GC Plasma Models in 3D and ELM Modelling

L. Sugiyama

Massachusetts Institute of Technology
Cambridge MA 02139-4307

CEMM Meeting APS-DPP
Nov. 11, 2007
Orlando FL

TOPICS

- **Guiding Center plasma models in 3D magnetic fields**

  3D geometrical conditions on validity of GC model

  Time-dependence

  Relation of Guiding Center to fluid picture

- **M3D code improved for long-time nonlinear ELM instability and saturation**

  2F long-time simulations starting

GC: Poster UP8, Thursday afternoon

Guiding Center plasma models in 3D plasmas

- **Guiding Center (GC) model for single charged particle separates particle motion into guiding center motion and faster gyration around magnetic field lines,**

$$\mathbf{x} = \mathbf{X} + \frac{\epsilon v_\perp}{\Omega}\hat{\boldsymbol{\rho}}. \tag{1}$$

"Tube" picture of particle motion: Guiding center trajectory $\mathbf{X}$ defines a tube in space, particle moves on tube. Slices through the tube define the gyro-orbit.

- **2D slab (straight uniform magnetic field lines) has exact GC expansion in small parameter ρ_i/L to all orders.**

- **3D plasma GC expansion still has problems after many years.**
  - **Good mathematical formalism (Hamiltonian/Lagrangian variational principle)**
  - **First order equations well known (Morozov-Solovév (1963), Northrup-Rome (1978) et al. direct derivation; later Hamiltonian/Lagrangian method) but have large $v_\parallel/v_\perp$ restriction, magnetic axis problem**

– **Lowest order drift picture (drift kinetic equation) does not match MHD in a torus $\longrightarrow$ reduced (r/R) approximation and no shear Alfvén wave.**

– **Does not match small gyro-radius fluid equation expansion (Ramos (2005))**

WHY??

GC expansion is not exact in most confined plasmas

- **Gyromotion $(\mathrm{x} - \mathrm{X})$ is described in local orthogonal coordinates tied to the magnetic field line, with unit vectors $(\hat{\mathrm{e}}_1,\ \hat{\mathrm{e}}_2,\ \hat{\mathrm{b}})$ where $\hat{\mathrm{b}} = \mathrm{B}/B$ along the magnetic field.**

  Phase space $(\mathrm{X},\ \mathrm{V},\ t)$, $\mathrm{V} = (v_\parallel, \theta, v_\perp)$ describes GC.

- **Two kinds of 3D geometrical effects on a vector field, eg B**

  Metric (local) — curvature $\boldsymbol{\kappa} = \hat{\mathrm{b}} \cdot \boldsymbol{\nabla}\hat{\mathrm{b}}$, metric tensor $g_{\mu\nu}$

  Topological (nonlocal) — field line twisting (torsion), affine connection, asymmetric Christoffel symbols $\Gamma^k_{ij} \neq \Gamma^k_{ji}$

- **Fluid dynamics: vorticity $\boldsymbol{\nabla} \times \mathrm{v}$ is important, theory well developed.**

- **In 3D, the component of the curl of a vector field in a given direction is twice the rate of rotation of the field about that axis see when moving in that direction.**

  Torsion $\boldsymbol{\tau} = \hat{\mathrm{b}} \cdot \boldsymbol{\nabla} \times \hat{\mathrm{b}} = (1/B^2)\mathrm{B} \cdot \boldsymbol{\nabla} \times \mathrm{B}$ is the twisting of the field line along itself.

$$\boldsymbol{\nabla} \times \hat{\mathrm{b}} = \boldsymbol{\tau}\hat{\mathrm{b}} + \boldsymbol{\kappa}\hat{\boldsymbol{\beta}}. \tag{2}$$

A 3D plasma typically has plasma current $\mathrm{J}_{\|} = \hat{\mathrm{b}} \cdot \nabla \times \mathrm{B}$, so torsion is nonzero!

