Continuum Kinetics in NIMROD
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E. Held, S. Kruger, C. Kim and NIMROD Team

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1 First-order DKE in the (ξ,s) velocity variables

Hazeltine's form for the drift kinetic equation (ϵ, μ) :

$$\partial_t f + (\mathbf{v}_{||} + v_D) \cdot \nabla f + \left(\mu \tfrac{\partial B}{\partial t} + e(\mathbf{v}_{||} + \mathbf{v}_D) \cdot \mathbf{E}\right) \partial_\epsilon f = C.$$

Using $\xi = v_{||}/v$ and $s = v/v_0$ yields

$$\begin{aligned} \partial_t f + (\mathbf{v}_{||} + v_D) \cdot \nabla f - (\mathbf{v}_{||} + v_D) \cdot \left[\frac{1-\xi^2}{2\xi} \nabla \ln B \partial_\xi + s \nabla \ln v_0 \partial_s\right] f \\ + \left(\frac{e}{2\epsilon_0 s^2} (\mathbf{v}_{||} + \mathbf{v}_D) \cdot \mathbf{E}\right) \left(s \partial_s f + \frac{1-\xi^2}{\xi} \partial_\xi f\right) \ = \ C \end{aligned}$$

with drift

$$\begin{aligned} v_D \ &= \ \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\epsilon_0 s^2}{eB} \left[\mathbf{b} \times \left(\left(1-\xi^2\right) \nabla \ln B + 2\xi^2 \kappa - \frac{v_0 s \xi}{e_0 B} \nabla \times \mathbf{E}\right) + (1-\xi^2) \frac{\mu_0 \mathbf{J}_{||}}{B}\right] \\ &\approx \ \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{\epsilon_0 s^2}{eB^2} \left[\left(\mathbf{1} + \xi^{\mathbf{2}}\right) \mathbf{b} \times \nabla B + 2\xi^2 \mu_0 \mathbf{J}_\perp\right]. \end{aligned}$$

2 Coulomb collision operator written in moment form.

- Full, linearized Coulomb collision operator taken from Ji and Held, PoP (2006):

$$
\begin{aligned}
C^{ab} &= C(f_a^{(0)}, f_b^{(0)}) + C(f_{a1}, f_b^{(0)}) + C(f_a^{(0)}, f_{b1}) \\
&= C(f_a^{(0)}, f_b^{(0)}) + \sum_{lk} \frac{f_a^{(0)}}{\sigma_k^l} P_l(v_{||}/v) \left(\nu_{ab}^{lk,0} M_{||a}^{lk}(\mathbf{r}, t) + \nu_{ab}^{0,lk} M_{||b}^{lk}(\mathbf{r}, t) \right)
\end{aligned}
$$

where $f_a^{(0)}$and $f_b^{(0)}$ are Maxwellians, $\nu_{ab}^{lk,0}$ and $\nu_{ab}^{0,lk}$'s are speed dependent collision frequency and

$$n_a M_{||a}^{lk} = \tfrac{l!}{(2l-1)!!} v_{Ta}^{l+2k} \int d\mathbf{v} L_k^{l+1/2}(s^2) s^l P_l(v_{||}/v) f_{a1}.$$

3 Velocity space representation.

- f_{a1} expanded in pitch-angle basis functions: $f_{a1} = \sum_l F_{al}(s_a)\phi_l(v_{||}/v)$.

- Coefficients, F_{al}, solved on grid in normalized speed, s_a.

- Moments in collision operator computed as:

$$
\begin{aligned}
n_a M_{||a}^{lk} &= \frac{l!}{(2l-1)!!} v_{Ta}^{l+2k} \int_0^\infty ds s^{2l+1} L_k^{l+1/2}(s^2) \sum_{l'} p_{ll'} F_{al'}(s) \\
&= \frac{l!}{(2l-1)!!} v_{Ta}^{l+2k} \sum_j w_j s_j^{2l+1} L_k^{l+1/2}(s_j^2) \sum_{l'} p_{ll'} F_{al'}(s_j)
\end{aligned}
$$

- Possible weight functions, $w(s)$, for s quadrature:

  1. Maxwellian drives: $w(s) = s^\alpha exp(-s^2)$ on $s\epsilon[0, \infty)$.

  2. Slowing down distribution: $w(s) = s^4/(1+s^3)$ on $s\epsilon[0, s_{max}]$.

4 Neoclassical transport benchmark.

Order $v_D << v_{||}$ and assume weak electric field:

$$\partial_t f_0 + \mathbf{v}_{||} \cdot \nabla f_0 - \mathbf{v}_{||} \cdot \left[\frac{1-\xi^2}{2\xi}\nabla \ln B \partial_\xi + s\nabla \ln v_0 \partial_s\right] f_0 \;=\; C(f_0).$$

f_0 is a stationary Maxwellian parameterized by $n_0(\psi)$ and $T_0(\psi)$.

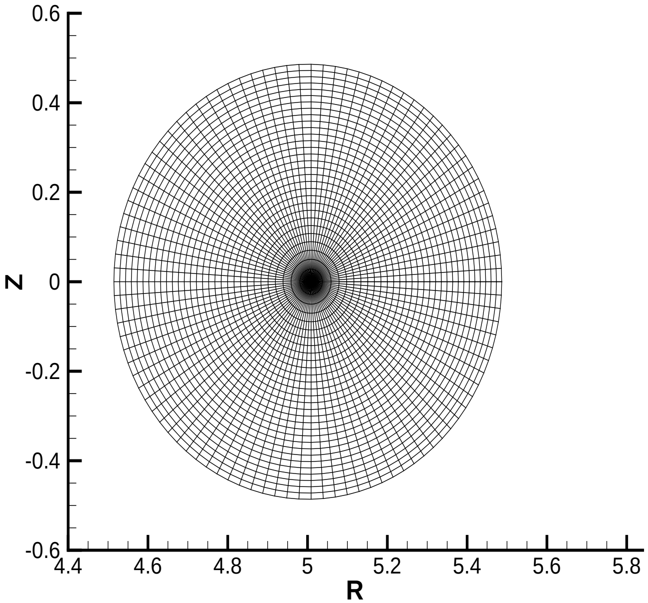
To next order :

