Are ghost surfaces and quadratic-flux-minimizing surfaces the same?

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Motivation

→ heat transport is anisotropic: $\nabla \cdot \left(\kappa_{\parallel} \, \mathbf{b} \, \mathbf{b} \cdot \nabla T + \kappa_{\perp} \nabla_{\perp} T \right) = 0$ with $\kappa_{\perp} / \kappa_{\parallel} = 10^{-10}$

→ if nested flux surfaces exist, then $T = T(\psi)$, where ψ labels invariant surfaces
→ if field is chaotic, goal is to adapt coordinates so that $T = T(s)$,

→ we need a fast, robust, simple construction of chaotic coordinates;
→ this talk will show that, despite their different definitions,
ghost-surfaces and **quadratic-flux minimizing surfaces** are almost identical

→ the "easy method" of constructing the "best surfaces" may be possible!

Part 1 Motivation

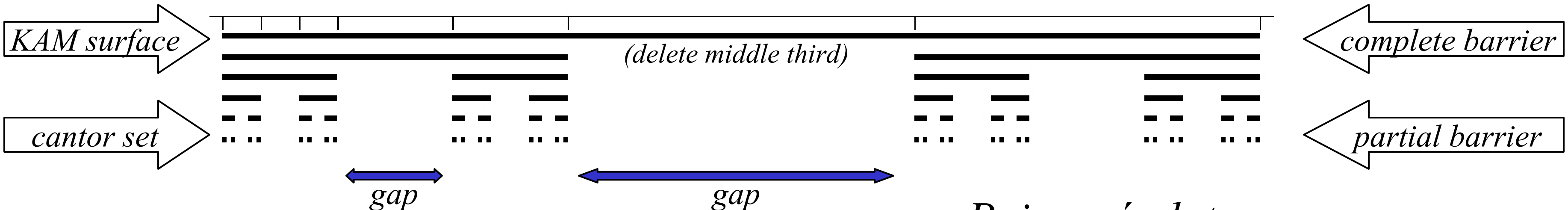
1) field line transport in chaos is restricted by cantori

2) construct ***chaotic-coordinates*** bit fitting coordinate surfaces to cantori

3) chaotic-coordinates allow simple solution for anisotropic transport

Field-line transport is restricted by irrational field-lines

→ *the irrational KAM surfaces disintegrate into invariant irrational sets ≡ cantori, which continue to restrict field-line transport even after the onset of chaos.*



→ *KAM surfaces* ***stop*** *radial field-line transport*

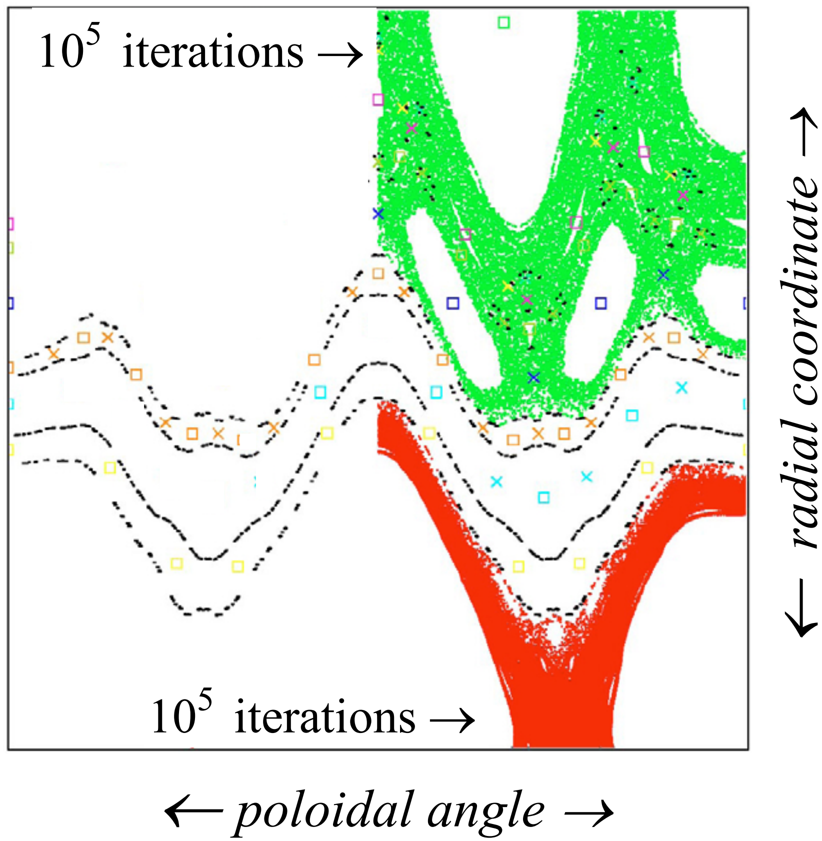
→ *broken KAM surfaces ≡cantori*
do not stop, but do slow down
radial field-line transport

