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LOW-COLLISIONALITY ORDERINGS IN EXTENDED MHD*

J. J. Ramos

Plasma Science and Fusion Center, Massachusetts Institute of Technology

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I. MOTIVATION

IT SHOULD BE DESIRABLE TO TREAT THE COLLISIONAL EFFECTS IN EXTENDED MHD CONSISTENT WITH THE REALISTIC COLLISIONALITY REGIMES OF FUSION PLASMAS.

THAT IS A VERY DIFFICULT TASK IF ONE TRIES TO COVER ALL THE PLASMA SPATIAL EXTENT AND ALL THE TIME SCALES FROM TRANSPORT TO MHD INSTABILITY EVOLUTION.

HERE, AN ORDERING SCHEME IS PROPOSED TO COVER THE BROAD LOW-COLLISIONALITY REGIME CHARACTERIZED BY ION COLLISION FREQUENCIES COMPARABLE OR SMALLER THAN THE DIAMAGNETIC DRIFT FREQUENCY ω_*, AND TIME SCALES COMPARABLE OR FASTER THAN ω_*.

Extended MHD uses as fundamental expansion parameter the ratio $\delta \sim \rho_\iota/L$ between the ion Larmor radius and the shortest characteristic length other than the gyroradii (typically a fluctuation perpendicular wavelength or a perpendicular gradient scale length). In the strictly collisionless limit, a consistent formulation is possible based on the single expansion parameter δ.

At finite collisionality, two more independent small parameters, namely the ratio m_e/m_ι between the electron and ion masses and the ratio $\nu_\iota/\Omega_{c\iota}$ between the ion collision and cyclotron frequencies must be considered.

THE BASIC WORKING HYPOTHESIS ADOPTED HERE ARE:

$$\left(\frac{m_e}{m_\iota}\right)^{1/2} \lesssim \delta \ll 1 \, , \qquad \frac{\nu_\iota}{\Omega_{c\iota}} \lesssim \delta \left(\frac{m_e}{m_\iota}\right)^{1/2} \qquad \textbf{and} \qquad \delta \frac{v_{th\iota}}{L} \lesssim \frac{\partial}{\partial t} \sim \frac{u_\iota}{L} \sim \frac{u_e}{L} \lesssim \frac{v_{th\iota}}{L} \, .$$

Besides, we shall assume quasineutrality with one ion species of unit charge, $n_\iota = n_e = n$, and comparable pressures with arbitrary anisotropies, $p_\iota \sim p_e \sim (p_{\iota\|} - p_{\iota\perp}) \sim (p_{e\|} - p_{e\perp})$.

II. THE COLLISIONAL MOMENTS

For each species, $\alpha, \beta \in (\iota, e)$**, the underlying kinetic description is assumed given by**

$$\frac{\partial f_\alpha(\mathbf{v},\mathbf{x},t)}{\partial t} \ + \ v_j \frac{\partial f_\alpha(\mathbf{v},\mathbf{x},t)}{\partial x_j} \ + \ \frac{e_\alpha}{m_\alpha}(E_j + \epsilon_{jkl} v_k B_l) \frac{\partial f_\alpha(\mathbf{v},\mathbf{x},t)}{\partial v_j} \ = \ C_\alpha(\mathbf{v},\mathbf{x},t),$$

with Fokker-Plank Coulomb collision operators (to be kept in their complete, quadratic form so that the analysis remains valid away from Maxwellians):

$$C_\alpha(\mathbf{v},\mathbf{x},t) \ = \ -\sum_\beta \ \frac{c^4 e_\alpha^2 e_\beta^2 \ln \Lambda_{\alpha\beta}}{8\pi m_\alpha} \ \Gamma_{\alpha\beta}(\mathbf{v},\mathbf{x},t) \ ,$$

where $\ln \Lambda_{\alpha\beta} = \ln \Lambda_{\beta\alpha}$ **are the Coulomb logarithms,**

$$\Gamma_{\alpha\beta}(\mathbf{v},\mathbf{x},t) \ = \ \frac{\partial}{\partial v_j} \int d^3\mathbf{w} \ \mathsf{U}_{jk}(\mathbf{v},\mathbf{w}) \Big[\frac{f_\alpha(\mathbf{v},\mathbf{x},t)}{m_\beta} \ \frac{\partial f_\beta(\mathbf{w},\mathbf{x},t)}{\partial w_k} \ - \ \frac{f_\beta(\mathbf{w},\mathbf{x},t)}{m_\alpha} \ \frac{\partial f_\alpha(\mathbf{v},\mathbf{x},t)}{\partial v_k} \Big]$$

and

$$\mathsf{U}_{jk}(\mathbf{v},\mathbf{w}) \ = \ \frac{|\mathbf{v}-\mathbf{w}|^2 \ \delta_{jk} \ - \ (v_j - w_j)(v_k - w_k)}{|\mathbf{v}-\mathbf{w}|^3} \ .$$