- **Torsion $\tau \neq 0$: the gyromotion mixes parallel and perpendicular directions! No global planes perpendicular to field line exist.**

  For a small closed curve C surrounding a field line that encloses a surface S that has normal direction n along $\hat{\mathrm{b}}$ at one point P on the field line,

$$\lim_{S \to 0} \frac{1}{S} \oint_C d\mathrm{l} \cdot \mathrm{B} = \lim_{S \to 0} \frac{1}{S} \int\int_S dS\, \mathrm{n} \cdot \nabla \times \hat{\mathrm{b}} = \hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}}|_P. \qquad (3)$$

  Holds for all paths C.

- **If plasma has magnetic surfaces, a second type of torsion can be defined, the geodesic torsion τ_g relative to the field line in the surface.**

  Torsion τ is rotation of normal and binormal axes around field line, where the normal is direction of curvature of field line $\kappa = \kappa \hat{\mathrm{n}}$, binormal $\hat{\mathrm{b}} \times \hat{\mathrm{n}}$. Geodesic torsion τ_g takes the normal to be the (inward) normal to the surface, eg $\hat{\mathrm{N}} = (-)\nabla\psi/|\nabla\psi|$ on torus.

  GC rotation parameter $\hat{\mathrm{b}} \cdot \mathrm{R} \equiv \hat{\mathrm{b}} \cdot (\nabla \hat{\mathrm{e}}_1) \cdot \hat{\mathrm{e}}_2 = (-)\tau_g$ if $\hat{\mathrm{e}}_1 = -\hat{\mathrm{N}}$.

- **GC expansion**

  Lowest order: $\langle \mathrm{x} \rangle = \mathrm{X}$, $\langle v \rangle = v_{\parallel}$

  First order: τ **and** τ_g **appear!**

$$\begin{aligned} \dot{\zeta} &= \Omega \left[1 + \frac{\epsilon w}{B} \frac{U_{\parallel}}{w} \left(\frac{1}{2} \hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}} + \hat{\mathrm{b}} \cdot \mathrm{R} \right) \right] \\ \langle v_{\parallel} \rangle &= U_{\parallel} \left[1 - \frac{\epsilon w}{B} \frac{w}{U_{\parallel}} \left(\frac{1}{2} \hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}} - \hat{\mathrm{b}} \cdot \mathrm{R} \right) \right] \end{aligned} \tag{4}$$

- **Nonuniform gyroangle $\rightarrow$ nonuniform gyroperiod, depending on parallel velocity $U_{\parallel}$ and its direction. (Gyroperiod/gyroaverage defined as $\oint d\zeta = 2\pi$.)**

  Particle sees longer or shorter gyroperiod depending on whether it moves parallel or anti-parallel to B, also how far it moves along B in gyroperiod. Due to torsion of field lines – the baseline direction for defining ζ rotates along B. (Northrup 1978)

- **The angle coordinate nonuniformity due to torsion is a real physical effect; appears in many areas (Aharonov-Bohm effect, Berry phase, related to Dirac magnetic monopole)**

- **First order problems affect small region of phase space, not too bad in practice.**

- **Second order.**
- **Second order GC equations from Hamiltonian or Lagrangian non-canonical phase-space variable methods, developed to extend the expansion to all orders (Littlejohn, Brizzard).**

  Whole point of this formulation is to eliminate the geometrical terms from the dynamical equations, keeping only in gyroangle time derivative $\dot{\zeta}$.

- **Add free gauge functions to the Lagrangian and define their gyroaverages to eliminate $\hat{\mathrm{b}} \cdot \nabla \times \hat{\mathrm{b}}$ and $\hat{\mathrm{b}} \cdot \mathrm{R}$: OK**
- **Effective magnetic vector potential A^* allows many terms to be eliminated by defining effective B^***

$$\mathbf{A}^{*}* = \mathbf{A} + \epsilon U' \hat{\mathrm{b}} - \epsilon^2 \mu \mathrm{R}. \tag{5}$$

- **Problem: Full R. The curl for $\nabla \times \mathrm{A}^*$ is taken in phase space coordinates X, but R is defined in local magnetic coordinates. No connection specified for local coordinate systems at different points. When is the curl well defined?**

  Can local magnetic coordinates be extended to a global coordinate system, so that the gradients of R are well defined?