→ *cantori are approximated by high-order, unstable, periodic-orbits;*

→ *chaotic-coordinates are fit to the cantori, we need to (i) locate the cantori*
(ii) fill-in-the-gaps to make surfaces

Poincaré plot (model field)

"noble" cantori (black dots)



Anisotropic transport is solved by chaotic-coordinates.

→ *ghost-surfaces for high-order periodic orbits "fill-in-the-gaps" in the irrational cantori;*
→ *ghost-surfaces and isotherms are almost indistinguishable; suggests T=T(s);*

- heat transport in plasmas is strongly anisotropic

$$\kappa_{\parallel}\nabla_{\parallel}^2 T + \kappa_{\perp}\nabla_{\perp}^2 T = S, \qquad (S \text{ is source})$$

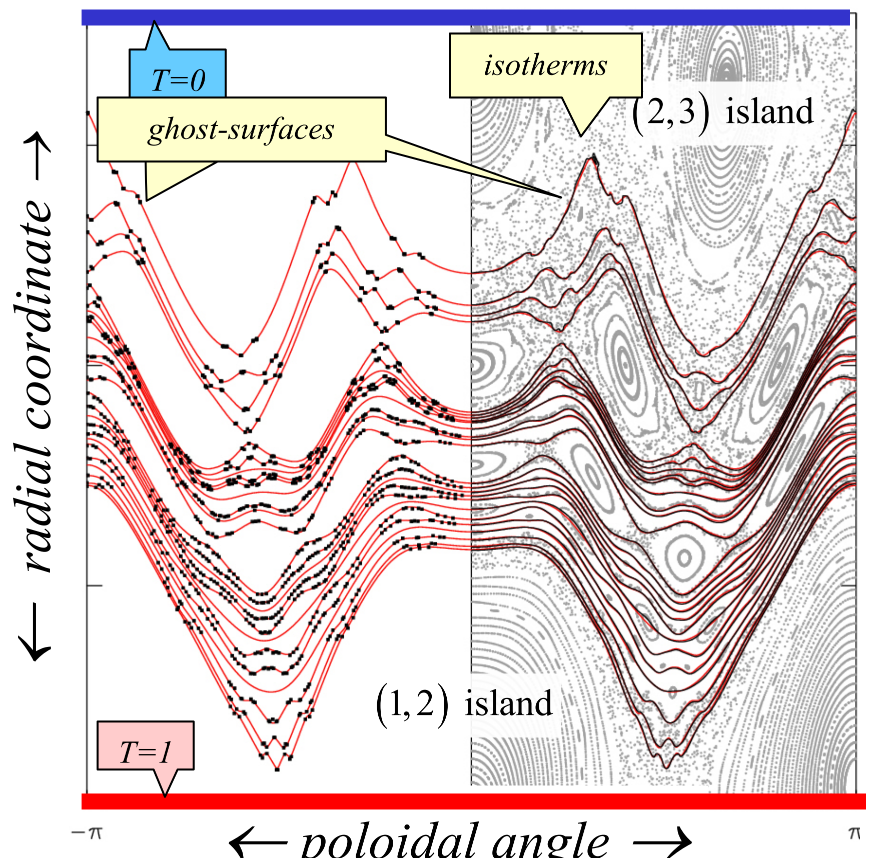
$\kappa_{\perp}/\kappa_{\parallel} \sim 10^{-10}$, solved numerically on grid $2^{12} \times 2^{12}$

→ parallel diffusion dominates perpendicular diffusion

- structure of temperature is dominated by the structure of the magnetic field;

→ structure of coordinates ≡ structure of field

- temperature adapts to almost-invariant surfaces; we obtain $T = T(s)$, where s labels ghost surfaces;



Hudson & Breslau, Phys. Rev. Lett. **100**, 095001 (2008)

