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HALL-MHD TEARING MODES IN A FINITE ASPECT RATIO CYLINDRICAL PLASMA*

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MOTIVATION

- RECENTLY DERIVED ANALYTIC DISPERSION RELATIONS FOR THE HALL-MHD RESISTIVE TEARING INSTABILITY IN SLAB GEOMETRY [1-3] APPLY TO RATHER GENERAL PARAMETER REGIMES, MAKING THEM USEFUL FOR TESTING THE EXTENDED-MHD CODES. THEY WERE USED IN A SUCCESSFUL BENCHMARK OF NIMROD [4].

- A GENERALIZATION TO CYLINDRICAL GEOMETRY HAS BEEN SUGGESTED IN ORDER TO FACILITATE COMPARISONS WITH M3D-C1 SIMULATIONS.

[1] V.V. Mirnov, C.C. Hegna and S.C. Prager, Phys. Plasmas 11, 4468 (2004).

[2] E. Ahedo and J.J. Ramos, Plasma Phys. Control. Fusion 51, 055018 (2009).

[3] E. Ahedo and J.J. Ramos, Phys. Plasmas 19, 072519 (2012).

[4] C.R. Sovinec, J.R. King and the NIMROD Team, J. Comp. Phys. 229, 5803 (2010).

TWO-FLUID, HALL-MHD MODEL

$$\frac{\partial n}{\partial t} \; + \; \nabla \cdot (n\mathbf{u}) \; = \; 0$$

$$\mathbf{j} \; = \; \nabla \times \mathbf{B}$$

$$\frac{\partial \mathbf{B}}{\partial t} \; + \; \nabla \times \mathbf{E} \; = \; 0$$

$$\mathbf{E} \; = \; -\mathbf{u} \times \mathbf{B} \; + \; \frac{1}{en}(\mathbf{j} \times \mathbf{B} - \nabla p_e) \; + \; \eta \mathbf{j}$$

$$m_\iota n \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right] \; + \; \nabla(p_\iota + p_e) \; - \; \mathbf{j} \times \mathbf{B} \; = \; 0$$

$$p_s n^{-\Gamma_s} = \text{constant} \qquad (s = \iota, e)$$

FORCE-FREE CYLINDRICAL EQUILIBRIUM

$$n_0 \ , \quad p_{\iota 0} \ , \quad p_{e0} \ = \ \text{constants}$$

$$\mathbf{u}_0 = 0$$

$$\mathbf{B}_0 \ = \ B_{0\theta}(r) \ \mathbf{e}_\theta \ + \ B_{0z}(r) \ \mathbf{e}_z$$

$$\mathbf{j}_0 \times \mathbf{B}_0 \ = \ 0 \quad \Rightarrow \quad \mathbf{j}_0 \ = \ \lambda_0 \mathbf{B}_0 \ , \qquad \lambda_0 \ = \ \frac{\mathbf{j}_0 \cdot \mathbf{B}_0}{B_0^2} \ = \ \frac{(rB_{0\theta})'}{rB_{0z}} \ = \ - \ \frac{B'_{0z}}{B_{0\theta}} \qquad \left[' \equiv \frac{d}{dr} \right]$$

NORMAL MODE LINEARIZED SYSTEM

- **CONSIDER NORMAL MODES**

$$f(\mathbf{x},t) \;=\; f_0(r) \;+\; f_1(r)\;\exp(im\theta + ik_z z + \gamma t)$$

- **DEFINE**

$$\mathbf{k}(r) \;\equiv\; \frac{m}{r}\mathbf{e}_\theta \;+\; k_z\mathbf{e}_z$$

$$F(r) \;\equiv \mathbf{k}\cdot\mathbf{B}_0 \;=\; \frac{m}{r}B_{0\theta} + k_z B_{0z}\;, \qquad F(r_s) \;=\; 0$$

$$G(r) \;\equiv \mathbf{e}_r\cdot(\mathbf{k}\times\mathbf{B}_0) \;=\; \frac{m}{r}B_{0z} - k_z B_{0\theta}$$

$$c_S^2 \equiv (m_\iota n_0)^{-1}\sum_s \Gamma_s p_{s0}\;, \qquad \xi \equiv \frac{u_{1r}}{\gamma}\;, \qquad Q \equiv B_{1z} + \frac{i}{rk^2}\Big[m\lambda_0 B_{1r} - k_z(rB_{1r})'\Big]$$

