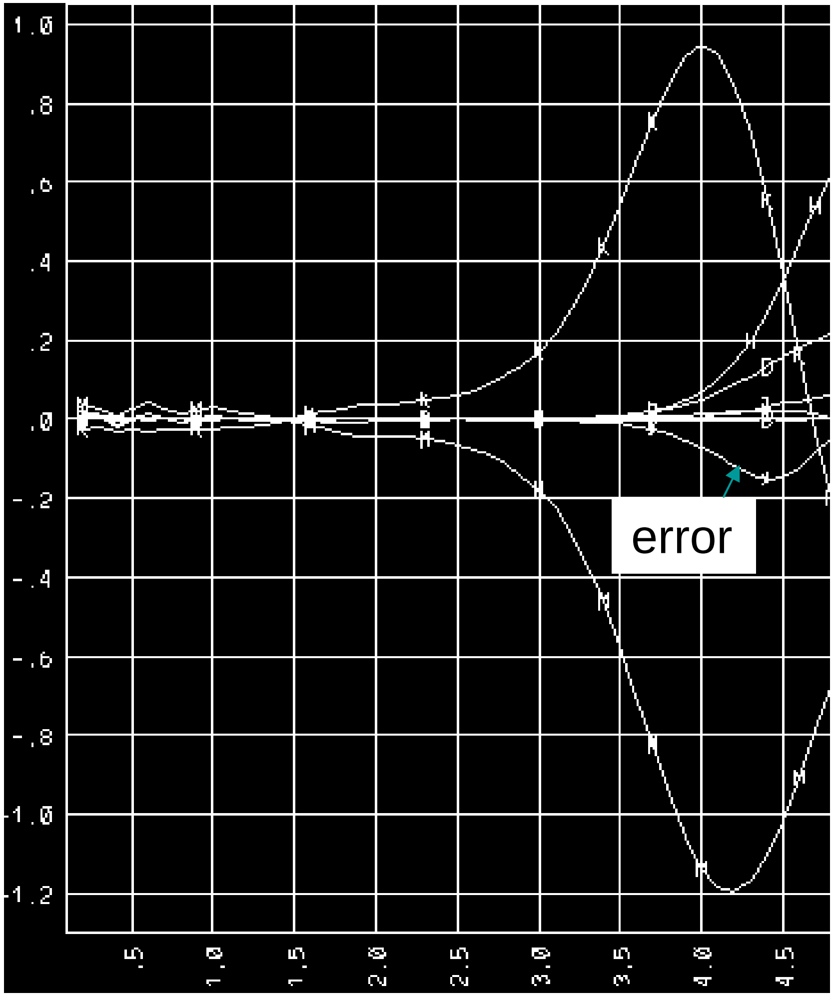


J: t=0.8 | J: t=3.2 | J: t=4.0 | J: t=4.8

Energy error decreases with increasing number of nodes for sequence with hyper coef. H = C $(\Delta x)^2$