Chaotic-coordinates simplifies temperature profile

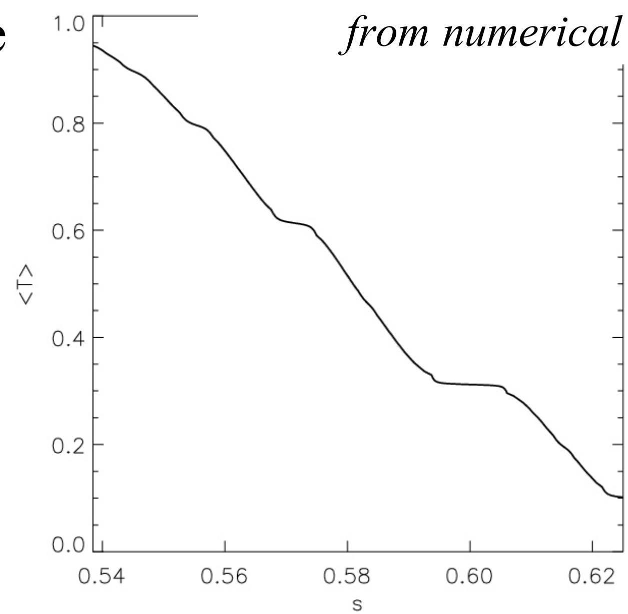
→ ghost-surfaces can be used as radial coordinate surfaces → chaotic-coordinates (s,θ,ϕ)

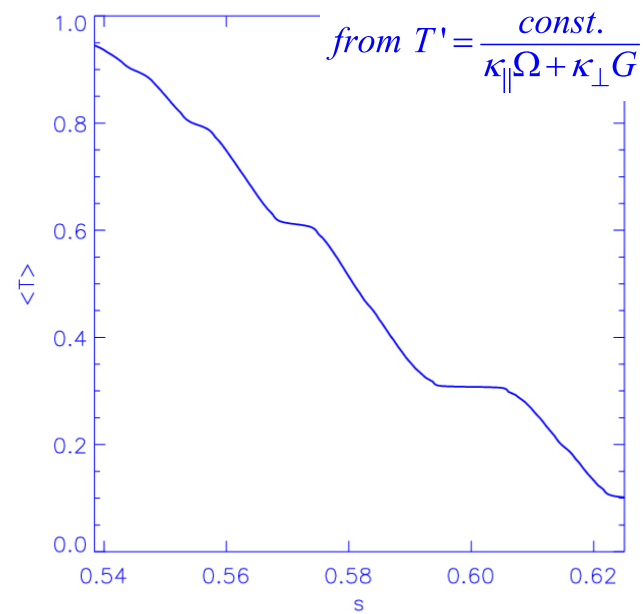
- From $0=\dfrac{\partial}{\partial s}\displaystyle\int_V \nabla\cdot\mathbf{q}\,dV=\dfrac{\partial}{\partial s}\displaystyle\int_{\partial V}\mathbf{q}\cdot\mathbf{n}\,d\sigma$ assume $T=T(s)$ to derive $T'=\dfrac{const.}{\kappa_{\|}\Omega+\kappa_{\perp}G}$

  for quadratic-flux $\Omega=\int d\sigma g^{ss}(B_n/B)^2$, and metric $G=\int d\sigma g^{ss}$, where $g^{ss}=\nabla s\cdot\nabla s$, $B_n=\mathbf{B}\cdot\nabla s/|\nabla s|$
- in the "ideal limit" $\kappa_{\perp}\to 0$, $T'\to\infty$ on irrational KAM surfaces where $\Omega=0$;
- non-zero $\kappa_{\perp}$ ensures $T(s)$ is smooth, T' peaks on minimal-Ω surfaces (noble cantori).

Temperature Profile

$(\kappa_{\perp}/\kappa_{\|}=10^{-10})$





Part 2 Almost invariant surfaces

1) two classes of almost-invariant surfaces have been suggested :

a) quadratic-flux minimizing (QFMin) surfaces, and
b) ghost-surfaces

2) an efficient, robust algorithm for constructing QFMin surfaces exists, but ghost-surfaces have attractive mathematical properties *(e.g. guaranteed non-intersection at strong-chaos, . . .)*

3) if the different classes of surfaces are in fact the same, improved numerical methods become available

Quadratic-flux minimizing (QFMin) surfaces are a natural extension of flux surfaces, defined for chaotic fields

- toroidal coordinates (ψ,θ,ζ), given magnetic field $\mathbf{B}=\nabla\times(\psi\,\nabla\theta-\chi\,\nabla\zeta)$, where $\chi=\chi(\psi,\theta,\zeta)$,