Define the following collisional moments:

$$F^{coll}_{\alpha,j}(\mathbf{x},t) \;=\; m_\alpha \int d^3\mathbf{v}\; (v_j - u_{\alpha,j})\; C_\alpha(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{G}^{coll}_{\alpha,jk}(\mathbf{x},t) \;=\; m_\alpha \int d^3\mathbf{v}\; (v_j - u_{\alpha,j})(v_k - u_{\alpha,k})\; C_\alpha(\mathbf{v},\mathbf{x},t),$$

$$\mathsf{H}^{coll}_{\alpha,jkl}(\mathbf{x},t) \;=\; m_\alpha \int d^3\mathbf{v}\; (v_j - u_{\alpha,j})(v_k - u_{\alpha,k})(v_l - u_{\alpha,l})\; C_\alpha(\mathbf{v},\mathbf{x},t)\;,$$

where $u_{\alpha,j}(\mathbf{x},t) \;=\; n^{-1} \int d^3\mathbf{v}\; v_j\; f_\alpha(\mathbf{v},\mathbf{x},t)$ **are the macroscopic flow velocities.**

Also define:

$$g^{coll}_{\alpha} = \mathsf{G}^{coll}_{\alpha,jj}/2\;, \qquad g^{coll}_{\alpha B} = \mathsf{G}^{coll}_{\alpha,jk} b_j b_k/2\;,$$

$h^{coll}_{\alpha} = \mathsf{H}^{coll}_{\alpha,jkk} b_j/2$ **and** $h^{coll}_{\alpha B} = \mathsf{H}^{coll}_{\alpha,jkl} b_j b_k b_l/2\;.$

The collisional contributions to the "perpendicular" (i.e. non-CGL) parts of the stress and stress-flux tensors are:

$$\hat{\mathsf{P}}^{coll}_{\alpha,jk} \;=\; \frac{1}{4}\epsilon_{[jlm}b_l\mathsf{K}^{coll}_{\alpha,mn}(\delta_{nk]} + 3b_nb_{k]})$$

and

$$\hat{\mathsf{Q}}^{coll}_{\alpha,jkl} \;=\; \frac{1}{3}\epsilon_{[jmn}b_m\mathsf{L}^{coll}_{\alpha,nkl]} \;-\; \frac{1}{12}\epsilon_{[jmn}b_kb_mb_p\mathsf{L}^{coll}_{\alpha,npl]} \;+\; \frac{2}{9}\epsilon_{[jmn}\epsilon_{kpq}\epsilon_{lrs]}b_mb_pb_r\mathsf{L}^{coll}_{\alpha,nqs} \;+\; \frac{5}{6}\epsilon_{[jmn}b_kb_{l]}b_mb_pb_q\mathsf{L}^{coll}_{\alpha,npq}\;,$$

where

$$\mathsf{K}^{coll}_{\alpha,jk} \;=\; -\,\frac{m_\alpha}{e_\alpha B}\;\mathsf{G}^{coll}_{\alpha,jk}\;,$$

$$\mathsf{L}^{coll}_{\alpha,jkl} \;=\; -\,\frac{m_\alpha}{e_\alpha B}\Big(\mathsf{H}^{coll}_{jkl} - \;\frac{1}{m_\alpha n}F^{coll}_{\alpha,[j}\;\mathsf{P}_{\alpha,kl]}\Big)$$

and $\mathbf{P}_{\alpha,jk} = m_\alpha \int d^3\mathbf{v}\;(v_j - u_{\alpha,j})(v_k - u_{\alpha,k})f_\alpha(\mathbf{v}) = \mathbf{P}^{CGL}_{\alpha,jk} + \hat{\mathbf{P}}^{gyr}_{\alpha,jk} + \hat{\mathbf{P}}^{coll}_{\alpha,jk}$ are the complete stress tensors.