31 x 31 nodes



61 x 61 nodes

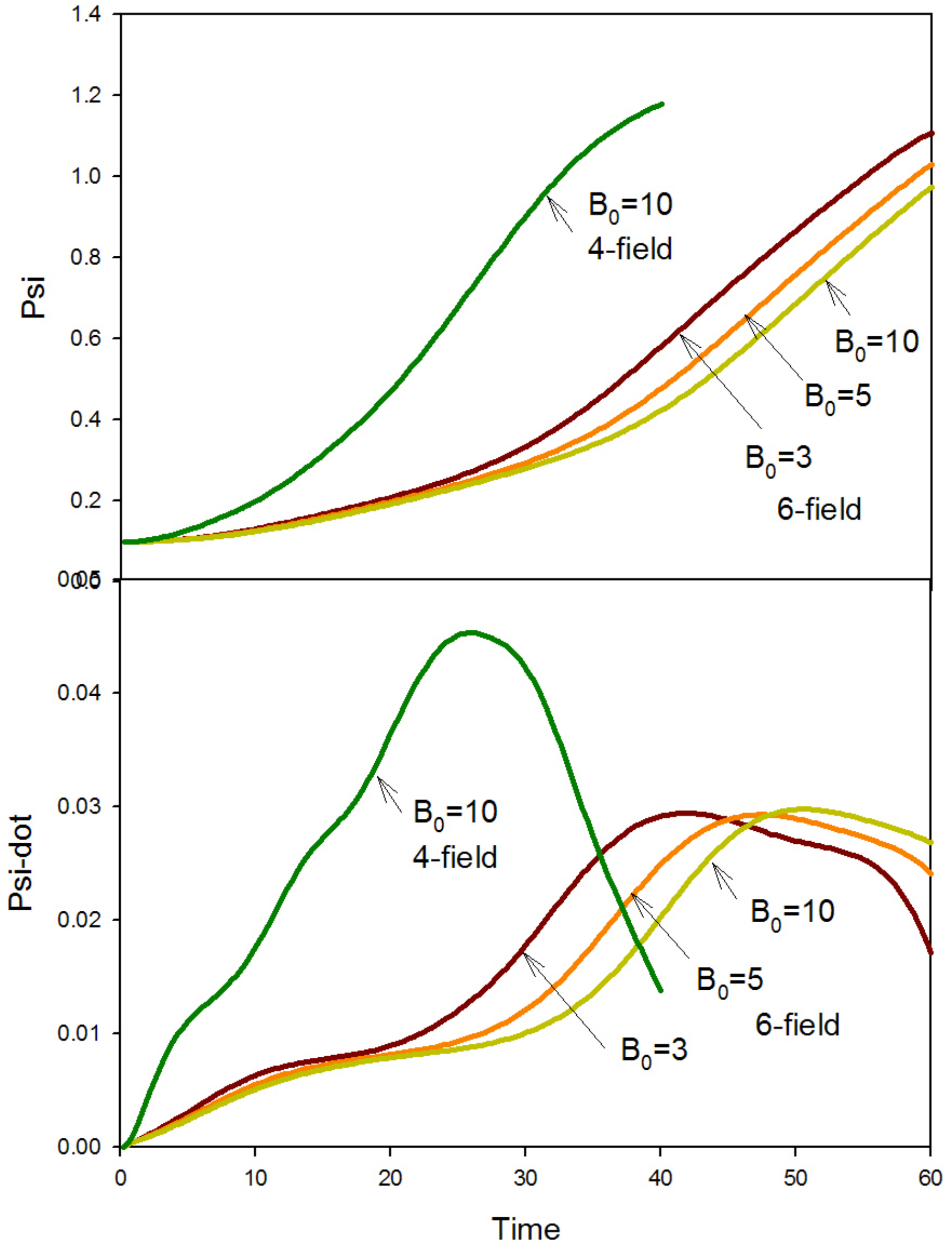
Magnetic Reconnection

Modified GEM Challenge Problem:

$$\Psi(z) = \tfrac{1}{2}\log\left[\cosh\left(2z\right)\right]$$
$$+\,\varepsilon\cos(k_x x)\cos\left(k_z z\right)$$

$$P_e(z) = 0.6 - \tfrac{1}{2}\tanh\left(2z\right)$$
$$P_i(z) = 0.$$
$$B_z = B_0$$

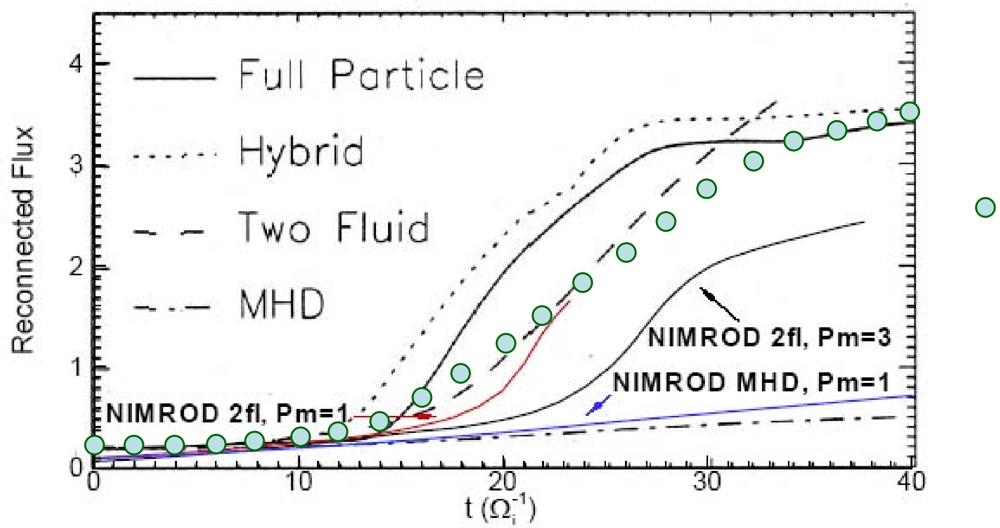
constant density n_0



Full 6-field model gives lower reconnection rates than does reduced 4-field Fitzpatrick-Porcelli model

M3D-*C*¹ has been tested against a number of test problems, including the GEM Challenge Problem (Birn, et al. JGR)