- a toroidal surface may be described $\psi=P(\theta,\zeta)$, normal $\mathbf{N}\equiv(\mathbf{e}_\theta+P_\theta\mathbf{e}_s)\times(\mathbf{e}_\zeta+P_\zeta\mathbf{e}_s)$, $\nu\equiv\mathbf{B}\bullet\mathbf{N}$

- tangential dynamics described according to
  angle dynamics from field, $\dot{\theta}=B^\theta/B^\zeta$, and radial dynamics constrained to lie on surface $\dot{\psi}=P_\theta\dot{\theta}+P_\zeta$
  i.e. *pseudo-field* $\mathbf{B}_\nu=\mathbf{B}-\nu\,\nabla\theta\times\nabla\zeta$

- quadratic flux $\varphi_2=\frac{1}{2}\int\int(\mathbf{B}\bullet\mathbf{N})^2\,d\theta d\zeta$,

Note → coordinate dependence;
extra Jacobian factor appears

- allowing the surface to vary to extremize φ_2, obtain Euler-Lagrange equation $\mathbf{B}_\nu\bullet\nabla\nu=0$
  pseudo-field-lines determined by following *pseudo*-field

1) normal-field, ν, constant along pseudo-field lines;
2) rational surface = family of periodic pseudo-field lines;
3) o.d.e. integration suitable for low-order periodic surfaces;

Dewar & Meiss, Physica D, ***57****, 476 (1992); Dewar, Hudson & Price, Phys. Lett. A* ***194****, 49 (1994); Dewar & Khorev, Physica D,* ***85****, 66 (1995); Hudson & Dewar, J. Plasma Phys.,* ***56****, 361 (1996); Hudson & Dewar, Phys. Lett. A (2009)*

Alternative construction of QFMin surfaces employs constrained action-integral techniques

- Magnetic field-lines are curves, $C: \theta = \theta(\zeta),\ \psi = \psi(\zeta)$, that extremize the action $S = \int_C \mathbf{A} \cdot d\mathbf{r}$

→ Euler-Lagrange equation $\mathbf{B} \times \delta \mathbf{r} = 0$

→ variational integration faster, robust to chaos; suited for finding high-order periodic orbits in chaos;

→ for numerical implementation: discretize infinite-dimensional curves, $S = S(\theta_0, \theta_1, \theta_2, \ldots, \theta_N)$

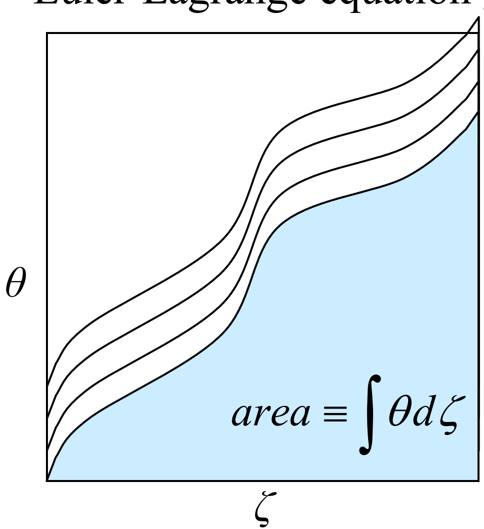
enforce periodicity constraint $\theta_N = \theta_0 + 2\pi p,\ \zeta_N = \zeta_0 + 2\pi q$

find zero of action-gradient vector, $\partial S / \partial \theta_i = 0$, using Hessian $\partial^2 S / \partial^2 \theta_{ij}$,

- A constrained variational principle for pseudo-field-lines $S = \int_C \mathbf{A} \cdot d\mathbf{r} - \nu \left(\int \theta \, \nabla \zeta \cdot d\mathbf{r} - a \right)$

recall, to find minimum of $f(x)$, subject to constraint $g(x) = g_0$, minimize $F(x, \lambda) = f(x) - \lambda\, [g(x) - g_0]$

- Euler-Lagrange equation gives pseudo-field $\mathbf{B}_\nu \equiv \mathbf{B} - \nu \, \nabla \theta \times \nabla \zeta$



↑ *increasing area constraint gives **family** of periodic pseudo-curves*

1) *Lagrangian variational methods suitable for high-order periodic orbits (q ~ 10,000) in strongly chaotic fields;*
2) *High-order periodic orbits approximate KAM surfaces and broken KAM surfaces (cantori);*

Ghost-surfaces constructed via action-gradient flow between the stable & unstable periodic orbits.