- **ELIMINATE ALGEBRAICALLY** E, **j**, $B_{1\theta}$, $u_{1\theta}$, u_{1z} $p_{\iota 1}$, p_{e1} **AND** n_1, **TO OBTAIN AN EXACT LINEARIZED SYSTEM FOR** ξ, B_{1r} **and** Q

$$\gamma^2 m_\iota n_0 \left\{\xi - \left[\frac{c_S^2}{\gamma^2 + c_S^2 k^2}\ \frac{(r\xi)'}{r}\right]'\right\} \ + \ \frac{\gamma}{\eta k^2} F^2 \xi \ = \ \frac{i}{k^2}\left[\lambda_0' G - \left(\lambda_0^2 + \frac{\gamma}{\eta}\right) F\right] B_{1r} \ -$$

$$- \ \left(\frac{\gamma^2}{\gamma^2 + c_S^2 k^2}\ \frac{r}{m}\ GQ\right)' \ + \ \left(\frac{ir}{\eta e n_0 m} F^2 - \frac{2k_z^2}{mk^2} G\right) Q$$

$$(\gamma + \eta k^2) B_{1r} \ - \ \eta k^2 \left[\frac{1}{rk^2}(rB_{1r})'\right]' \ - \ \frac{2\eta m k_z}{r^2 k^2} \lambda_0 B_{1r} \ = \ F\left(i\gamma\xi + \frac{rk^2}{en_0 m} Q\right) \ + \ \frac{2i\eta k_z}{r} Q$$

$$\left[\gamma + \frac{1}{\gamma m_\iota n_0}\left(F^2 + \frac{\gamma^2 G^2}{\gamma^2 + c_S^2 k^2}\right) + \frac{F^2}{\eta e^2 n_0^2}\right] Q \ + \ \eta\left[k^2 Q - \frac{1}{r}(rQ')'\right] \ + \ \frac{ik_z}{en_0 r}\left[\frac{2k_z}{k^2} GQ - \frac{1}{m}(r^2 FQ)'\right] \ =$$

$$= \ \frac{i}{rk^2}(\gamma + \eta k^2)\Big[m\lambda_0 B_{1r} - k_z (rB_{1r})'\Big] \ - \ \frac{i\eta}{r}\left\{r\left(\frac{1}{rk^2}\Big[m\lambda_0 B_{1r} - k_z (rB_{1r})'\Big]\right)'\right\}' \ -$$

$$- \ \frac{m}{en_0 rk^2}\left[\lambda_0' G - \left(\lambda_0^2 + \frac{\gamma}{\eta}\right) F\right] B_{1r} \ + \ \frac{\gamma}{r}\left[\frac{c_S^2 m/r}{\gamma^2 + c_S^2 k^2} G(r\xi)' - (B_{0z} r\xi)'\right] \ - \ \frac{i\gamma m}{\eta e n_0 rk^2} F^2 \xi$$

MARGINALLY STABLE IDEAL-MHD SOLUTION

$$\eta = 0 \ , \qquad \gamma = 0$$

$$Q = 0 \ , \qquad B_{1r} = iF\xi$$

$$\left[\frac{1}{rk^2}(rB_{1r})'\right]' - \left(1 + \frac{\lambda_0' G}{k^2 F} - \frac{\lambda_0^2}{k^2} - \frac{2mk_z\lambda_0}{r^2k^4}\right) B_{1r} = 0$$

- **APPLICABLE TO THE "OUTER" REGION AWAY FROM** $r = r_s$

- **DISCONTINUITY AT** $r = r_s$ **DEFINES THE TEARING STABILITY INDEX**

$$\Delta' = \frac{B_{1r}'(r_{s+}) - B_{1r}'(r_{s-})}{B_{1r}(r_s)}$$

ASYMPTOTIC EXPANSIONS IN THE MODE SINGULAR LAYER

- **IN A LAYER WHERE $x \equiv (r - r_s) \ll r_s$, USE THE LOWEST-ORDER TAYLOR EXPANSION OF EQUILIBRIUM QUANTITIES:**

$$r = r_s \ , \qquad k = k(r_s) \ , \qquad B_0 = B_0(r_s)$$

$$B_{0\theta} = -\frac{k_z}{k(r_s)} B_0(r_s) \ , \qquad B_{0z} = \frac{m}{r_s k(r_s)} B_0(r_s) \ , \qquad G = k(r_s) \ B_0(r_s)$$

$$F = \left[k(r_s) \ \lambda_0(r_s) + \frac{2mk_z}{r_s^2 k(r_s)} \right] B_0(r_s) \ (r - r_s) \ \equiv \ \frac{k(r_s) \ B_0(r_s)}{L_B} \ x$$