Taking now $\alpha \neq \beta$, we obtain after integrations by parts:

$$F^{coll}_{\alpha,j} \;=\; -F^{coll}_{\beta,j} \;=\; -\frac{c^4 e^4}{4\pi} \ln \Lambda_{\alpha\beta} \Big(\frac{1}{m_\alpha} + \frac{1}{m_\beta}\Big) \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\beta(\mathbf{w})\; \frac{v_j - w_j}{|\mathbf{v}-\mathbf{w}|^3} \,,$$

$$\begin{aligned}
\mathsf{G}^{coll}_{\alpha,jk} \;=\; & \frac{c^4 e^4}{4\pi} \Big[\frac{\ln \Lambda_{\alpha\alpha}}{m_\alpha} \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\alpha(\mathbf{w})\; \frac{|\mathbf{v}-\mathbf{w}|^2\, \delta_{jk} - 3(v_j - w_j)(v_k - w_k)}{|\mathbf{v}-\mathbf{w}|^3} + \\
& + \frac{\ln \Lambda_{\alpha\beta}}{m_\alpha} \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\beta(\mathbf{w})\; \frac{|\mathbf{v}-\mathbf{w}|^2\, \delta_{jk} - (v_j - w_j)(v_k - w_k)}{|\mathbf{v}-\mathbf{w}|^3} - \\
& - \ln \Lambda_{\alpha\beta} \Big(\frac{1}{m_\alpha} + \frac{1}{m_\beta}\Big) \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\beta(\mathbf{w})\; \frac{(v_{[j} - w_{[j})(v_{k]} - u_{\alpha,k]})}{|\mathbf{v}-\mathbf{w}|^3} \Big]
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{H}^{coll}_{\alpha,jkl} = & \frac{c^4 e^4}{4\pi} \Big\{ \frac{\ln \Lambda_{\alpha\alpha}}{m_\alpha} \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\alpha(\mathbf{w})\; \frac{(v_{[j} - u_{\alpha,[j})[|\mathbf{v}-\mathbf{w}|^2\, \delta_{kl]} - 3(v_k - w_k)(v_{l]} - w_{l]})]}{|\mathbf{v}-\mathbf{w}|^3} + \\
& + \frac{\ln \Lambda_{\alpha\beta}}{m_\alpha} \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\beta(\mathbf{w})\; \frac{(v_{[j} - u_{\alpha,[j})[|\mathbf{v}-\mathbf{w}|^2\, \delta_{kl]} - (v_k - w_k)(v_{l]} - w_{l]})]}{|\mathbf{v}-\mathbf{w}|^3} - \\
& - \ln \Lambda_{\alpha\beta} \Big(\frac{1}{m_\alpha} + \frac{1}{m_\beta}\Big) \int\!\!\int d^3\mathbf{v}\; d^3\mathbf{w}\; f_\alpha(\mathbf{v})\; f_\beta(\mathbf{w})\; \frac{(v_{[j} - w_{[j})[(v_k - u_{\alpha,k})(v_{l]} - u_{\alpha,l]})]}{|\mathbf{v}-\mathbf{w}|^3} \Big\} \,.
\end{aligned}$$

Define the thermal velocities as $v_{th\alpha} = \sqrt{p_\alpha/(m_\alpha n)}$**, and the collision frequencies as**

$$\nu_e \;=\; \frac{c^4 e^4 n \ln \Lambda_{e\iota}}{4\pi m_e^2 v_{the}^3} \;=\; \frac{c^4 e^4 n \ln \Lambda_{ee}}{4\pi m_e^2 v_{the}^3} \qquad \textbf{and} \qquad \nu_\iota \;=\; \frac{c^4 e^4 n \ln \Lambda_{\iota\iota}}{4\pi m_\iota^2 v_{th\iota}^3} \;.$$

Introduce also the dimensionless velocity space coordinates $\boldsymbol{\xi}$ **and the dimensionless distribution functions** $\hat{f}_\alpha(\boldsymbol{\xi})$ **defined by**

$$\mathbf{v} = \mathbf{u}_\alpha + v_{th\alpha}\,\boldsymbol{\xi} \qquad \textbf{and} \qquad f_\alpha(\mathbf{v}) \;=\; f_\alpha(\mathbf{u}_\alpha + v_{th\alpha}\,\boldsymbol{\xi}) \;=\; \frac{n}{v_{th\alpha}^3}\hat{f}_\alpha(\boldsymbol{\xi}) \;, \qquad \textbf{such that}$$

$$\int d^3\boldsymbol{\xi}\;\hat{f}_\alpha(\boldsymbol{\xi}) = 1, \qquad \int d^3\boldsymbol{\xi}\;\xi_j\;\hat{f}_\alpha(\boldsymbol{\xi}) = 0, \qquad \textbf{and} \qquad \int d^3\boldsymbol{\xi}\;\xi_j\xi_k\;\hat{f}_\alpha(\boldsymbol{\xi}) = \frac{1}{p_\alpha}\,\mathsf{P}_{\alpha,jk}\;.$$