C. Golé, J. Differ. Equations **97**, 140 1992., R. S. MacKay and M. R. Muldoon, Phys. Lett. A **178**, 245, 1993.

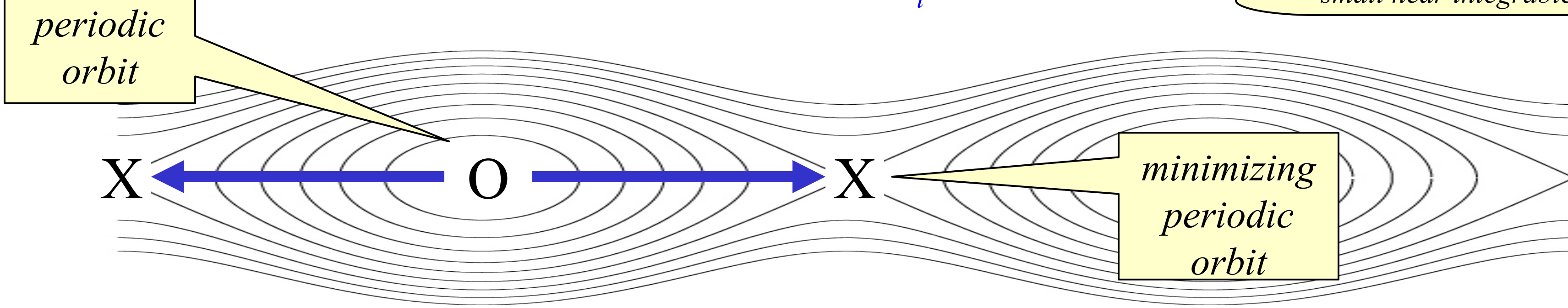
- At the minimax (stable) periodic orbit, the eigenvector of the Hessian, $\partial^2 S / \partial^2 \theta_{ij}$, with negative eigenvalue indicates the direction in which the action integral decreases.

- Pushing trial curve from minimax (stable) p/q orbit down action-gradient flow to minimizing (unstable) p/q orbit defines *ghost-surfaces*,

- Ghost-surfaces may be thought of as rational coordinate surfaces that pass through island chains.

Action Gradient Flow $\dfrac{\partial \theta_i}{\partial \alpha} = -\dfrac{\partial S}{\partial \theta_i}$

problem: action-gradient flow is small near integrable limit



Ghost-surfaces are almost identical to QFMin surfaces!

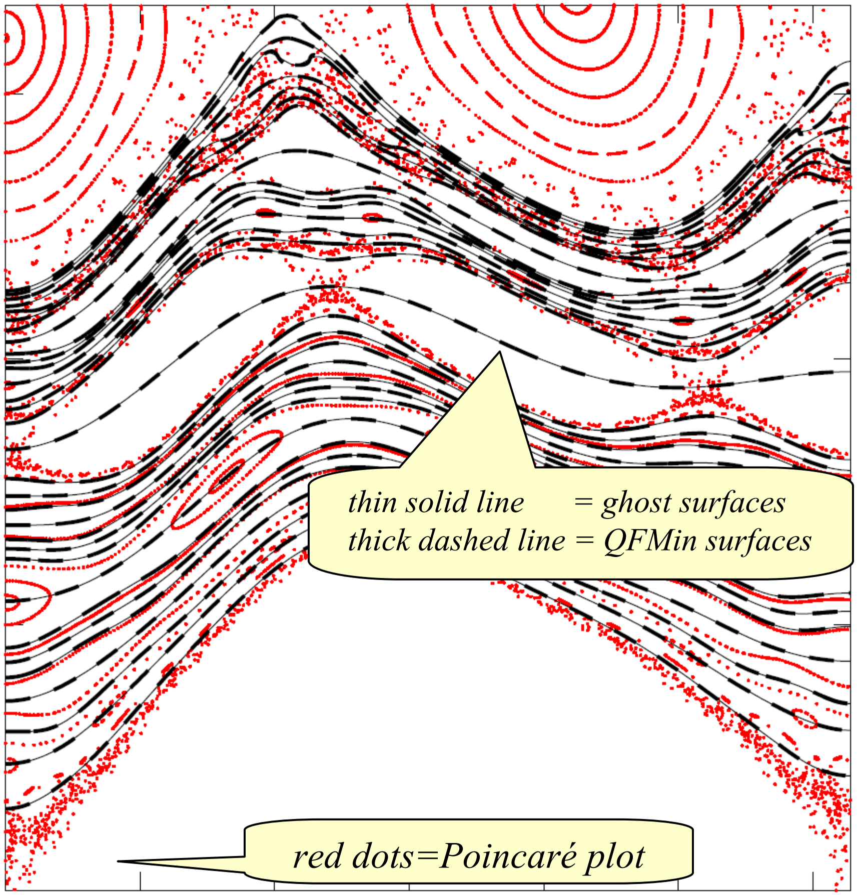
!!Ghost surfaces are defined by action-gradient flow;
!! QFMin surfaces defined by minimizing quadratic-flux;
→ no obvious reason why these different definitions should give the same surfaces

- Numerical evidence suggests ghost-surfaces and QFMin surfaces are *almost* the same;

- confirmed to1st-order using perturbation theory;

→ to higher order, need to exploit coordinate dependence of QFMin surfaces and ghost surfaces . . .