- **USE THE CONSTANT-ψ APPROXIMATION:** $(rB_{1r})' \simeq 0, \ (rB_{1r})'' \neq 0$

- **NEGLECT ηk^2 COMPARED TO γ**

$$\gamma B_{1r} \;-\; \eta B_{1r}'' \;=\; \frac{kB_0 x}{L_B}\left(i\gamma\xi + \frac{rk^2}{en_0 m}Q\right) \;+\; \frac{2i\eta k_z}{r}Q$$

$$\gamma^2 m_\iota n_0\left(\xi - \frac{c_S^2}{\gamma^2 + c_S^2 k^2}\;\xi''\right) \;+\; \frac{\gamma B_0^2 x^2}{\eta L_B^2}\;\xi \;=\; \frac{iB_0}{k}\left(\lambda_0' - \frac{\gamma x}{\eta L_B}\right)B_{1r} \;-$$
$$-\;\frac{\gamma^2 rkB_0}{m(\gamma^2 + c_S^2 k^2)}\;Q' \;+\; \frac{rkB_0}{m}\left(\frac{ikB_0 x^2}{\eta e n_0 L_B^2} - \frac{2k_z^2}{rk^2}\right)Q$$

$$\left[\gamma + \frac{k^2 B_0^2(\gamma^2 + c_S^2 k^2 x^2/L_B^2)}{\gamma m_\iota n_0(\gamma^2 + c_S^2 k^2)} + \frac{k^2 B_0^2 x^2}{\eta e^2 n_0^2 L_B^2}\right]Q \;-\; \eta Q'' \;-\; \frac{iB_0}{en_0}\left(\frac{\lambda_0 kx}{L_B} - \frac{2k_z^2}{rk}\right)Q \;=$$
$$=\;\frac{mB_0}{en_0 rk}\left(\frac{\gamma x}{\eta L_B} - \lambda_0'\right)B_{1r} \;-\; i\left\{\eta m\left[r\left(\frac{\lambda_0}{r^2k^2}\right)'\right]' - \frac{2\gamma k_z m^2}{r^3 k^4}\right\}B_{1r} \;-$$
$$-\;\frac{\gamma^3 mB_0}{rk(\gamma^2 + c_S^2 k^2)}\;\xi' \;-\; \frac{\gamma mB_0}{rk}\left(\frac{ikB_0 x^2}{\eta e n_0 L_B^2} - \frac{2k_z^2}{rk^2}\right)\xi$$

INFINITE ASPECT RATIO LIMIT

$$\frac{rk_z}{m} = -\frac{nr}{mR} \to 0$$

$$\gamma B_{1r} - \eta B_{1r}'' = \frac{kB_0 x}{L_B}\left(i\gamma\xi + \frac{k}{en_0}Q\right)$$

$$\gamma^2 m_\imath n_0\left(\xi - \frac{c_S^2}{\gamma^2 + c_S^2 k^2}\,\xi''\right) + \frac{\gamma B_0^2 x^2}{\eta L_B^2}\,\xi = \frac{iB_0}{k}\left(\lambda_0' - \frac{\gamma x}{\eta L_B}\right)B_{1r} - \frac{\gamma^2 B_0}{\gamma^2 + c_S^2 k^2}\,Q' + \frac{ikB_0^2 x^2}{\eta e n_0 L_B^2}\,Q$$

$$\left[\gamma + \frac{k^2 B_0^2(\gamma^2 + c_S^2 k^2 x^2/L_B^2)}{\gamma m_\imath n_0(\gamma^2 + c_S^2 k^2)} + \frac{k^2 B_0^2 x^2}{\eta e^2 n_0^2 L_B^2}\right]Q - \eta Q'' - \frac{ikB_0 x}{en_0 L_B^2}\,Q =$$

$$= \frac{B_0}{en_0}\left(\frac{\gamma x}{\eta L_B} - \lambda_0'\right)B_{1r} - \frac{i\eta k}{r}\left[r\left(\frac{\lambda_0}{r^2k^2}\right)'\right]' B_{1r} - \frac{\gamma^3 B_0}{\gamma^2 + c_S^2 k^2}\,\xi' - \frac{i\gamma k B_0 x^2}{\eta e n_0 L_B^2}\,\xi$$