For the magnetized plasmas under consideration:

$$\hat{f}_\alpha(\boldsymbol{\xi}) \;=\; \hat{f}_\alpha^{(0)}(\xi,\xi_\parallel) \;+\; \hat{f}_\alpha^{(1)}(\boldsymbol{\xi}) \;+\; O(\delta_\alpha^2)\;,$$

where $\xi_\parallel = \xi_j b_j\;,\quad \hat{f}_\alpha^{(0)}(\xi,\xi_\parallel) = O(1)\;,\quad \hat{f}_\alpha^{(1)}(\boldsymbol{\xi}) = O(\delta_\alpha)$ **and** $\delta_\alpha \sim \rho_\alpha/L \sim \delta\;(m_\alpha/m_\iota)^{1/2}\;.$

PERTURBATIVE EXPANSION OF THE FRICTION FORCE

Keeping $O(\nu_e p_e/v_{the}) + O(\delta_e \nu_e p_e/v_{the})$ under our assumed orderings:

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\Big[\Big(1+\frac{m_e}{m_\iota}\Big)b_j \int d^3\boldsymbol{\xi}\;\frac{\xi_\parallel}{\xi^3}\hat{f}^{(0)}_e(\xi,\xi_\parallel) \;+\int d^3\boldsymbol{\xi}\;\frac{\xi_j}{\xi^3}\hat{f}^{(1)}_e(\boldsymbol{\xi})\;+$$

$$+\;\frac{1}{env_{the}}j_k\int\frac{d^3\boldsymbol{\xi}}{\xi}\;\frac{\partial^2\hat{f}^{(0)}_e(\xi,\xi_\parallel)}{\partial\xi_j\partial\xi_k}\;+\;\frac{m_e}{2m_\iota p_e}\mathsf{P}^{CGL}_{\iota,kl}\int\frac{d^3\boldsymbol{\xi}}{\xi}\;\frac{\partial^3\hat{f}^{(0)}_e(\xi,\xi_\parallel)}{\partial\xi_j\partial\xi_k\partial\xi_l}\;+\;O(\delta^3)\Big]\,.$$

If the zeroth-order electron distribution function were isotropic, $\hat{f}^{(0)}_e = \hat{f}^{(0)}_e(\xi)$, the leading order collisional friction force would reduce to

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\Big[\int d^3\boldsymbol{\xi}\;\frac{\xi_j}{\xi^3}\hat{f}^{(1)}_e(\boldsymbol{\xi})\;-\;\frac{4\pi\hat{f}^{(0)}_e(0)}{3env_{the}}\;j_j\Big]\;=\;O\Big(\delta_e\frac{\nu_e p_e}{v_{the}}\Big)\,.$$

In principle, for the low-collisionality regime of interest here, we have in the leading order:

$$F^{coll}_{\iota,j} \;=\; -F^{coll}_{e,j} \;=\; \frac{\nu_e p_e}{v_{the}}\;b_j\int d^3\boldsymbol{\xi}\;\frac{\xi_\parallel}{\xi^3}\hat{f}^{(0)}_e(\xi,\xi_\parallel)\;=\;O\Big(\frac{\nu_e p_e}{v_{the}}\Big)\,,$$

the remaining terms giving corrections of order $\delta_e\nu_e p_e/v_{the} \lesssim \delta^2\nu_e p_e/v_{the}$ or higher.

PERTURBATIVE EXPANSION OF THE HIGHER COLLISIONAL MOMENTS

Keeping $O(\nu_e p_e)$ and $O(\nu_\iota p_\iota)$, but neglecting $O(\delta_e \nu_e p_e) \sim O(\delta \nu_\iota p_\iota)$:

$$\mathsf{G}^{coll}_{e,jk} \;=\; \frac{1}{2}\,\nu_e p_e\,(3b_j b_k - \delta_{jk})\int d^3\boldsymbol{\xi}\,\Big(\frac{\xi^2 - 3\xi_\parallel^2}{\xi^3}\Big)\Big[\Im^{(0)}_e(\xi,\;\xi_\parallel) + \hat{f}^{(0)}_e(\xi,\;\xi_\parallel)\Big] \;+\; O(\delta_e \nu_e p_e)$$