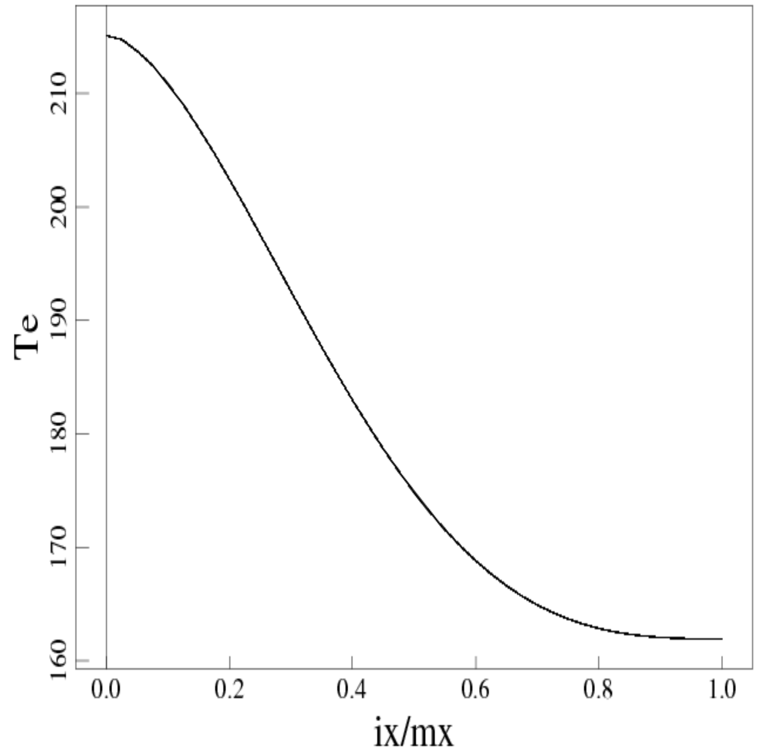
$$\begin{aligned}\partial_t f_1 + \mathbf{v}_{||} \cdot \nabla f_1 - \left(\mathbf{v}_{||} \cdot \nabla \ln B\right) \frac{1-\xi^2}{2\xi}\partial_\xi f_1 \;&=\\ -\mathbf{v}_D \cdot \nabla f_0 + s v_D \cdot \nabla \ln v_0 \partial_s f_0 - \frac{e}{2\epsilon_0 s}\mathbf{v}_{||} \cdot \left(\mathbf{E}^A - \nabla\phi_1\right) \partial_s f_0 \;&+\; C^{aa} + C^{ab}\end{aligned}$$

Using $g = f_1 - (e\phi_1/T_0) f_0$ yields (compare with Eq. 23 of Belli and Candy, 51 PPCF 2009):

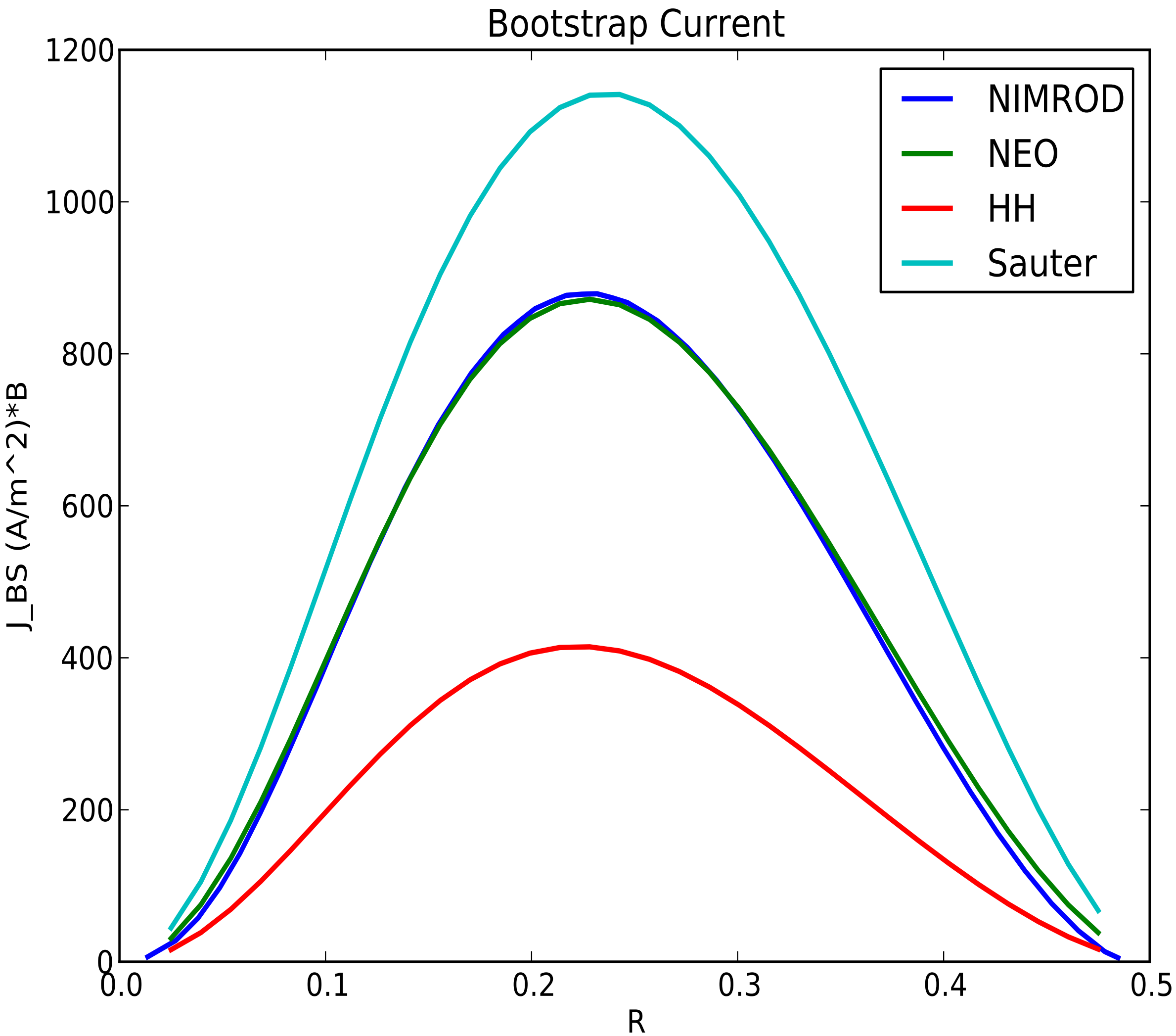
$$\begin{aligned}\partial_t g + \mathbf{v}_{||} \cdot \nabla g - \mathbf{v}_{||} \cdot \nabla \ln B \left(\frac{1-\xi^2}{2\xi}\right) \partial_\xi g - C^{aa} - C^{ab} \qquad &=\\ -\mathbf{v}_D \cdot \left[\nabla n_0 - (\frac{3}{2} - s^2)\nabla T_0\right] f_0 &\\ -\frac{e}{2\epsilon_0 s}\mathbf{v}_{||} \cdot \mathbf{E}^A \partial_s f_0 \;\; &+(e f_0/T_0)\partial_t \phi_1\end{aligned}$$