WHICH IS THE SAME AS THE SLAB GEOMETRY SYSTEM

- **IN THE INFINITE ASPECT RATIO LIMIT, THE RESULTS IN CYLINDRICAL GEOMETRY ARE THE SAME AS THOSE IN SLAB GEOMETRY WITH THE IDENTIFICATIONS**

$$\frac{m}{r_s} \;\rightarrow\; k \;, \qquad\qquad -\,\frac{B'_{0z}(r_s)}{B_{0\theta}(r_s)} \;=\; \lambda_0(r_s) \;\rightarrow\; \frac{1}{L_B}$$

- **DISPERSION RELATIONS FOR THE ALFVEN-NORMALIZED GROWTH RATE**

$$\hat{\gamma} \;=\; \frac{\gamma}{kc_A}$$

AS FUNCTION OF THE DIMENSIONLESS PARAMETERS

$$\frac{\Delta'}{k}\;, \qquad kL_B\;, \qquad \epsilon_\eta \;=\; S^{-1} \;=\; \frac{\eta k}{c_A}\;, \qquad \alpha \;=\; kd_\iota \;=\; \frac{km_\iota^{1/2}}{en_0^{1/2}}\;, \qquad \beta \;=\; \frac{c_S^2}{c_A^2}$$

SLAB GEOMETRY (INFINITE ASPECT RATIO) DISPERSION RELATIONS

PR1: $\hat{\gamma} = \epsilon_\eta^{3/5} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{2/5}, \; C = \frac{2\pi\Gamma(3/4)}{\Gamma(1/4)}$ **[1]**

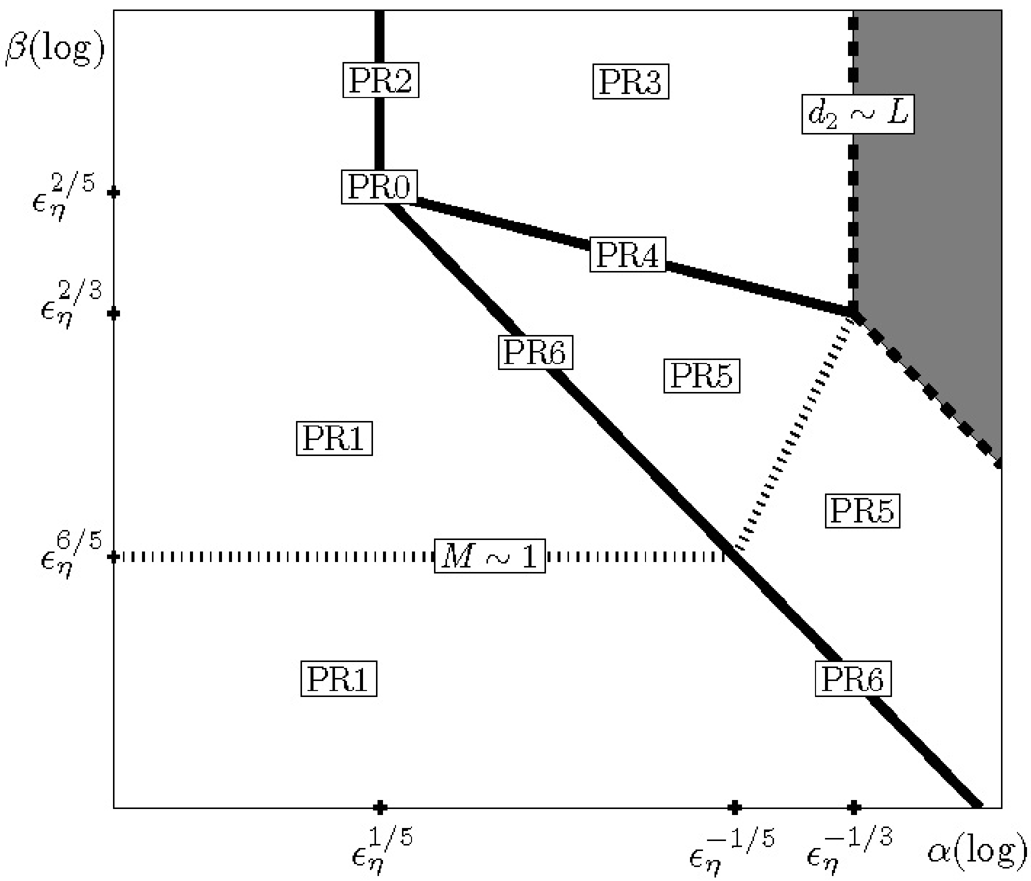
PR2: $\hat{\gamma} = \epsilon_\eta^{3/5} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{2/5} f_2^{-4/5} \left(\frac{\hat{\gamma}^{1/2}\alpha}{\epsilon_\eta^{1/2}} \right)$ **[2]**