and

$$\mathsf{G}^{coll}_{\iota,jk} \;=\; \frac{1}{2}\,\nu_\iota p_\iota\,(3b_j b_k - \delta_{jk})\int d^3\boldsymbol{\xi}\,\Big(\frac{\xi^2 - 3\xi_\parallel^2}{\xi^3}\Big)\,\Im^{(0)}_\iota(\xi,\;\xi_\parallel) \;+\; O(\delta \nu_\iota p_\iota)\;,$$

where

$$\Im^{(0)}_\alpha(\xi,\;\xi_\parallel) \;=\; \int d^3\boldsymbol{\zeta}\;\;\hat{f}^{(0)}_\alpha\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\;\xi_\parallel+\zeta_\parallel\Big)\;\;\hat{f}^{(0)}_\alpha(\zeta,\;\zeta_\parallel) \;=\; O(1)\;.$$

Therefore, within this accuracy, $\mathsf{G}^{coll}_{e,jk}$ and $\mathsf{G}^{coll}_{\iota,jk}$ do not contribute to either g^{coll}_e, g^{coll}_ι, $\hat{\mathsf{P}}^{coll}_{e,jk}$ or $\hat{\mathsf{P}}^{coll}_{\iota,jk}$, and contribute only to g^{coll}_{eB} and $g^{coll}_{\iota B}$.

Keeping $O(\nu_e p_e v_{the})$, but neglecting $O(\delta_e \nu_e p_e v_{the})$:

$$\mathsf{H}^{coll}_{e,jkl} \; - \; \frac{1}{m_e n} F^{coll}_{e,[j} \; \mathsf{P}_{e,kl]} \; =$$

$$= \; \nu_e p_e v_{the} \left\{ \, b_{[j}\delta_{kl]} \left[\frac{1}{2} \int\!\!\int d^3\boldsymbol{\xi} \; d^3\boldsymbol{\zeta} \;\; \hat{f}^{(0)}_e\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|, \; \xi_\parallel + \zeta_\parallel\Big) \;\; \hat{f}^{(0)}_e(\zeta, \; \zeta_\parallel) \; \Big(\frac{9\xi_\parallel^2 \; \zeta_\parallel \; - \; \xi^2 \; \zeta_\parallel \; - \; 6\xi_\parallel \; \boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\Big) + \right.\right.$$

$$+ \; \int d^3\boldsymbol{\xi} \; \hat{f}^{(0)}_e(\xi, \; \xi_\parallel) \; \frac{\xi_\parallel \; (3\xi_\parallel^2 \; - \; 2\xi^2)}{\xi^3} \; + \frac{p_{e\perp}}{p_e} \int d^3\boldsymbol{\xi} \; \hat{f}^{(0)}_e(\xi, \; \xi_\parallel) \; \frac{\xi_\parallel}{\xi^3} \; \Big] +$$

$$+ \; b_j b_k b_l \Big[\frac{9}{2} \int\!\!\int d^3\boldsymbol{\xi} \; d^3\boldsymbol{\zeta} \;\; \hat{f}^{(0)}_e\Big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|, \; \xi_\parallel + \zeta_\parallel\Big) \;\; \hat{f}^{(0)}_e(\zeta, \; \zeta_\parallel) \; \Big(\frac{\xi^2 \; \zeta_\parallel \; - \; 5\xi_\parallel^2 \; \zeta_\parallel \; + \; 2\xi_\parallel \; \boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\Big) +$$

$$\left.\left. + \; 3 \int d^3\boldsymbol{\xi} \; \hat{f}^{(0)}_e(\xi, \; \xi_\parallel) \; \frac{\xi_\parallel \; (3\xi^2 \; - \; 5\xi_\parallel^2)}{\xi^3} \; + \; \frac{3(p_{e\parallel} - p_{e\perp})}{p_e} \int d^3\boldsymbol{\xi} \; \hat{f}^{(0)}_e(\xi, \; \xi_\parallel) \; \frac{\xi_\parallel}{\xi^3} \; \right] \right\} \; + \; O(\delta_e \nu_e p_e v_{the}) \; .$$

Within this accuracy, $\mathbf{H}^{coll}_{e,jkl} - F^{coll}_{e,[j} \; \mathbf{P}_{e,kl]}/(m_e n)$ does not contribute to $\hat{\mathbf{Q}}^{coll}_{e,jkl}$.