Reconnected magnetic flux in the GEM Challenge problem as a function of time

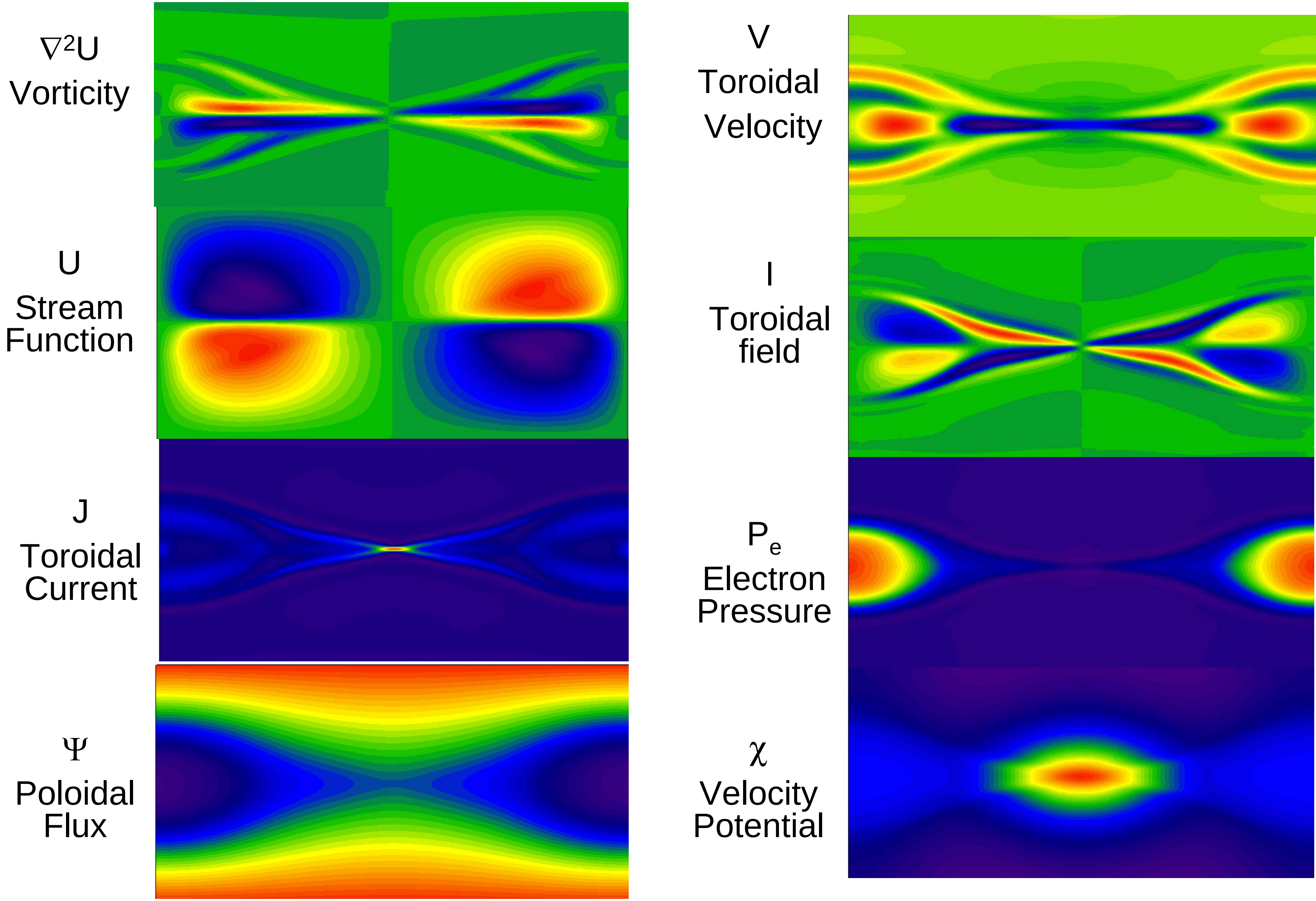


M3D-*C*¹, *zero ion pressure, uniform density,* $\eta=\mu=0.001$

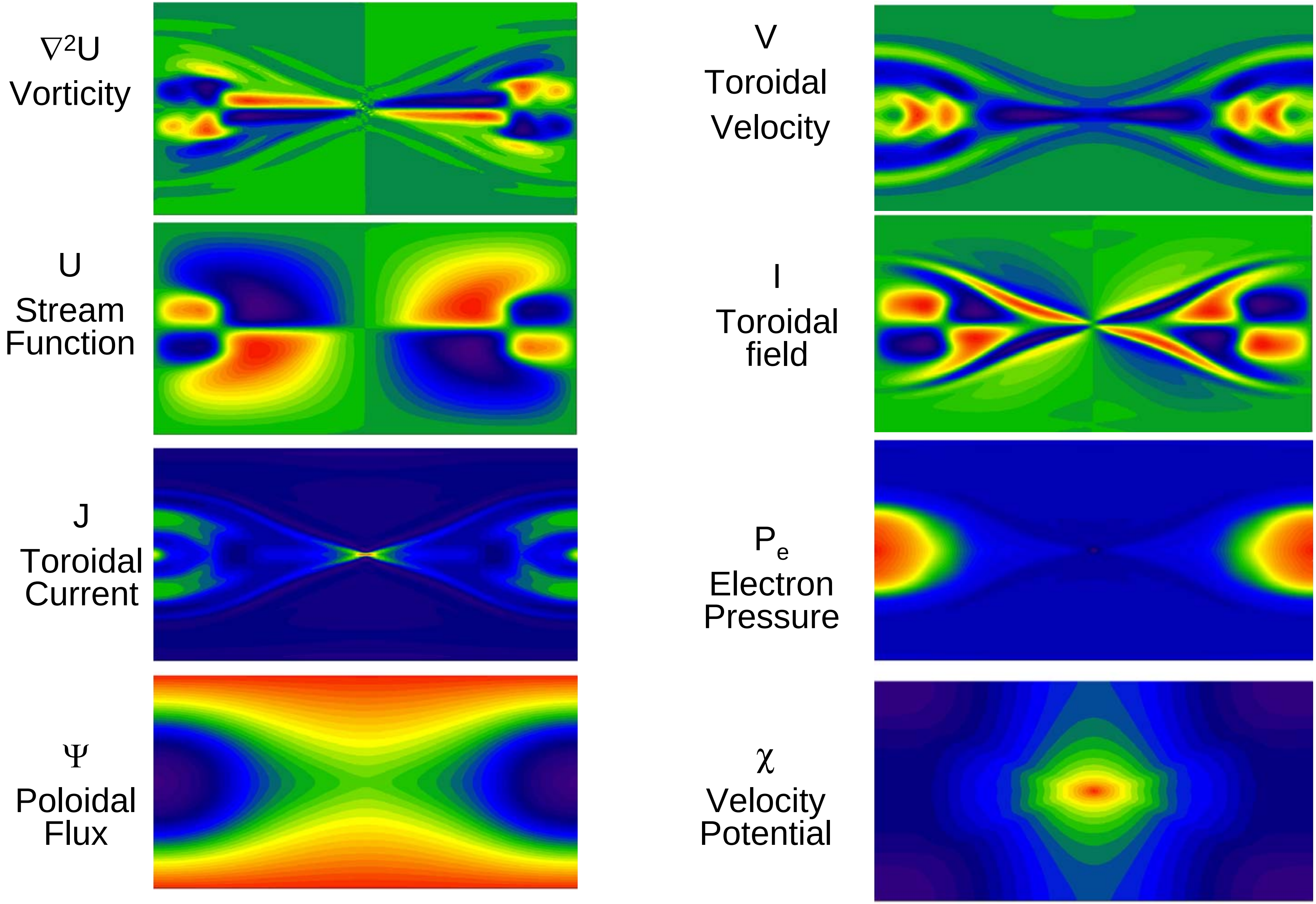
- The calculations presented in this figure all have different physics models and slightly different initial conditions and parameters
- However, all models with two-fluid physics (ie, the Hall Term) show fast reconnection. Single fluid resistive MHD does not
- **M3D-*C*¹** also shows fast reconnection
- Demanding problem with no guide field, and in which Alfvenic, highly compressible flows develop

M3D-C^1 61x61 triangles, no symmetry imposed: t=30

GEM Magnetic Reconnection 6-field 2-fluid model: t=30, $V_{MAX} \sim 0.8\ V_A$



M3D-C^1 61x61 triangles, no symmetry imposed: t=40
GEM Magnetic Reconnection 6-field 2-fluid model: t=40, $V_{MAX} \sim 0.8\ V_A$



Wave Propagation Tests

Linearize about an equilibrium with: $B_P / B_T \sim 0.1$

$$I = I_0, \quad \psi^0 = \delta z, \quad p = p_0,$$

Assume perturbations parallel to the poloidal field of the form:

$$I = I\cos(kx - \omega t), \;\; \psi = \cos(kx - \omega t), \quad p = \varepsilon\cos(kx - \omega t),$$

$$\chi = \chi\sin(kx - \omega t), \;\; U = U\cos(kx - \omega t), \quad V_z = V_z\cos(kx - \omega t)$$

$$\left\{\left[\left(\omega^2 - k^2\delta^2\right)^2 - \omega^2 k^4 d_\beta^2 \delta^2\right]\left(\omega^2 - \gamma p_0 k^2\right) - I_0^2\omega^2 k^2(\omega^2 - k^2\delta^2)\right\} I = 0$$