PR3: $\hat{\gamma} = \epsilon_\eta^{1/2} \alpha^{1/2} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{1/2}$ **[3]**

PR4: $\hat{\gamma} = \epsilon_\eta^{1/2} \alpha^{1/2} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{1/2} f_4^{-1} \left(\frac{\hat{\gamma} k L_B}{\alpha\beta} \right)$ **[4]**

PR5: $\hat{\gamma} = \epsilon_\eta^{1/3} \alpha^{2/3} \beta^{1/3} \left(\frac{\Delta'}{\pi k^2 L_B} \right)^{2/3}$ **[5]**

PR6: $\hat{\gamma} = \epsilon_\eta^{3/5} \left(\frac{\Delta'^2}{C^2 k^3 L_B} \right)^{2/5} f_6^{-4/5} \left(\frac{\alpha^2 \beta}{\epsilon_\eta^{1/2} \hat{\gamma}^{1/2} k L_B} \right)$ **[2,6]**



[1] H. Furth, J. Killeen and M. Rosenbluth, Phys. Fluids 6, 459 (1963).

[2] E. Ahedo and J.J. Ramos, Plasma Phys. Control. Fusion 51, 055018 (2009).

[3] S. Bulanov, F. Pegoraro and A. Sakharov, Phys. Fluids B 4, 2499 (1992).

[4] V.V. Mirnov, C.C. Hegna and S.C. Prager, Phys. Plasmas 11, 4468 (2004).

[5] J. Drake and Y. Lee, Phys. Fluids 20, 1341 (1977).

[6] E. Ahedo and J.J. Ramos, Phys. Plasmas 19, 072519 (2012).

EFFECTIVELY HIGH-β REGIME ($\beta \gg S^{-2/5}$) AT FINITE ASPECT RATIO

- **INCOMPRESSIBLE PERTURBATION WITH GROWTH RATE INDEPENDENT OF β**

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- INCOMPRESSIBLE PERTURBATION WITH GROWTH RATE INDEPENDENT OF β

- **INTRODUCE SCALED VARIABLES**

$$\bar{x} \;=\; x\,(\epsilon_\eta \hat{\gamma})^{-1/4}\,|k/L_B|^{1/2}$$

$$\bar{\xi} \;=\; \frac{i\xi B_0}{B_{1r}}\,(\epsilon_\eta \hat{\gamma})^{1/4}\,\mathrm{sign}(L_B)\,|k/L_B|^{1/2}$$

$$\bar{Q} \;=\; \frac{rQ}{mB_{1r}}\,\epsilon_\eta^{1/4}\hat{\gamma}^{-3/4}\,\alpha\,\mathrm{sign}(L_B)\,|k/L_B|^{1/2}$$

EFFECTIVELY HIGH-β REGIME ($\beta \gg S^{-2/5}$) AT FINITE ASPECT RATIO

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$$\bar{x} = x \, (\epsilon_\eta \hat{\gamma})^{-1/4} \, |k/L_B|^{1/2}$$

$$\bar{\xi} = \frac{i\xi B_0}{B_{1r}} \, (\epsilon_\eta \hat{\gamma})^{1/4} \, \mathrm{sign}(L_B) \, |k/L_B|^{1/2}$$

$$\bar{Q} = \frac{rQ}{mB_{1r}} \, \epsilon_\eta^{1/4} \hat{\gamma}^{-3/4} \, \alpha \, \mathrm{sign}(L_B) \, |k/L_B|^{1/2}$$