Keeping $O(\nu_\iota p_\iota v_{th\iota})$, but neglecting $O(\delta\nu_\iota p_\iota v_{th\iota})$:

$$\mathsf{H}^{coll}_{\iota,jkl} \;-\; \frac{1}{m_\iota n} F^{coll}_{\iota,[j} \,\mathsf{P}_{\iota,kl]} \;=$$

$$= \;\nu_\iota p_\iota v_{th\iota} \Big[\,\frac{1}{2}\, b_{[j}\delta_{kl]} \int\!\!\int d^3\boldsymbol{\xi}\; d^3\boldsymbol{\zeta}\;\; \hat{f}^{(0)}_\iota\big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\; \xi_\parallel+\zeta_\parallel\big)\;\; \hat{f}^{(0)}_\iota(\zeta,\; \zeta_\parallel)\;\Big(\frac{9\xi_\parallel^2\,\zeta_\parallel \;-\; \xi^2\,\zeta_\parallel \;-\; 6\xi_\parallel\,\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\Big) \;+$$

$$+\,\frac{9}{2}\, b_j b_k b_l \int\!\!\int d^3\boldsymbol{\xi}\; d^3\boldsymbol{\zeta}\;\; \hat{f}^{(0)}_\iota\big(|\boldsymbol{\xi}+\boldsymbol{\zeta}|,\; \xi_\parallel+\zeta_\parallel\big)\;\; \hat{f}^{(0)}_\iota(\zeta,\; \zeta_\parallel)\;\Big(\frac{\xi^2\,\zeta_\parallel \;-\; 5\xi_\parallel^2\,\zeta_\parallel \;+\; 2\xi_\parallel\,\boldsymbol{\xi}\cdot\boldsymbol{\zeta}}{\xi^3}\Big)\Big] \;+\; O(\delta\nu_\iota p_\iota v_{th\iota})\;.$$

Within this accuracy, $\mathsf{H}^{coll}_{\iota,jkl} - F^{coll}_{\iota,[j}\,\mathsf{P}_{\iota,kl]}/(m_\iota n)$ does not contribute to $\hat{\mathsf{Q}}^{coll}_{\iota,jkl}$.

IN SUMMARY, THE COLLISIONAL MOMENTS DO NOT CONTRIBUTE TO THE PERPENDICULAR STRESS TENSORS $\hat{\mathsf{P}}_{\alpha,jk}$ WITHIN $O(\delta p_\alpha u_\alpha/v_{th\iota})$, TO THE PERPENDICULAR STRESS-FLUX TENSORS $\hat{\mathsf{Q}}_{\alpha,jkl}$ WITHIN $O(\delta p_\alpha v_{th\iota})$, OR TO THE HEAT EXCHANGE RATES g^{coll}_α WITHIN $O(\delta p_\alpha v_{th\iota}/L)$.

III. COLLISIONAL CONTRIBUTIONS IN THE EXTENDED MHD TWO-FLUID EQUATIONS

PARALLEL OHM'S LAW

From the parallel component of the electron momentum equation, keeping the accuracy of $O(\delta v_{th\iota} B) + O(\delta^2 v_{th\iota} B)$**:**

$$E_{\|} \; = \; \frac{1}{en}\Big[-\mathbf{b}\cdot\nabla p_{e\|} \; + \; (p_{e\|} - p_{e\perp})\mathbf{b}\cdot\nabla(\ln B) \; + \; F_{e\|}^{coll}\Big] \; ,$$

with

$$F_{e\|}^{coll} \; = \; -\,\frac{\nu_e p_e}{v_{the}}\int d^3\boldsymbol{\xi}\,\frac{\xi_{\|}}{\xi^3}\hat{f}_e^{(0)}(\xi,\xi_{\|}) \; \sim \; \nu_e p_e/v_{the} \; \lesssim \; \delta\, p_e/L \; .$$

ION PRESSURE EQUATIONS

Keeping $O(p_\iota v_{th\iota}/L) + O(\delta p_\iota v_{th\iota}/L)$:

$$\frac{3}{2}\Big[\frac{\partial p_\iota}{\partial t} + \nabla\cdot\Big(p_\iota \mathbf{u}_\iota\Big)\Big] \;+\; p_\iota \nabla\cdot\mathbf{u}_\iota \;+\; (p_{\iota\|} - p_{\iota\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big] - \nabla\cdot\mathbf{u}_\iota/3\Big\} \;+\; \nabla\cdot\Big(q_{\iota\|}\mathbf{b}\Big) \;+$$