$$\Omega^3 + A\Omega^2 + B\Omega + C = 0$$

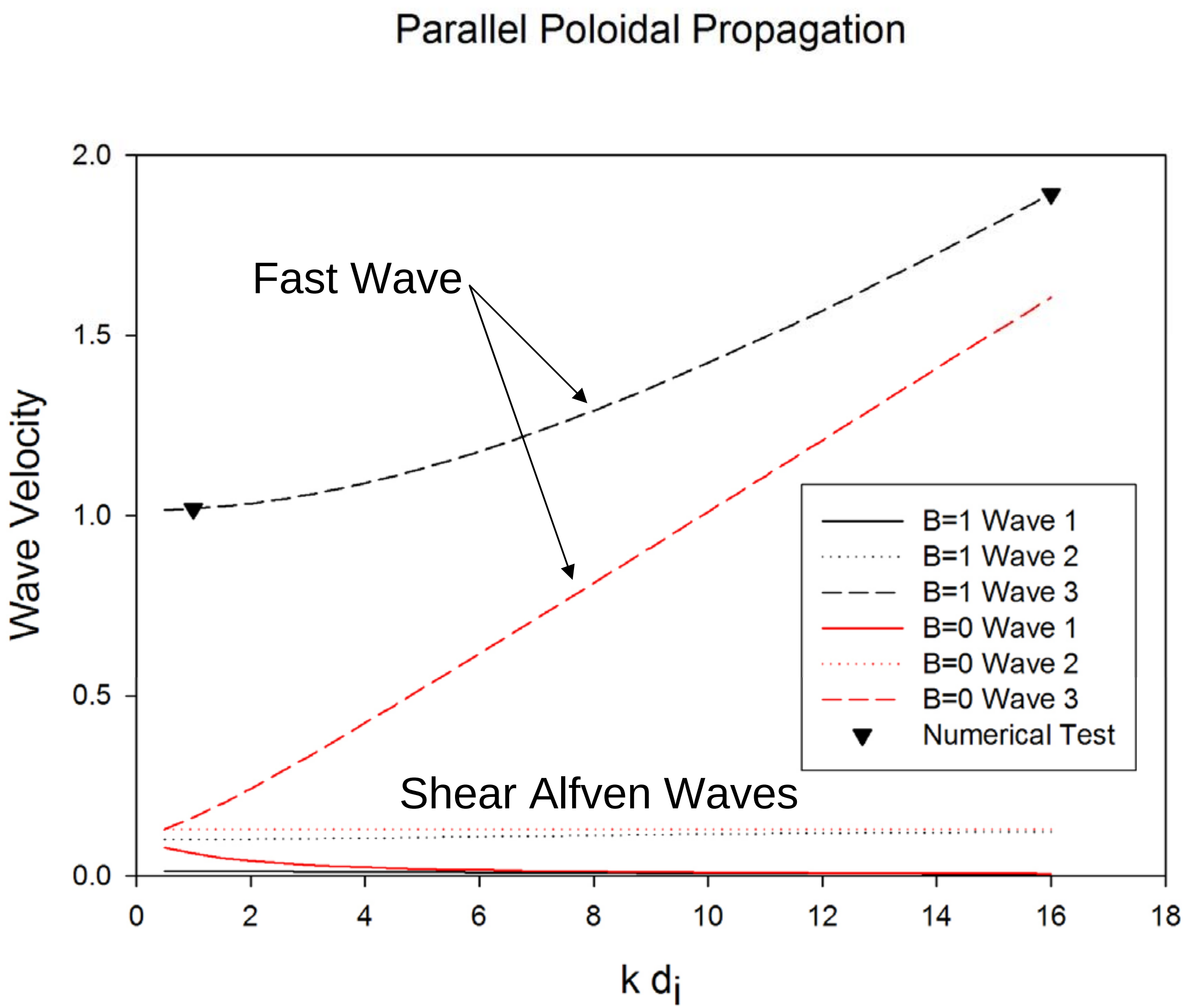
$$A = -k^2\left[2\delta^2 + \gamma p_0 + I_0^2 + k^2\delta^2 d_\beta^2\right]; \quad B = k^4\left[\delta^4 + \gamma p_0\left(2\delta^2 + k^2\delta^2 d_\beta^2\right) + \delta^2 I_0^2\right];$$

$$C = -k^6\gamma p_0\delta^4$$

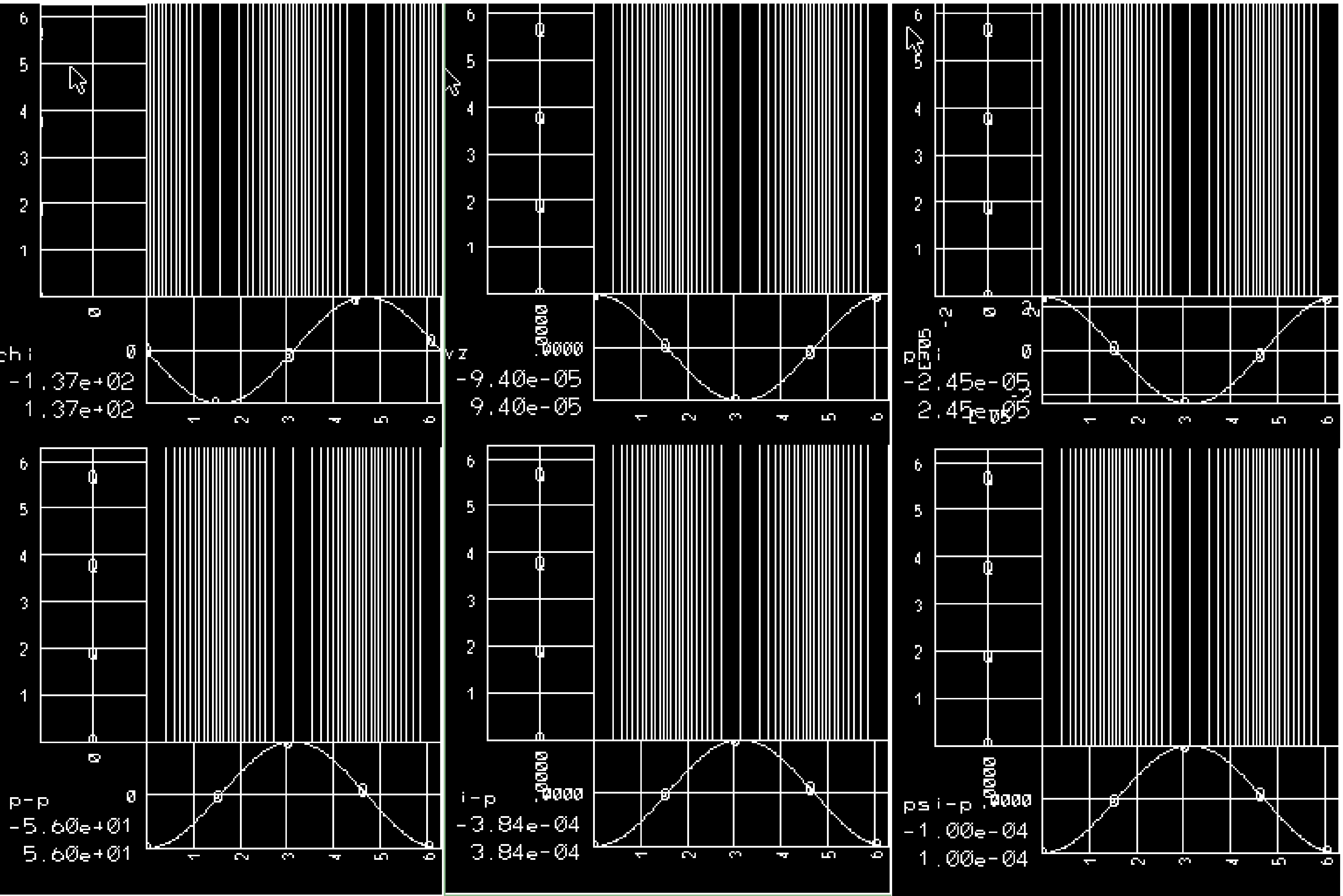
Evaluation of the 6-field dispersion relation for δ=0.1, d_β=1