- **Answer is closely related to τ and τ_g, ie, twisting of field lines**
- **General 3D configuration requires $\tau = \tau_g = 0$.**
- **Axisymmetric torus (2D) has second order GC expansion for nonzero τ and τ_g, valid to same degree as first order.**
- **Extending local coordinates over a flux surface is possible. In GC theory, done via the generalized gyrogauge that broke some of the Poisson bracket symmetries, to avoid using the gyrophase gauge function in gyroangle.**
- **Extending local coordinates across flux surfaces is tricky and requires either 2D restrictions, such as axisymmetry, or no torsion.**
- **Conclusions based on results from 3D differential geometry. (details in poster)**
- **General n-dimensional results (using manifolds, differential forms) show that the problem is one of linking the twisting of the different coordinate systems (the affine connections) and that it is solvable given restrictions on the exterior derivatives of the general torsion and curvature two-forms.**

Drift kinetic equation and MHD in torus

- Drift kinetic equation requires the time derivative of $\partial \mathrm{A}/\partial t$ in the $\mathrm{E}\times\mathrm{B}$ drift at lowest order to match MHD in a toroidal plasma.

- Electrostatic drift is equivalent to dropping parallel gradients; for equilibrium quantities, this drops some higher order terms in inverse aspect ratio.

  Take the divergence of Ohm's law $\mathrm{E} + (1/c)\,(\mathrm{v}\times\mathrm{B}) = 0$,

$$c\nabla^2\phi = \nabla\cdot(\mathrm{v}\times\mathrm{B})\,. \tag{6}$$

  Compare to divergence of lowest order drift $\mathrm{v}\times\mathrm{B}\simeq \mathrm{v}_E\times\mathrm{B}$,

$$\nabla\cdot(\mathrm{v}\times\mathrm{B}) \simeq \nabla\cdot(\mathrm{v}_E\times\mathrm{B}) \simeq c\nabla\cdot\Big(\nabla\phi - \hat{\mathrm{b}}(\hat{\mathrm{b}}\cdot\nabla\phi)\Big)\,, \tag{7}$$

  Since $\hat{\mathrm{b}}\cdot\nabla\phi = 0$ already from parallel Ohm's law, $(\hat{\mathrm{b}}\cdot\nabla)(\hat{\mathrm{b}}\cdot\nabla B) = 0$.

- No shear Alfvén wave exists. Polarization drift derivation used $\nabla\cdot(\mathrm{J}_{GC}\simeq \mathrm{J}_{GC\|}\simeq \mathrm{J}_\|)$, but divergence is constrained by charge neutrality and $\nabla\cdot\mathrm{J}_{GC}$ is one order smaller than $\nabla\cdot\mathrm{J}_\|$.

- Adding back electromagnetic part of drift reproduces full toroidal geometry and shear Alfvén wave, full MHD w/o compressional Alfvén wave.

GC Model in 3D: conclusions

- 3D imposes strong constraints on the validity of the GC idea compared to 2D.
- Gyromotion is non-uniform in phase space due to 3D twisting of field lines

  First order GC equations give useful model; 3D inconsistencies involve restricted regions of phase space.

- Strong 3D constraint required to be able extend the local magnetic coordinate systems at each point to a global coordinate system in which cross-flux-surface derivatives can be defined.

  Second order GC equations require axisymmetry (2D) or $\tau = \tau_g = 0$. (Cross-field drifts appear at first order, so need gradients in next order.)

- First order GC (lowest order $\mathrm{E} \times \mathrm{B}$ drift) must be electromagnetic to reproduce full toroidal geometry and the MHD shear Alfvén wave. Leads to potential ordering problems in second order.
- In "guiding center tube" picture of particle motion, a gyro-orbit is not well-described by slices through the tube, even in axisymmetry. In 3D, each point on slice corresponds to different GC trajectory.