- opens possibility of using fast, robust construction of *unified* almost-invariant surfaces & chaotic coordinates



Hudson & Dewar, Phys. Lett. A, 2009.

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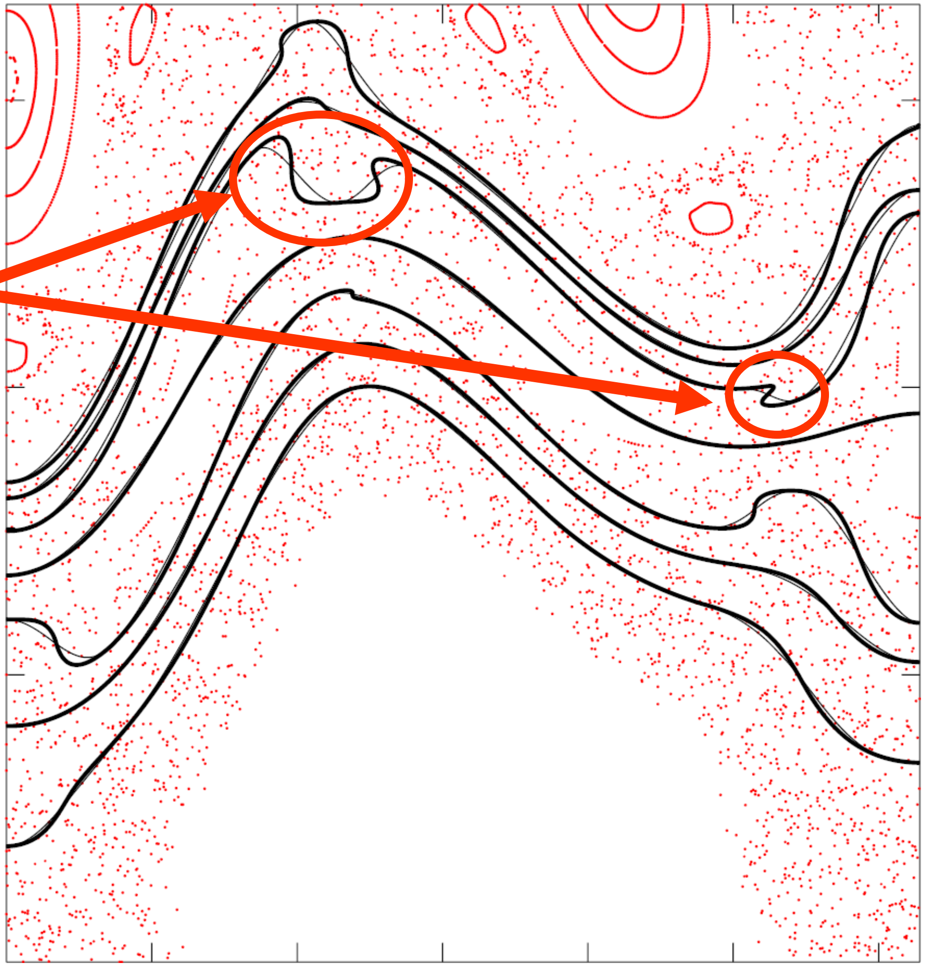
- For stronger chaos, ghost-surfaces and QFMin surfaces are no longer the same;

- hopefully, can re-define QFMin surfaces so as to agree with ghost-surfaces yet keep efficient, robust numerical algorithm



Hudson & Dewar, Phys. Lett. A, 2009.

Summary

→ in chaotic fields, anisotropic heat transport is restricted by irrational field-lines ≡ cantori;

→ ghost-surfaces are closely related to quadratic-flux minimizing surfaces;

→ a simple numerical construction has been introduced;

→ the temperature takes the form $T=T(s)$, where s labels the chaotic coordinate surfaces;

→ an expression for the temperature gradient in chaotic fields is derived;