Case	k	λ	B_0	P_0	$\omega/k(1)$	$\omega/k(2)$	$\omega/k(3)$
1a	1.0	6.28318	1	.01	0.012639	0.10033	1.0181
1b	1.0	6.28318	0	.01	0.06180	0.12910	0.16180
1c	1.0	6.28318	0	.10	0.06180	0.1618	0.40825
2a	2.0	3.14159	1	01	0.012345	0.10127	1.0326
2b	2.0	3.14159	0	.01	0.041421	0.12910	0.24142
2c	2.0	3.14159	0	.10	0.041421	0.24142	0.40825
3a	4.0	1.57079	1	.01	0.011346	0.10449	1.0889
3b	4.0	1.57079	0	.01	0.023607	0.12910	0.42361
3c	4.0	1.57079	0	.10	0.023607	0.40825	0.42361
4a	8.0	0.78539	1	.01	0.008916	0.11225	1.2900
4b	8.0	0.78539	0	.01	0.012311	0.12910	0.81231
4c	8.0	0.78539	0	.10	0.012311	0.40825	0.81231
5a	16.	.39269	1	.01	0.005608	0.12163	1.8926
5b	16.	.39269	0	.01	0.0062258	0.12910	1.6062
5c	16.	.39269	0	.10	0.0062258	0.40825	1.6062

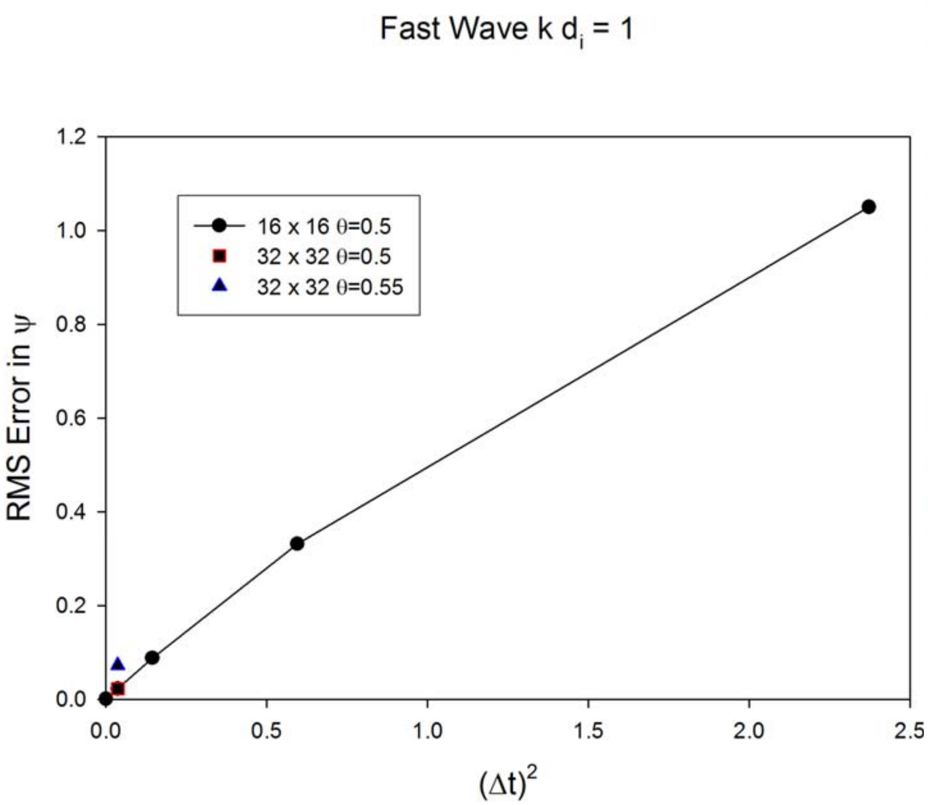
Propagation in the poloidal plane, with and without a guide field