- **INTEGRATE B''_{1r} THROUGH SINGULAR LAYER TO MATCH Δ'**

- **IN ITS LEADING ORDER, THE DISPERSION RELATION REDUCES TO**

$$\frac{\Delta'}{k} = \frac{\hat{\gamma}^{5/4}}{\epsilon_\eta^{3/4}} \, |kL_B|^{1/2} \, D(\rho, \sigma)$$

where

$$D(\rho, \sigma) = \int_{-\infty}^{\infty} d\bar{x} \, \left[1 - \bar{x}(\bar{\xi} + \bar{Q})\right]$$

$$\frac{d^2\bar{\xi}}{d\bar{x}^2} = \bar{x}^2\bar{\xi} + \left(\bar{x}^2 + \frac{i\rho}{\sigma}\right)\bar{Q} - \bar{x}$$

$$\frac{d^2\bar{Q}}{d\bar{x}^2} = \left[(1+\sigma^2)\bar{x}^2 + i\rho\sigma\right]\bar{Q} + \left(\sigma^2\bar{x}^2 + i\rho\sigma\right)\bar{\xi} - \sigma^2\bar{x}$$

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$$\frac{d^2\bar{\xi}}{d\bar{x}^2} \;=\; \bar{x}^2\bar{\xi} \;+\; \left(\bar{x}^2 + \frac{i\rho}{\sigma}\right)\bar{Q} \;-\; \bar{x}$$

$$\frac{d^2\bar{Q}}{d\bar{x}^2} \;=\; \left[(1+\sigma^2)\bar{x}^2 + i\rho\sigma\right]\bar{Q} \;+\; \left(\sigma^2\bar{x}^2 + i\rho\sigma\right)\bar{\xi} \;-\; \sigma^2\bar{x}$$

- **FINITE ASPECT RATIO AND HALL EFFECTS IN THE PARAMETERS ρ AND σ**

$$\rho \;=\; \frac{2k_z^2|L_B|}{k^2 r} \;=\; \frac{2}{|Rq_s'|}$$

$$\sigma \;=\; \epsilon_\eta^{-1/2}\hat{\gamma}^{1/2} \; \alpha$$

• **THE SYSTEM FOR $(\bar{\xi}, \bar{Q})$ HAS THE SOLUTION**

$$\bar{\xi} \;=\; \frac{\mu_+ \;|\mu_-|^{1/2}\; W_-\left(|\mu_-|^{1/2}\bar{x}\right) \;-\; \mu_- \;|\mu_+|^{1/2}\; W_+\left(|\mu_+|^{1/2}\bar{x}\right)}{2(1+\sigma^2/4)^{1/2}}$$

$$\bar{Q} \;=\; \frac{\sigma\left[|\mu_+|^{1/2}\; W_+\left(|\mu_+|^{1/2}\bar{x}\right) \;-\; |\mu_-|^{1/2}\; W_-\left(|\mu_-|^{1/2}\bar{x}\right)\right]}{2(1+\sigma^2/4)^{1/2}}$$

where

$$\mu_\pm \;=\; \sigma/2 \;\pm (1+\sigma^2/4)^{1/2}$$

$$\frac{d^2 W_\pm(y)}{dy^2} \;=\; y^2\; W_\pm(y) \;\pm\; i\rho\; W_\pm(y) \;-\; y$$

• **THE SYSTEM FOR $(\bar{\xi}, \bar{Q})$ HAS THE SOLUTION**

$$\bar{\xi} \;=\; \frac{\mu_+ \; |\mu_-|^{1/2} \; W_- \left(|\mu_-|^{1/2}\bar{x}\right) \; - \; \mu_- \; |\mu_+|^{1/2} \; W_+ \left(|\mu_+|^{1/2}\bar{x}\right)}{2(1+\sigma^2/4)^{1/2}}$$

$$\bar{Q} \;=\; \frac{\sigma \left[|\mu_+|^{1/2} \; W_+ \left(|\mu_+|^{1/2}\bar{x}\right) \; - \; |\mu_-|^{1/2} \; W_- \left(|\mu_-|^{1/2}\bar{x}\right)\right]}{2(1+\sigma^2/4)^{1/2}}$$

where

$$\mu_\pm \;=\; \sigma/2 \;\pm (1+\sigma^2/4)^{1/2}$$

$$\frac{d^2W_\pm(y)}{dy^2} \;=\; y^2 \; W_\pm(y) \;\pm\; i\rho \; W_\pm(y) \;-\; y$$