$$+\; \hat{\mathsf{P}}_\iota^{gyr} : (\nabla\mathbf{u}_\iota) \;+\; \nabla\cdot\mathbf{q}_{\iota\perp}^{gyr} \;=\; 0\;,$$

and

$$\frac{1}{2}\Big[\frac{\partial p_{\iota\|}}{\partial t} + \nabla\cdot\Big(p_{\iota\|}\mathbf{u}_\iota\Big)\Big] \;+\; p_{\iota\|}\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big] \;+\; \nabla\cdot\Big(q_{\iota B\|}\mathbf{b}\Big) \;+\; q_{\iota T\|}\mathbf{b}\cdot\nabla(\ln\; B) \;+$$

$$+\; \mathbf{b}\cdot\hat{\mathsf{P}}_\iota^{gyr}\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota) \;+\; \nabla\cdot\mathbf{q}_{\iota B\perp}^{gyr} \;-\; 2\mathbf{q}_{\iota B\perp}^{gyr}\cdot\boldsymbol{\kappa} \;+\; q_{\iota T\|}\sigma_\iota^{gyr} \;-\; g_{\iota B}^{coll} \;=\; 0\;,$$

where the first line of each equation contains the classic CGL terms of order $(p_\iota v_{th\iota}/L)$, and the terms involving $\hat{\mathsf{P}}_\iota^{gyr}$, $\mathbf{q}_{\iota\perp}^{gyr}$, $\mathbf{q}_{\iota B\perp}^{gyr}$ and σ_ι^{gyr} are the collision-independent FLR corrections of order $(\delta p_\iota v_{th\iota}/L)$.

Consider the linear combination that yields the evolution of the ion pressure anisotropy:

$$\frac{\partial(p_{\iota\|}-p_{\iota\perp})}{\partial t} + \nabla\cdot\Big[(p_{\iota\|}-p_{\iota\perp})\mathbf{u}_\iota\Big] + (p_{\iota\|}-p_{\iota\perp})\Big\{\mathbf{b}\cdot\Big[(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]+\nabla\cdot\mathbf{u}_\iota/3\Big\} +$$

$$+ p_\iota\Big\{\mathbf{b}\cdot\Big[3(\mathbf{b}\cdot\nabla)\mathbf{u}_\iota\Big]-\nabla\cdot\mathbf{u}_\iota\Big\} + \nabla\cdot\Big[(3q_{\iota B\|}-q_{\iota\|})\mathbf{b}\Big] + 3q_{\iota T\|}\mathbf{b}\cdot\nabla(\ln\ B) +$$

$$+ 3\mathbf{b}\cdot\hat{\mathsf{P}}_\iota^{gyr}\cdot(\mathbf{b}\times\boldsymbol{\omega}_\iota) - \hat{\mathsf{P}}_\iota^{gyr}:(\nabla\mathbf{u}_\iota) + \nabla\cdot(3\mathbf{q}_{\iota B\perp}^{gyr}-\mathbf{q}_{\iota\perp}^{gyr}) - 6\mathbf{q}_{\iota B\perp}^{gyr}\cdot\boldsymbol{\kappa} + 3q_{\iota T\|}\sigma_\iota^{gyr} - 3g_{\iota B}^{coll} = 0\,,$$

with

$$g_{\iota B}^{coll} = \nu_\iota p_\iota \int d^3\boldsymbol{\xi}\,\Big(\frac{\xi^2-3\xi_\|^2}{2\xi^3}\Big)\,\mathfrak{F}_\iota^{(0)}(\xi,\ \xi_\|) \sim \nu_\iota(p_{\iota\|}-p_{\iota\perp})\,.$$

Since $\nu_\iota \lesssim \delta^2\,\Omega_{c\iota}$, collisions cannot force $(p_{\iota\|}-p_{\iota\perp})$ to be much smaller than p_ι on either the sonic $(\partial/\partial t \sim \delta\ \Omega_{c\iota})$ or diamagnetic drift $(\partial/\partial t \sim \delta^2\ \Omega_{c\iota})$ time scales.

The same argument holds for the electrons, with $\nu_e \lesssim \delta\ \Omega_{c\iota}$, on the sonic time scale.