5 Test on high-aspect ratio Grad-Shafranov equilibrium

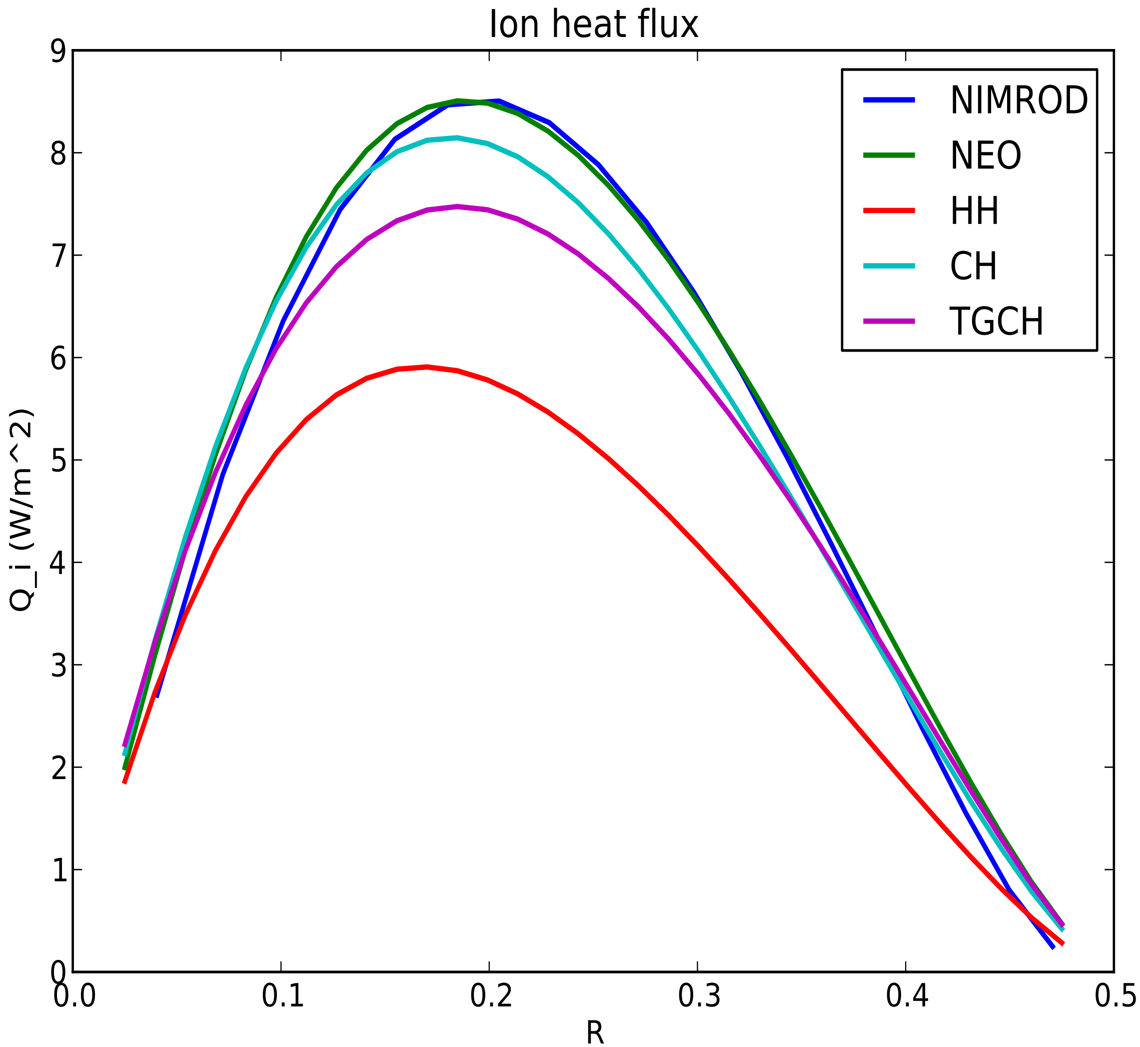