• **FROM THIS ONE CAN EVALUATE**

$$D(\rho,\sigma) \;=\; \frac{|\mu_+|^{1/2} \; C_+(\rho) \; + \; |\mu_-|^{1/2} \; C_-(\rho)}{2(1+\sigma^2/4)^{1/2}}$$

where

$$C_\pm(\rho) \;=\; \int_{-\infty}^{\infty} dy \;\; [1 - yW_\pm(y)]$$

- **HERMITE SERIES SOLUTION FOR $W_\pm(y)$**

$$W_\pm(y) \;=\; \exp(-y^2/2)\;\sum_{n=0}^{\infty} \frac{2^{-2n+1/2}}{(4n+3\pm i\rho)\; n!}\; H_{2n+1}(y)$$

which yields

$$C_\pm(\rho) \;=\; 2\pi\;\frac{\Gamma(3/4\pm i\rho/4)}{\Gamma(1/4\pm i\rho/4)}$$

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which yields

$$C_\pm(\rho) = 2\pi\, \frac{\Gamma(3/4 \pm i\rho/4)}{\Gamma(1/4 \pm i\rho/4)}$$

• **FINAL FORM OF THE HIGH β, FINITE ASPECT RATIO DISPERSION RELATION**

$$\frac{\Delta'}{k} = \frac{\hat{\gamma}^{5/4}}{\epsilon_\eta^{3/4}}\, |kL_B|^{1/2}\; D\left(\frac{2}{|Rq_s'|},\; \frac{\hat{\gamma}^{1/2} k d_\iota}{\epsilon_\eta^{1/2}}\right)$$

where

$$D(\rho,\sigma) = \frac{\pi}{(1+\sigma^2/4)^{1/2}} \left\{ \left[(1+\sigma^2/4)^{1/2} + \sigma/2\right]^{1/2} \frac{\Gamma(3/4+i\rho/4)}{\Gamma(1/4+i\rho/4)} + \left[(1+\sigma^2/4)^{1/2} - \sigma/2\right]^{1/2} \frac{\Gamma(3/4-i\rho/4)}{\Gamma(1/4-i\rho/4)} \right\}$$

$$k = \left(\frac{m^2}{r_s^2} + k_z^2\right)^{1/2}, \qquad |L_B| = \frac{r_s k^2}{|Rq_s'| k_z^2}$$

SPECIAL LIMITS

- **FOR** $\sigma \to 0$

$$D(\rho, 0) \;=\; \pi \left[\frac{\Gamma(3/4 + i\rho/4)}{\Gamma(1/4 + i\rho/4)} \;+\; \frac{\Gamma(3/4 - i\rho/4)}{\Gamma(1/4 - i\rho/4)}\right]$$

IS THE CYLINDRICAL, SINGLE-FLUID RESULT OF B. Coppi, J.M. Greene and J. Johnson, Nucl.Fusion 6**, 101 (1966).**

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- FOR $\sigma \rightarrow 0$

$$D(\rho, 0) \;=\; \pi \left[\frac{\Gamma(3/4 + i\rho/4)}{\Gamma(1/4 + i\rho/4)} + \frac{\Gamma(3/4 - i\rho/4)}{\Gamma(1/4 - i\rho/4)} \right]$$

IS THE CYLINDRICAL, SINGLE-FLUID RESULT OF B. Coppi, J.M. Greene and J. Johnson, Nucl.Fusion 6, 101 (1966).

- FOR $\sigma \rightarrow \infty$

$$D(\rho, \sigma \gg 1) \;=\; 2\pi \, \frac{\Gamma(3/4 + i\rho/4)}{\Gamma(1/4 + i\rho/4)} \, \sigma^{-1/2}$$

IS THE CYLINDRICAL GENERALIZATION OF THE SLAB GEOMETRY, ELECTRON-MHD RESULT OF S. Bulanov, F. Pegoraro and A. Sakharov, Phys. Fluids B 4, 2499 (1992).

- **FOR** $\rho \to 0$

$$D(0,\sigma) \;=\; \pi\,\frac{\Gamma(3/4)}{\Gamma(1/4)}\,(1+\sigma^2/4)^{-1/2}\left\{\left[(1+\sigma^2/4)^{1/2}+\sigma/2\right]^{1/2}+\left[(1+\sigma^2/4)^{1/2}-\sigma/2\right]^{1/2}\right\}$$

IS THE SLAB GEOMETRY, TWO-FLUID RESULT OF E. Ahedo and J.J. Ramos, Plasma Phys. Control. Fusion 51**, 055018 (2009), USED IN THE NIMROD CODE BENCHMARK OF C.R. Sovinec, J.R. King and the NIMROD Team, J. Comp. Phys.** 229**, 5803 (2010).**