Recent Progress on the DCON Code

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Outline

1. Rigorous proof of the generalized Newcomb criterion.

   A. H. Glasser,
   "The direct critierion of Newcomb for the ideal MHD stability of an axisymmetric toroidal plasma,"
   Phys. Plasmas 23, 072505 (2016).

2. Extensive verification of the resistive DCON code.

   A. H. Glasser, Z. R. Wang, and J.-K. Park,
   "Computation of resistive instabilities by matched asymptotic expansions,"
   Accepted for publication in Phys. Plasmas, November, 2016.

Ideal MHD Stability

1. I. B. Bernstein, E. A. Frieman, M. D. Kruskal and R. L. Kulsrud, "An energy principle for hydromagnetic stability problems", Proc. Roy. Soc. (London) 244, 17 (1958).
   Proves that ideal MHD stability can be formulated in terms of an energy principle δW. If a perturbation satisfying boundary conditions makes the potential energy δW negative, then there exists an instability.

2. W. A. Newcomb, "Hydromagnetic Stability of a Diffuse Linear pinch," Ann. Phys. 10, 232 (1960).
   For a cylindrical plasma, Newcomb reduces δW to a 1D integral of the radial displacement ξ over the radius r. Calculus of variations, Euler-Lagrange equation, 2nd-order ODE, $(f\xi')' - g\,\xi = 0$. Proved that an unstable perturbation exists iff the solution changes sign between singular points where $m = n\,q$.

3. G. A. Bliss, *Calculus of Variations* (Open Court Publishing Co., La Salle, IL, 1925).
   Reference used by Newcomb to prove necessary and sufficient conditions, using Hilbert invariant integral.

4. I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Prentice-Hall, Englewood Cliffs, NJ, 1963).
   More recent, thorough, and rigorous reference of the same material. Uses completion of the square in the integrand as alternative to Hilbert invariant integral.

5. A. H. Glasser, "The direct criterion of Newcomb for the ideal MHD stability of an axisymmetric toroidal plasma," Phys. Plasmas 23, 072505 (2016).
   Generalizes Newcomb criterion to axisymmetric toroidal plasma, uses Ref. 4 instead of Ref. 3, completion of the square rather than Hilbert invariant integral, uses Hilbert space $|\delta W[\Xi]| < \infty$.

Ideal MHD Energy Principle

Energy Principle, Original Form

$$\delta W = \frac{1}{2\mu_0}\int_\Omega d\mathbf{x}[Q^2 + \mathbf{J}\cdot\boldsymbol{\xi}\times\mathbf{Q} + \mu_0(\boldsymbol{\xi}\cdot\nabla P)(\nabla\cdot\boldsymbol{\xi}) + \mu_0\gamma P(\nabla\cdot\boldsymbol{\xi})^2]$$

Perturbed Displacement and Magnetic Field Vectors

$$\boldsymbol{\xi} = \mathcal{J}\left(\xi_\psi\nabla\theta\times\nabla\zeta + \xi_\theta\nabla\zeta\times\nabla\psi + \xi_\zeta\nabla\psi\times\nabla\theta\right), \quad \xi_s \equiv \mathcal{J}\left(\boldsymbol{\xi}\times\mathbf{B}\right)\cdot\left(\nabla\theta\times\nabla\zeta\right) = \chi'(q\xi_\theta - \xi_\zeta)$$

$$\mathbf{Q} = \nabla\times(\boldsymbol{\xi}\times\mathbf{B}) = \mathcal{J}(Q_\psi\nabla\theta\times\nabla\zeta + Q_\theta\nabla\zeta\times\nabla\psi + Q_\zeta\nabla\psi\times\nabla\theta)$$

$$Q_\psi = \frac{\chi'}{\mathcal{J}}\left(\frac{\partial}{\partial\theta} + q\frac{\partial}{\partial\zeta}\right)\xi_\psi, \quad Q_\theta = -\frac{1}{\mathcal{J}}\frac{\partial}{\partial\psi}(\chi'\xi_\psi) + \frac{1}{\mathcal{J}}\frac{\partial\xi_s}{\partial\zeta}, \quad Q_\zeta = -\frac{1}{\mathcal{J}}\frac{\partial}{\partial\psi}(q\chi'\xi_\psi) - \frac{1}{\mathcal{J}}\frac{\partial\xi_s}{\partial\theta}$$