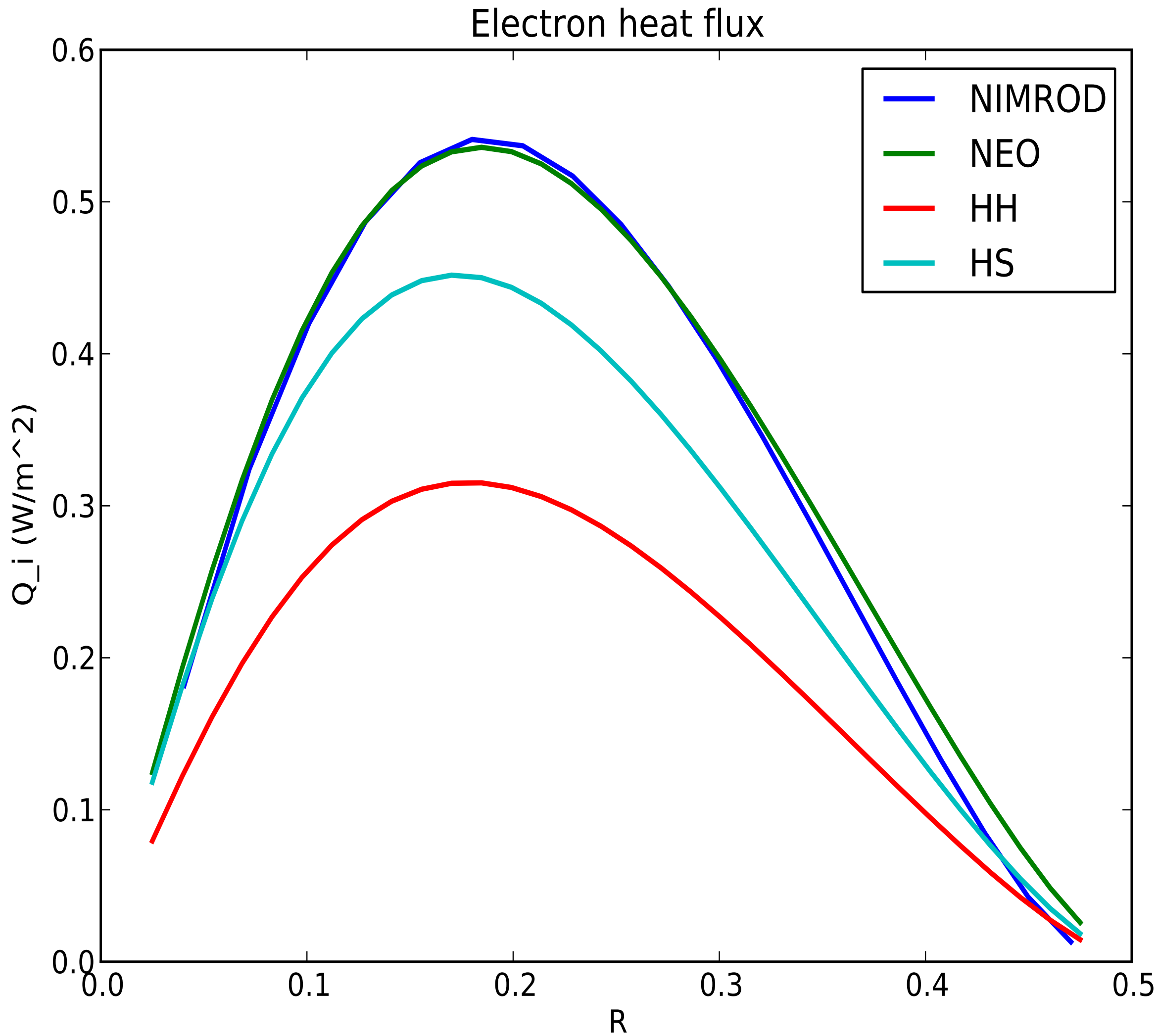
6 Comparison of Bootstrap Currents



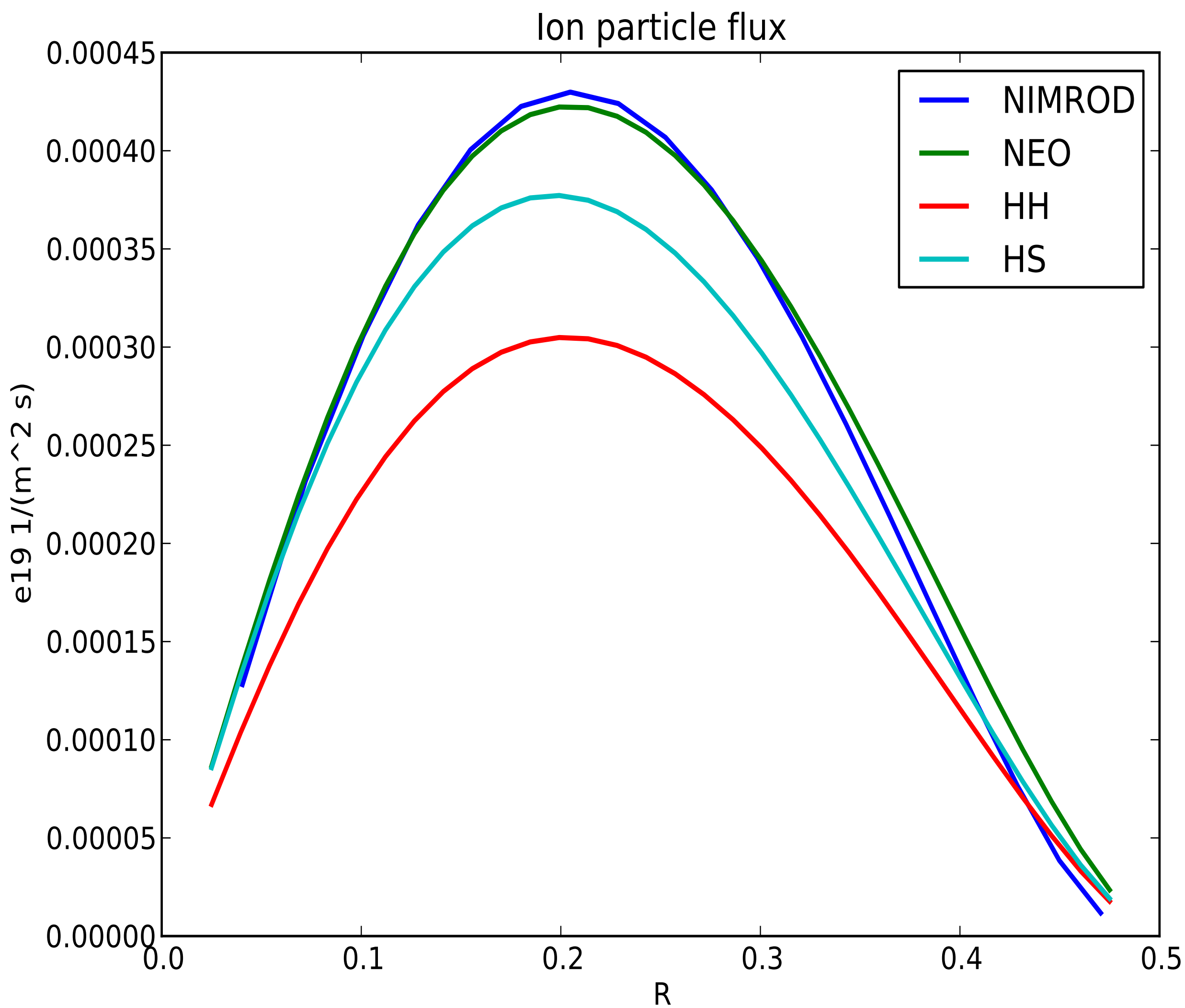