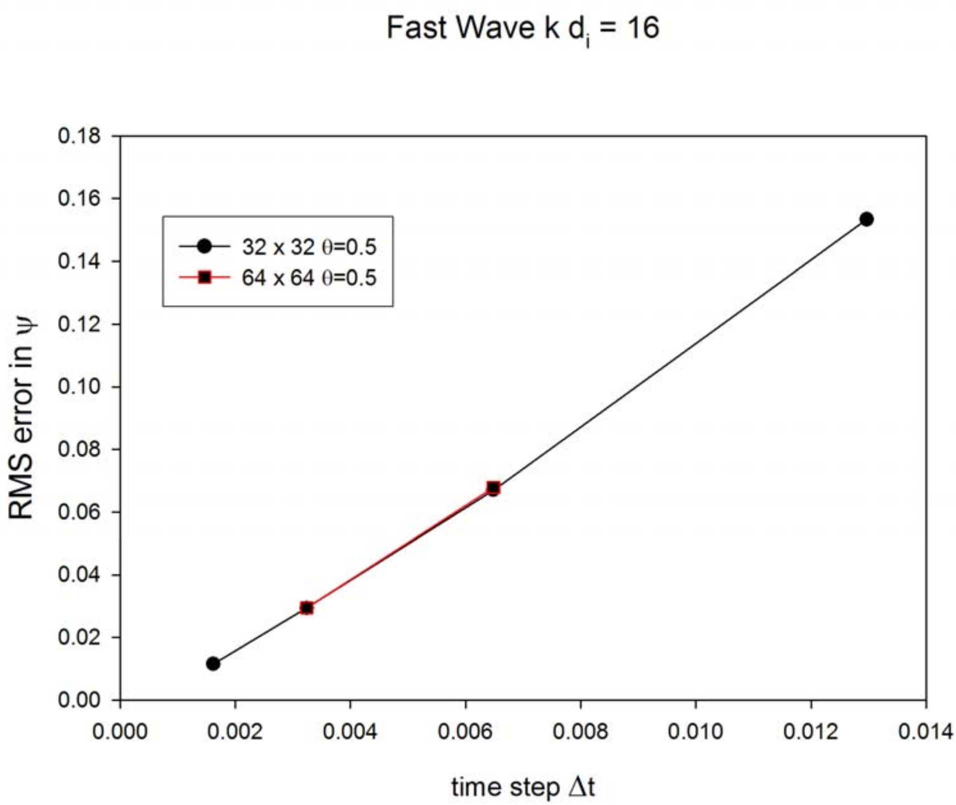


Start system in an eigenmode for a propagating wave, let it propagate 1 wavelength, and calculate RMS error between final and initial state. Doubly periodic



Quadratic convergence in time for small k, linear for large k





θ is implicit parameter: θ=0.5 is time centered

Braginskii gyro-viscosity:

$$\nabla\cdot\vec{\Pi}=\left\{\left[\nabla\times\left(\frac{mp}{eB^2}\vec{B}\right)\right]\cdot\nabla\right\}\vec{V}-\nabla\left[\frac{mp}{2eB^2}\vec{B}\cdot\left(\nabla\times\vec{V}\right)\right]-\nabla\times\left\{\frac{mp}{eB^2}\left[(B\cdot\nabla)\vec{V}+\frac{1}{2}\left(\nabla\cdot\vec{V}-\frac{3}{B^2}\vec{B}\cdot\left[\left(\vec{B}\cdot\nabla\right)\vec{V}\right]\right)\vec{B}\right]\right\}$$
$$+(B\cdot\nabla)\left\{\frac{mp}{eB^2}\left(\frac{3}{B^2}\vec{B}\times\left[\left(\vec{B}\cdot\nabla\right)\vec{V}\right]+\frac{3}{2B^2}\left[\vec{B}\cdot\left(\nabla\times\vec{V}\right)\right]\vec{B}-\nabla\times\vec{V}\right)\right\}$$

Ramos

$$\hat{z}\cdot\quad\rightarrow\quad[V_z,\alpha I]-\alpha\left\{\left[\psi,\nabla_\perp^2U\right]+\left[\frac{\partial\psi}{\partial x},\frac{\partial U}{\partial x}-\frac{\partial\chi}{\partial y}\right]+\left[\frac{\partial\psi}{\partial y},\frac{\partial U}{\partial y}+\frac{\partial\chi}{\partial x}\right]\right\}-\frac{\partial\alpha}{\partial x}\left[\psi,\frac{\partial U}{\partial x}-\frac{\partial\chi}{\partial y}\right]-\frac{\partial\alpha}{\partial y}\left[\psi,\frac{\partial U}{\partial y}+\frac{\partial\chi}{\partial x}\right]$$
$$+\frac{1}{2}\alpha\nabla_\perp^2\chi\nabla_\perp^2\psi+\frac{1}{2}\left(\alpha\nabla_\perp^2\chi,\psi\right)-\frac{1}{2}\left[(\gamma\kappa,\psi)+\gamma\kappa\nabla_\perp^2\psi\right]+[\gamma\xi_z,\psi]+\frac{1}{2}[\lambda I,\psi]+\left[\alpha\nabla_\perp^2U,\psi\right]$$