Energy Principle, Component Form

$$\begin{aligned}\delta W = \frac{1}{2\mu_0}\int_\Omega d\psi d\theta d\zeta\mathcal{J}\Big\{&\left[\nabla\xi_s\times\nabla\psi + (\nabla\theta\times\nabla\zeta)\mathcal{J}\mathbf{B}\cdot\nabla\xi_\psi - \mathbf{B}\xi'_\psi - [\chi''\nabla\zeta\times\nabla\psi + (q\chi')'\nabla\psi\times\nabla\theta]\,\xi_\psi\right]^2 \\ &+ \xi_\psi\mathbf{J}\cdot\nabla\xi_s - \xi_s\mathbf{J}\cdot\nabla\xi_\psi + [J_\theta(q\chi')' - J_\zeta\chi'']\,\xi_\psi^2 + \frac{\mu_0 P'}{\mathcal{J}}(\mathcal{J}\xi_\psi^2)' \\ &+ \frac{1}{\mathcal{J}}\frac{\partial}{\partial\theta}(\mathcal{J}\mu_0 P'\xi_\psi\xi_\theta) + \frac{1}{\mathcal{J}}\frac{\partial}{\partial\zeta}(\mathcal{J}\mu_0 P'\xi_\psi\xi_\zeta) + \gamma\mu_0 P(\nabla\cdot\boldsymbol{\xi})^2\Big\}\end{aligned}$$

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Complex Vector and Matrix Representation

Poloidal Fourier Representation

$$\begin{pmatrix} \xi_s \\ \xi_\psi \end{pmatrix} (\psi, \theta, \zeta) = \sum_{m=-\infty}^{\infty} \left. \begin{pmatrix} \bar{\xi}_s \\ \bar{\xi}_\psi \end{pmatrix} \right|_{m,n} (\psi) \exp\left[2\pi i \left(m\theta - n\zeta\right)\right]$$

Complex Vectors of Fourier Components

$$\begin{pmatrix} \Xi_s \\ \Xi_\psi \end{pmatrix} (\psi) \equiv \left\{ \left. \begin{pmatrix} \bar{\xi}_s \\ \bar{\xi}_\psi \end{pmatrix} \right|_{m,n} (\psi), \; m_{\text{low}} \leq m \leq m_{\text{high}} \right\}$$

Energy Principle in Terms of Vectors and Matrices

$$\delta W = \frac{1}{2\mu_0} \int_a^b d\psi [\Xi_s^\dagger \mathbf{A} \Xi_s + \Xi_s^\dagger (\mathbf{B} \Xi_\psi' + \mathbf{C} \Xi_\psi) + (\Xi_\psi'^\dagger \mathbf{B}^\dagger + \Xi_\psi^\dagger \mathbf{C}^\dagger) \Xi_s + \Xi_\psi'^\dagger \mathbf{D} \Xi_\psi' + \Xi_\psi'^\dagger \mathbf{E} \Xi_\psi + \Xi_\psi^\dagger \mathbf{E}^\dagger \Xi_\psi' + \Xi_\psi^\dagger \mathbf{H} \Xi_\psi]$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{H}$ are complex matrices, known in terms of equilibrium quantities.

$\mathbf{A}, \mathbf{D}, \mathbf{H}$ are self-adjoint, $\mathbf{A}$ is positive-definite.

Minimization of δW

Energy Principle

$$\delta W = \frac{1}{2\mu_0}\int_a^b d\psi [\Xi_s^\dagger \mathbf{A}\Xi_s + \Xi_s^\dagger(\mathbf{B}\Xi_\psi' + \mathbf{C}\Xi_\psi) + (\Xi_\psi'^\dagger \mathbf{B}^\dagger + \Xi_\psi^\dagger \mathbf{C}^\dagger)\Xi_s + \Xi_\psi'^\dagger \mathbf{D}\Xi_\psi' + \Xi_\psi'^\dagger \mathbf{E}\Xi_\psi + \Xi_\psi^\dagger \mathbf{E}^\dagger \Xi_\psi' + \Xi_\psi^\dagger \mathbf{H}\Xi_\psi]$$

Minimization with Respect to Surface Displacements

$$\mathbf{A}\Xi_s + \mathbf{B}\Xi_\psi' + \mathbf{C}\Xi_\psi = 0, \quad \Xi_s = -\mathbf{A}^{-1}(\mathbf{B}\Xi_\psi' + \mathbf{C}\Xi_\psi)$$

Energy Principle in Terms of Normal Displacements

$$\delta W = \frac{1}{2\mu_0}\int_a^b d\psi \left[\Xi'^\dagger \mathbf{F}\Xi' + \Xi'^\dagger \mathbf{K}\Xi + \Xi^\dagger \mathbf{K}^\dagger \Xi' + \Xi^\dagger \mathbf{G}\Xi\right]$$