7 Comparison of Ion Radial Heat Fluxes



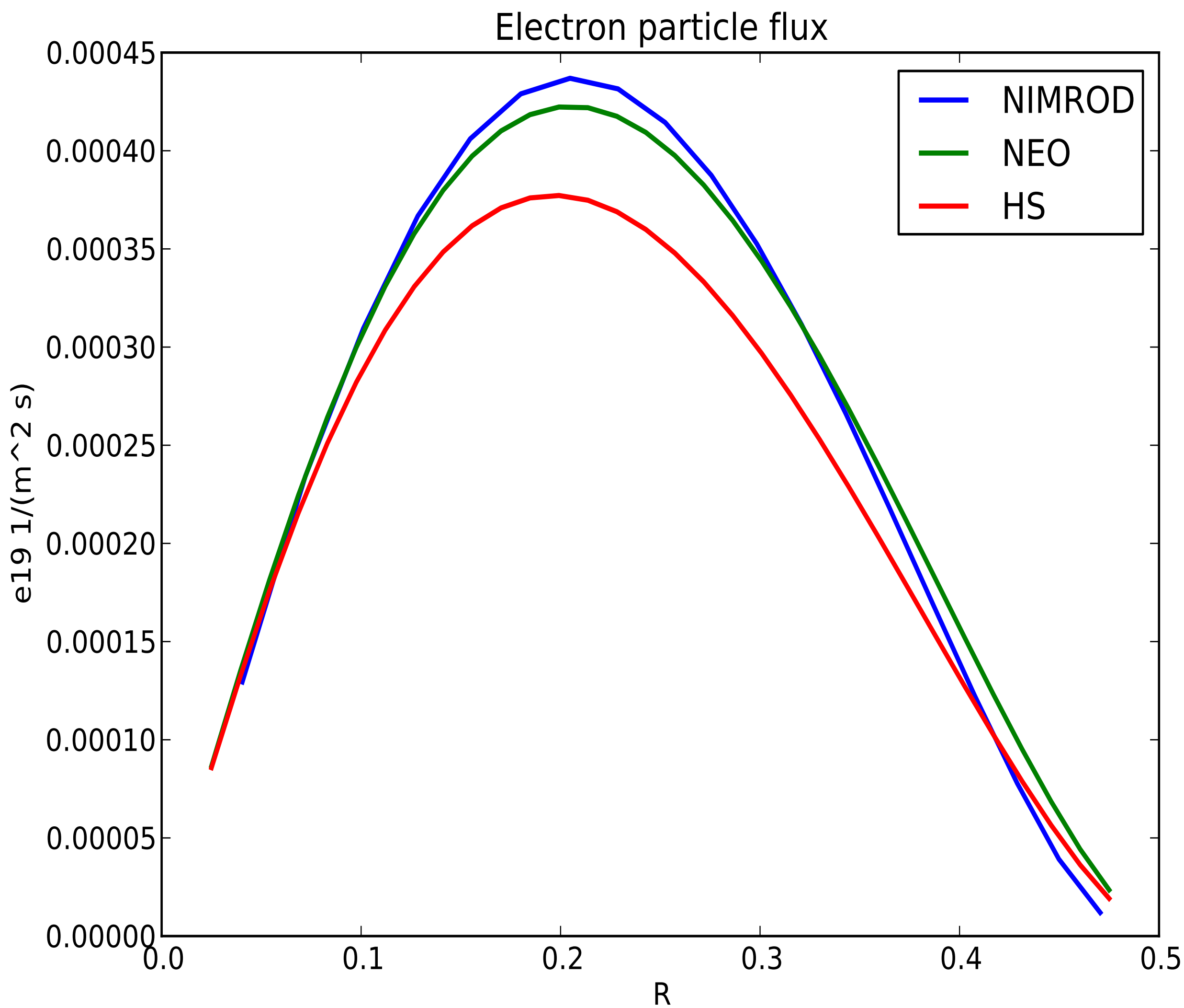
8 Comparison of Electron Radial Heat Fluxes