$$-\hat{z}\cdot\nabla\times\quad\rightarrow\quad\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\frac{\partial(\alpha I)}{\partial y}\right]-\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\frac{\partial(\alpha I)}{\partial x}\right]+\left[\nabla_\perp^2U,\alpha I\right]+\nabla_\perp^2\left\{\alpha[\psi,V_z]\right\}-\frac{1}{2}\nabla_\perp^2\left(\alpha I\nabla_\perp^2\chi\right)+\frac{1}{2}\nabla_\perp^2(\gamma\kappa I)$$
$$+\frac{\partial}{\partial y}[\gamma\xi_x,\psi]-\frac{\partial}{\partial x}[\gamma\xi_y,\psi]+\frac{1}{2}\left\{\frac{\partial\lambda}{\partial x}\left[\frac{\partial\psi}{\partial x},\psi\right]+\frac{\partial\lambda}{\partial y}\left[\frac{\partial\psi}{\partial y},\psi\right]+([\lambda,\psi],\psi)+\left[\lambda\nabla_\perp^2\psi,\psi\right]\right\}+\frac{\partial}{\partial x}\left[\psi,\alpha\frac{\partial V_z}{\partial x}\right]+\frac{\partial}{\partial y}\left[\psi,\alpha\frac{\partial V_z}{\partial y}\right]$$

$$\nabla\cdot\quad\rightarrow\quad\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\frac{\partial(\alpha I)}{\partial x}\right]+\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\frac{\partial(\alpha I)}{\partial y}\right]+\left[\nabla_\perp^2\chi,\alpha I\right]+\frac{1}{2}\nabla_\perp^2\left\{\alpha\left[I\nabla_\perp^2U-(\psi,V_z)\right]\right\}+\frac{\partial}{\partial x}[\gamma\xi_x,\psi]+\frac{\partial}{\partial y}[\gamma\xi_y,\psi]$$
$$+\frac{1}{2}[[\lambda,\psi],\psi]+\lambda\left[\frac{\partial\psi}{\partial y},\frac{\partial\psi}{\partial x}\right]+\frac{1}{2}\frac{\partial\lambda}{\partial x}\left[\frac{\partial\psi}{\partial y},\psi\right]-\frac{1}{2}\frac{\partial\lambda}{\partial y}\left[\frac{\partial\psi}{\partial x},\psi\right]+\frac{\partial}{\partial x}\left\{\alpha\left[\psi,\frac{\partial V_z}{\partial y}\right]\right\}-\frac{\partial}{\partial y}\left\{\alpha\left[\psi,\frac{\partial V_z}{\partial x}\right]\right\}+[V_z,[\alpha,\psi]]$$

$$\alpha\equiv\frac{ep}{mB^2}=\frac{ep/m}{\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+I^2}\qquad\gamma=\frac{3ep/m}{\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2+I^2\right]^2}$$

$$\vec{\xi}=\left\{\frac{\partial\psi}{\partial x}[\psi,V_z]+I\left[\psi,\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x}\right]\right\}\hat{x}+\left\{\frac{\partial\psi}{\partial y}[\psi,V_z]-I\left[\psi,\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y}\right]\right\}\hat{y}-\left\{\frac{\partial\psi}{\partial x}\left[\psi,\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y}\right]+\frac{\partial\psi}{\partial y}\left[\psi,\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x}\right]\right\}\hat{z}$$

$$\lambda=\gamma\left[(\psi,V_z)-I\nabla_\perp^2U\right]$$

$$\kappa\equiv\frac{\partial\psi}{\partial y}\left[\frac{\partial\chi}{\partial x}+\frac{\partial U}{\partial y},\psi\right]-\frac{\partial\psi}{\partial x}\left[\frac{\partial\chi}{\partial y}-\frac{\partial U}{\partial x},\psi\right]+I[V_z,\psi]$$

Breslau

Near-Term Plans

- Add ion pressure equation
- Add electron thermal conductivity
- Braginskii ion-gyroviscosity with ion pressure
- Mesh adaptation (Bauer, RPI)
- Physics studies of reconnection with different 2-fluid models and quantify the effect of compressibility (Ferraro)
  - Sugiyama-Park model
  - Full gyroviscous + Hall term
- Extend to toroidal system, and non-axisymmetric modes

Summary and Conclusions

- *M3D-C^1* finite element method has been extended to 7-field 2-fluid MHD
- Applied to GEM reconnection, shows that compressibility reduces the reconnection rate
- Incorporated full ion gyroviscosity tensor
- Linear wave tests show convergence as $(\Delta t)^2$ for small $k \times d_i$, (Δt) for large $k \times d_i$