Composite Complex Matrices

$$\mathbf{F} \equiv \mathbf{D} - \mathbf{B}^\dagger \mathbf{A}^{-1}\mathbf{B} = \mathbf{F}^\dagger, \quad \mathbf{K} \equiv \mathbf{E} - \mathbf{B}^\dagger \mathbf{A}^{-1}\mathbf{C} \neq \mathbf{K}^\dagger, \quad \mathbf{G} \equiv \mathbf{H} - \mathbf{C}^\dagger \mathbf{A}^{-1}\mathbf{C} = \mathbf{G}^\dagger$$

Euler-Lagrange Equation

$$-(\mathbf{F}\Xi' + \mathbf{K}\Xi)' + (\mathbf{K}^\dagger \Xi' + \mathbf{G}\Xi) = 0$$

This is a condition that $\delta W[\Xi]$ be stationary.
It is a necessary but not sufficient condition that it be a minimum.

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Matrix Form and Symmetries

Euler-Lagrange Equation as 2nd-Order System

$$-\left(\mathbf{F}\Xi' + \mathbf{K}\Xi\right)' + \left(\mathbf{K}^\dagger \Xi' + \mathbf{G}\Xi\right) = 0$$

Equivalent 1st-Order System

$$\mathrm{u} \equiv \begin{pmatrix} \Xi \\ \mathbf{F}\Xi' + \mathbf{K}\Xi \end{pmatrix}, \quad \mathbf{L} \equiv \begin{pmatrix} -\mathbf{F}^{-1}\mathbf{K} & \mathbf{F}^{-1} \\ \mathbf{G} - \mathbf{K}^\dagger\mathbf{F}^{-1}\mathbf{K} & \mathbf{K}^\dagger\mathbf{F}^{-1} \end{pmatrix}, \quad \mathrm{u}' = \mathbf{L}\mathrm{u}$$

Hamiltonian Symmetry

$$\mathbf{L} = \begin{pmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ tL_{11} & \mathbf{L}_{12} \end{pmatrix}, \quad \mathbf{L}_{22} = -\mathbf{L}_{11}^\dagger, \quad \mathbf{L}_{12} = \mathbf{L}_{12}^\dagger, \quad \mathbf{L}_{21} = \mathbf{L}_{21}^\dagger$$

$$\mathbf{J} \equiv \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{JLJ} = \mathbf{L}^\dagger$$

Fundamental Matrix of Solutions

$$\mathbf{U} = \begin{pmatrix} \mathbf{U}_{11} & \mathbf{U}_{12} \\ \mathbf{U}_{21} & \mathbf{U}_{22} \end{pmatrix}, \quad \mathbf{U}' = \mathbf{LU}, \quad \mathbf{U}(0) = \mathbf{I}$$

Symplectic Symmetry

$$\mathbf{UJU} = \mathbf{J}$$

$$\mathbf{U}_{11}^\dagger\mathbf{U}_{21} - \mathbf{U}_{21}^\dagger\mathbf{U}_{11} = \mathbf{0}, \quad \mathbf{U}_{12}^\dagger\mathbf{U}_{22} - \mathbf{U}_{22}^\dagger\mathbf{U}_{12} = \mathbf{0}, \quad \mathbf{U}_{11}^\dagger\mathbf{U}_{22} - \mathbf{U}_{21}^\dagger\mathbf{U}_{12} = \mathbf{I}$$

Sufficient Condition for $\delta W > 0$

Energy Principle

$$\delta W = \frac{1}{2\mu_0}\int_0^1 d\psi\left[\Xi'^\dagger \mathbf{F}\Xi' + \Xi'^\dagger \mathbf{K}\Xi + \Xi^\dagger \mathbf{K}^\dagger \Xi' + \Xi^\dagger \mathbf{G}\Xi\right]$$

Perfect Derivative

For any $\mathbf{W} = \mathbf{W}^\dagger$, $\displaystyle \frac{1}{2\mu_0}\int_0^1 (\Xi^\dagger \mathbf{W}\Xi)' d\psi = \frac{1}{2\mu_0}\int_0^1 (\Xi'^\dagger \mathbf{W}\Xi + \Xi^\dagger \mathbf{W}\Xi' + \Xi^\dagger \mathbf{W}'\Xi) d\psi = 0$

$$\delta W = \frac{1}{2\mu_0}\int_0^1 \left[\Xi'^\dagger \mathbf{F}\Xi' + \Xi'^\dagger(\mathbf{K}-\mathbf{W})\Xi + \Xi^\dagger(\mathbf{K}^\dagger - \mathbf{W})\Xi' + \Xi^\dagger(\mathbf{G}-\mathbf{W}')\Xi\right] d\psi$$