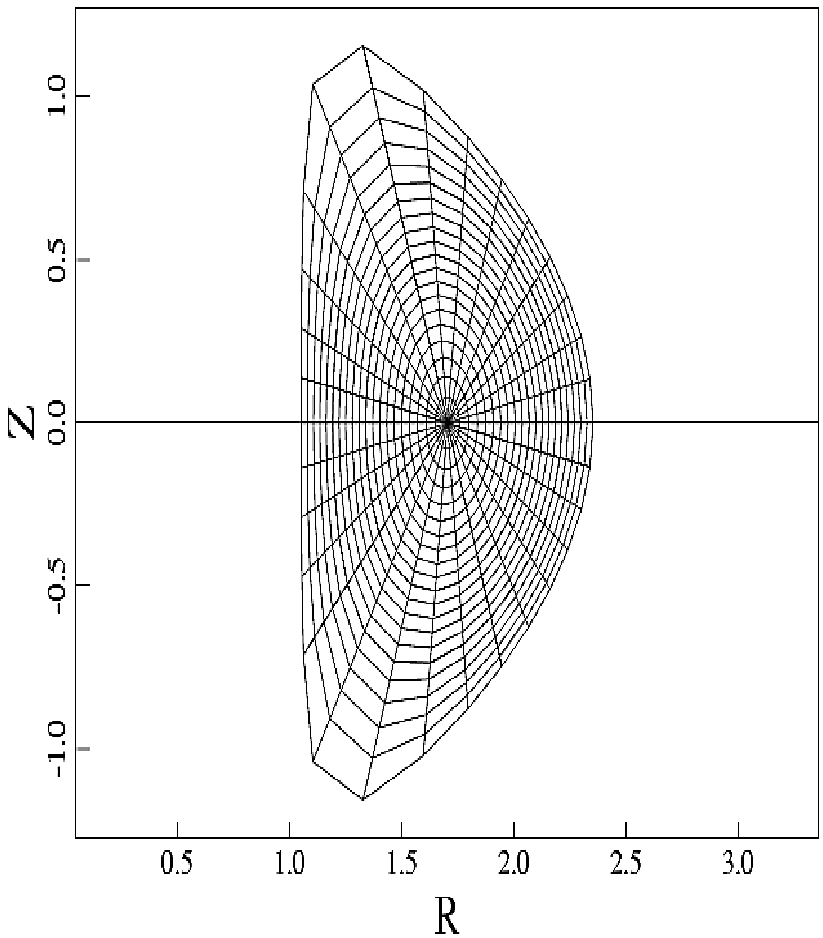
9 Comparison of Ion Radial Particle Fluxes

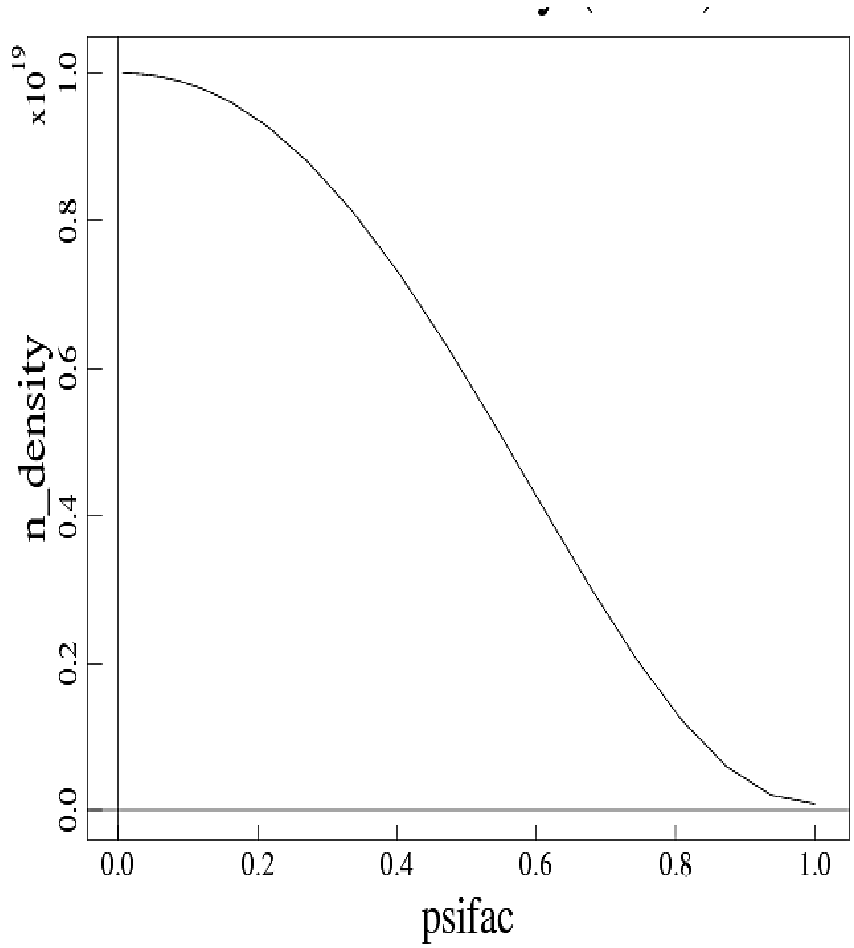


10 Comparison of Electron Radial Particle Fluxes



11 Next test: high-β, shaped equilibrium





12 Continuum method for hot particles in NIMROD

- Portion of MHD pressure replaced by isotropic hot particle pressure:
  $\int d\mathbf{v}(mv'^2/3)f_0 = \left(\frac{\beta_{hot}}{\beta_{hot}+\beta_{MHD}}\right) p_{eq}(\psi)$.

- Two f_0's implemented in NIMROD:

  - slowing-down distribution: $f_0 = A \exp[-P_\phi/\psi_n]/(\epsilon^{3/2} + \epsilon_c^{3/2})$, and
  - Maxwellian: $f_0 = A(\psi) \exp[-s^2]$.

- Solve drift kinetic equation for δf and couple hot particle pressure tensor back into NIMROD's plasma momentum evolution equation.