ELECTRON PRESSURE EQUATIONS

Keeping $O(p_e v_{th\iota}/L) + O(\delta p_e v_{th\iota}/L)$**:**

$$\frac{3}{2}\Big[\frac{\partial p_e}{\partial t} + \nabla\cdot\big(p_e\mathbf{u}_e\big)\Big] \;+\; p_e\nabla\cdot\mathbf{u}_e \;+\; (p_{e\|} - p_{e\perp})\Big\{\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\big] - \nabla\cdot\mathbf{u}_e/3\Big\} \;+\; \nabla\cdot\big(q_{e\|}\mathbf{b} + \mathbf{q}_{e\perp}^{gyr}\big) \;=\; 0$$

and

$$\frac{1}{2}\Big[\frac{\partial p_{e\|}}{\partial t} + \nabla\cdot\big(p_{e\|}\mathbf{u}_e\big)\Big] + p_{e\|}\mathbf{b}\cdot\big[(\mathbf{b}\cdot\nabla)\mathbf{u}_e\big] + \nabla\cdot\big(q_{eB\|}\mathbf{b} + \mathbf{q}_{eB\perp}^{gyr}\big) + q_{eT\|}\mathbf{b}\cdot\nabla(\ln\ B) - 2\mathbf{q}_{eB\perp}^{gyr}\cdot\boldsymbol{\kappa} - g_{eB}^{coll} \;=0,$$

or

$$\frac{\partial(p_{e\|} - p_{e\perp})}{\partial t} \;+\; \nabla\cdot[(p_{e\|} - p_{e\perp})\mathbf{u}_e] \;+\; (p_{e\|} - p_{e\perp})\{\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}_e] + \nabla\cdot\mathbf{u}_e/3\} \;+$$

$$+\; p_e\{3\mathbf{b}\cdot[(\mathbf{b}\cdot\nabla)\mathbf{u}_e] - \nabla\cdot\mathbf{u}_e\} \;+\; \nabla\cdot[(3q_{eB\|} - q_{e\|})\mathbf{b}] \;+\; 3q_{eT\|}\mathbf{b}\cdot\nabla(\ln\ B) \;+$$

$$+\; \nabla\cdot(3\mathbf{q}_{eB\perp}^{gyr} - \mathbf{q}_{e\perp}^{gyr}) \;-\; 6\mathbf{q}_{eB\perp}^{gyr}\cdot\boldsymbol{\kappa} \;-\; 3g_{eB}^{coll} \;=\; 0\;,$$

where

$$g_{eB}^{coll} \;=\; \nu_e p_e \int d^3\boldsymbol{\xi}\,\Big(\frac{\xi^2 - 3\xi_\|^2}{2\xi^3}\Big)\Big[\Im_e^{(0)}(\xi,\ \xi_\|) + \hat{f}_e^{(0)}(\xi,\ \xi_\|)\Big] \;\sim\; \nu_e(p_{e\|} - p_{e\perp})\;.$$

IV. SUMMARY

Assuming $\nu_i \lesssim \omega_* \lesssim \partial/\partial t$ and considering the parallel heat fluxes as closure variables to be determined kinetically, the only collisional terms that need be retained in the Extended MHD two-fluid equations are:

$$F_{e\parallel}^{coll} = -\frac{\nu_e p_e}{v_{the}} \int d^3\boldsymbol{\xi}\, \frac{\xi_\parallel}{\xi^3} \hat{f}_e^{(0)}(\xi, \xi_\parallel) \sim \nu_e p_e / v_{the}$$

in the parallel component of Ohm's law,

$$g_{\iota B}^{coll} = \nu_\iota p_\iota \int d^3\boldsymbol{\xi} \left(\frac{\xi^2 - 3\xi_\parallel^2}{2\xi^3}\right) \Im_\iota^{(0)}(\xi,\ \xi_\parallel) \sim \nu_\iota (p_{\iota\parallel} - p_{\iota\perp})$$

in the ion pressure anisotropy evolution equation,

and $$g_{eB}^{coll} = \nu_e p_e \int d^3\boldsymbol{\xi} \left(\frac{\xi^2 - 3\xi_\parallel^2}{2\xi^3}\right) \left[\Im_e^{(0)}(\xi,\ \xi_\parallel) + \hat{f}_e^{(0)}(\xi,\ \xi_\parallel)\right] \sim \nu_e (p_{e\parallel} - p_{e\perp})$$

in the electron pressure anisotropy evolution equation,

with $$\Im_\alpha^{(0)}(\xi,\ \xi_\parallel) = \int d^3\boldsymbol{\zeta}\ \ \hat{f}_\alpha^{(0)}\big(|\boldsymbol{\xi} + \boldsymbol{\zeta}|,\ \xi_\parallel + \zeta_\parallel\big)\ \ \hat{f}_\alpha^{(0)}(\zeta,\ \zeta_\parallel)\ .$$