13 Solve for δf using continuum approach

- Assuming $\mathbf{E}_0 = 0$ (for consistency with original kink benchmark test), solve

$$
\begin{aligned}
\partial_t f_1 + (\mathbf{v}_{||} + v_D)_0 \cdot \nabla f_1 - (\mathbf{v}_{||} + v_D)_0 \cdot \frac{1-\xi^2}{2\xi} \nabla \ln B_0 \partial_\xi f_1 \; &= \\
-(\mathbf{v}_{||} + v_D)_1 \cdot \nabla f_0 - (\mathbf{v}_{||} + v_D)_1 \cdot \frac{1-\xi^2}{2\xi} \nabla \ln B_0 \partial_\xi f_0 & \\
-\left(\frac{e}{2\epsilon_0 s^2} (\mathbf{v}_{||} + \mathbf{v}_D)_0 \cdot \mathbf{E}_1 \right) (s\partial_s f_0 + \frac{1-\xi^2}{\xi} \partial_\xi f_0) &
\end{aligned}
$$

- For kink benchmark:

$(\mathbf{v}_{||} + v_D)_0 = v_\| \mathbf{b}_0 + \frac{\epsilon_0 s^2}{e B_0^2} \left[\left(\mathbf{1} + \xi^{\mathbf{2}} \right) \mathbf{b}_0 \times \nabla B_0 + 2\xi^2 \mu_0 \mathbf{J}_{\perp 0} \right]$,

$(\mathbf{v}_{||} + v_D)_1 = v_\| \frac{\mathbf{B}_1}{B_0} + \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2}$. (Kim, Phys. Plasmas 15, 072507, 2008)

14 Preconditioning issues for continuum DKE solutions

NIMROD preconditioning diagonal in Fourier index, n:

$$\begin{bmatrix} [n=0] & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & [n=1] & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & [n=2] & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

N pitch angle basis coefficients coupled in each matrix:

$$\begin{bmatrix} [NxN]_{n=0} & \mathbf{0} & \mathbf{0} & \cdots \\ \mathbf{0} & [NxN]_{n=1} & \mathbf{0} & \cdots \\ \mathbf{0} & \mathbf{0} & [NxN]_{n=2} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Additional speed coupling due to collision operator and nonlinear acceleration:

$$[NxN]_1 = \begin{bmatrix} [s_{11}] & [s_{12}] & [s_{13}] & \cdots \\ [\mathbf{s_{21}}] & [s_{22}] & [\mathbf{s_{23}}] & \cdots \\ [\mathbf{s_{31}}] & [\mathbf{s_{32}}] & [s_{33}] & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

15 Kink benchmark equilibrium but with Maxwellian hot particle distribution

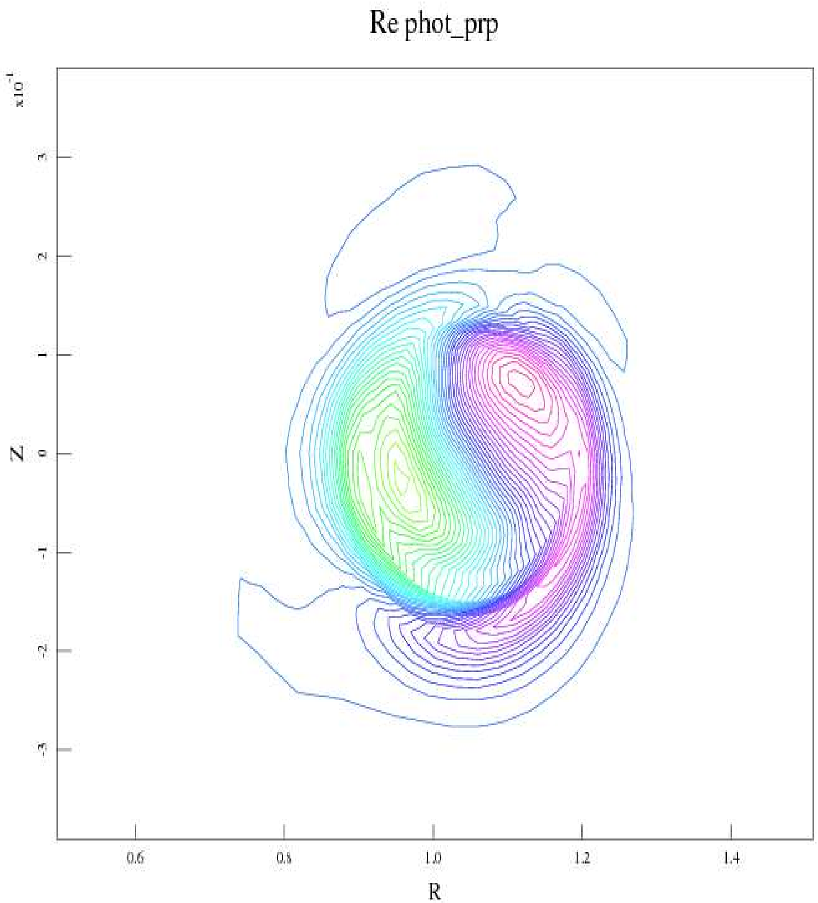
- Continuum results presented here are for a Maxwellian hot particle distribution.

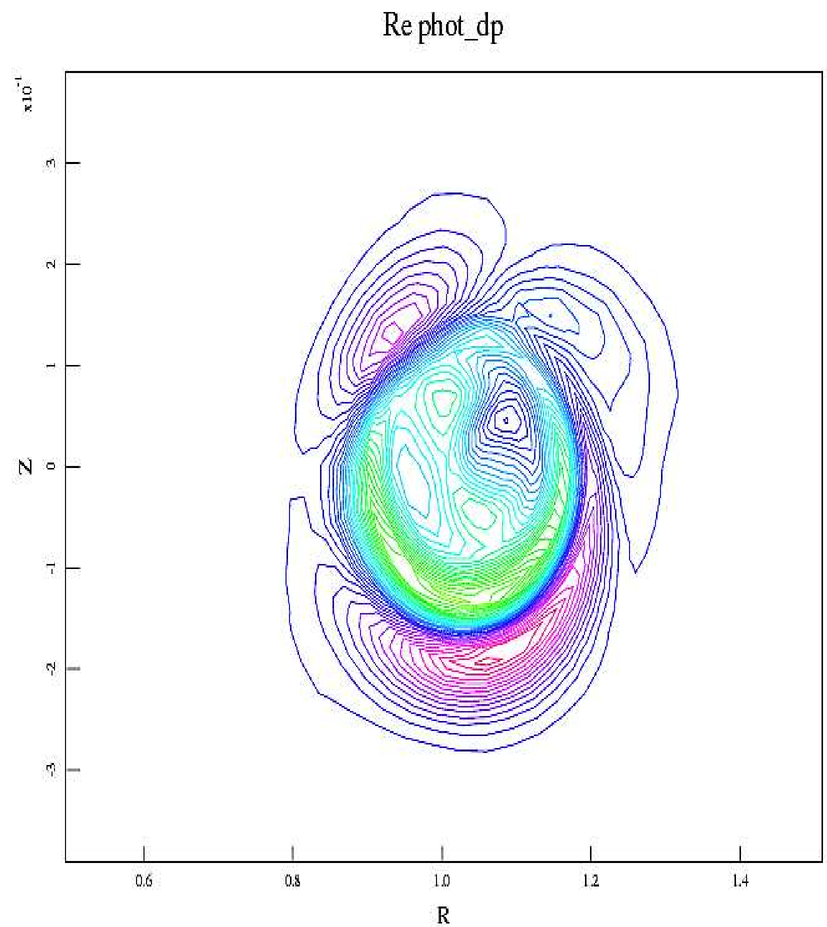
- Original benchmark problem used slowing down distribution with critical energy, $\epsilon_c = 10$KeV.

- Maxwellian f_0 with $T_0 = 10$ KeV leads to enhanced hot particle effects:

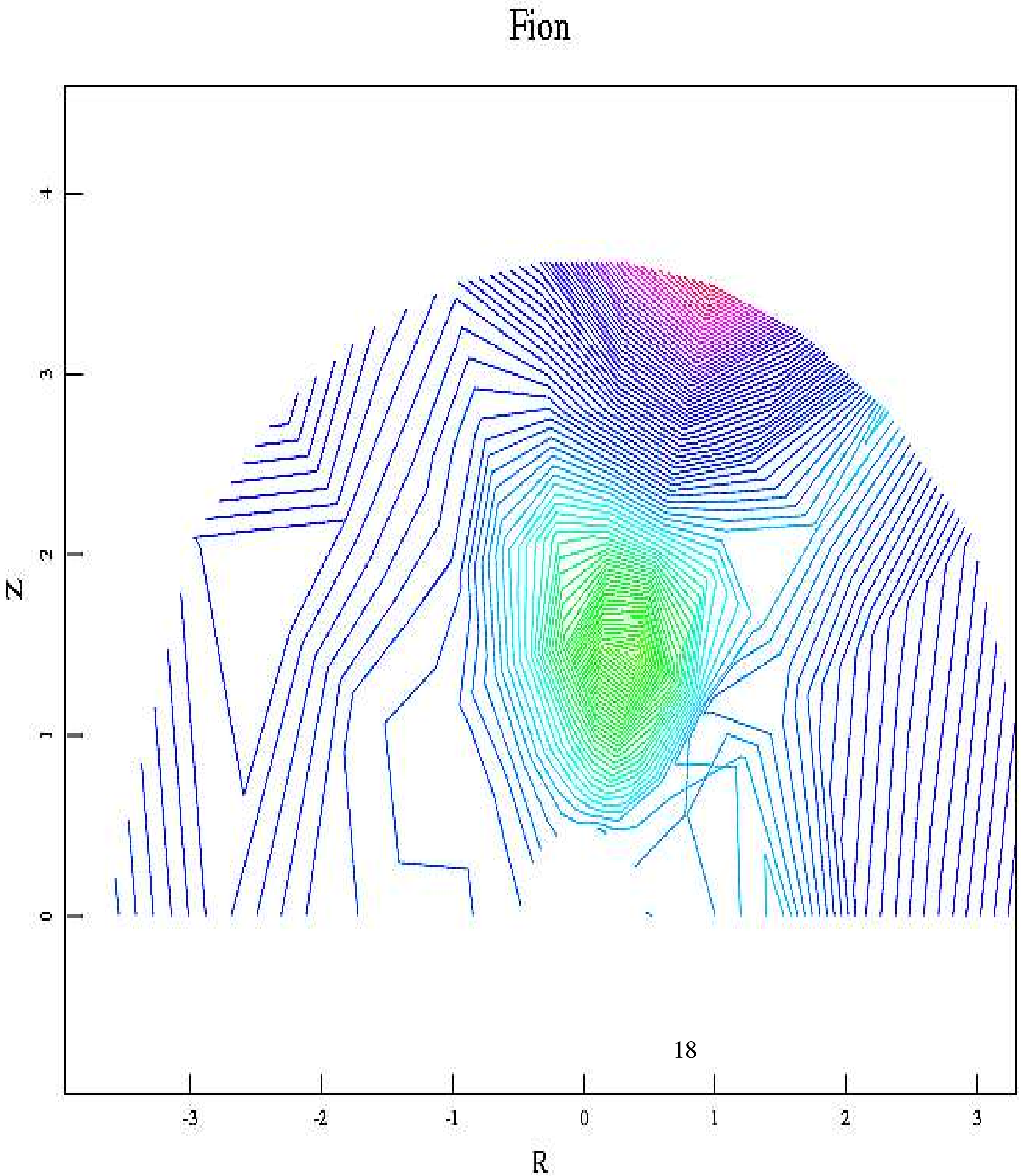
hot particle β fraction	$\gamma\tau_A$ slowing down	$\gamma\tau_A$ Maxwellian
0.00	0.0255	0.0239
0.25	0.0213	0.0162
0.50	0.0220	0.0238
0.75	0.0270	0.0542

16 Contours of isotropic and anisotropic hot particle pressures





17 Contours of hot particle distribution



18 Future Work

1. Parallelize over speed points to reduce memory constraints.

2. Improve preconditioning for collision terms in steady-state calculations.

3. Continue NEO and hot particle benchmark calculations.

4. Related work (at Utah State) on continuum solutions of kinetic equations:

   (a) Andy Spencer is developing a Fokker-Planck code using NIMROD's 2D finite-element/Fourier representation for velocity space (submitted JCP in July on test particle part).

   (b) Jeong-Young Ji's higher-order moment equations to be implemented in NIMROD.

   (c) Mukta Sharma's studies of heat flow along magnetic fields testing effects of particle trapping on parallel transport.