Completing the Square

Choose $\mathbf{W} \equiv \mathbf{U}_{22}\mathbf{U}_{12}^{-1}$, $\quad D_C \equiv \det \mathbf{W}^{-1}$

$\mathbf{W}$ is self-adjoint by the symplectic symmetry of $\mathbf{U}$.
It exists if $D_C \neq 0$ for $\psi \in (0,1)$.

$$\mathbf{W}' = \mathbf{G} - \mathbf{K}^\dagger \mathbf{F}^{-1}\mathbf{K} - \mathbf{W}\mathbf{F}^{-1}\mathbf{W} + \mathbf{K}^\dagger \mathbf{F}^{-1}\mathbf{W} + \mathbf{W}\mathbf{F}^{-1}\mathbf{K}$$

$$\delta W = \frac{1}{2\mu_0}\int_0^1 \left[\Xi'^\dagger + \Xi^\dagger(\mathbf{K}^\dagger - \mathbf{W})\mathbf{F}^{-1}\right]\mathbf{F}\left[\Xi' + \mathbf{F}^{-1}(\mathbf{K}-\mathbf{W})\Xi\right] d\psi$$

This is a real quadratic form in $\mathbf{F}$, which is Hermitian-positive-definite. δW is therefore positive definite if $D_C \neq 0$. This is the sufficient condition for δW to be a minimum.

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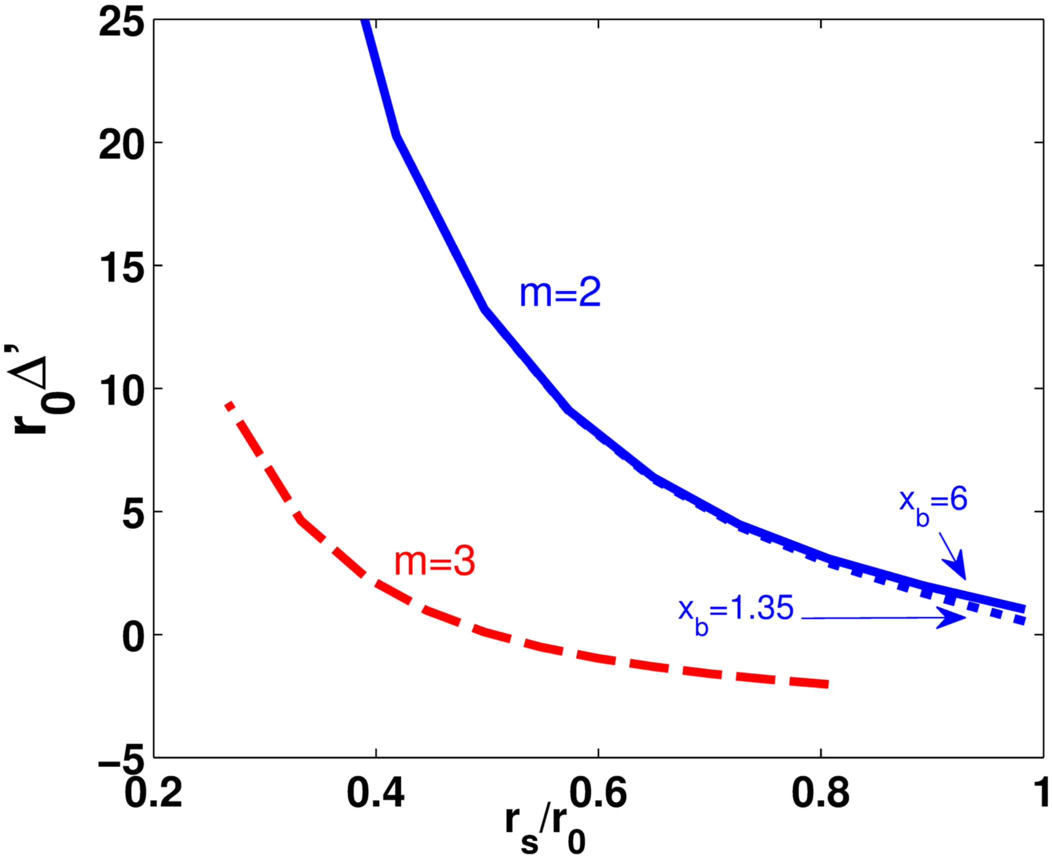
Comparison of Resistive DCON to Furth, Rutherford and Selberg



Furth, Rutherford, and Selberg, Phys. Fluids 16, 7, 1054 (1974).
Their results are $\beta_0 = 0$, cylindrical plasma.
Our results are for $\beta_0 = 4.5 \times 10^{-3}$, R/a =10

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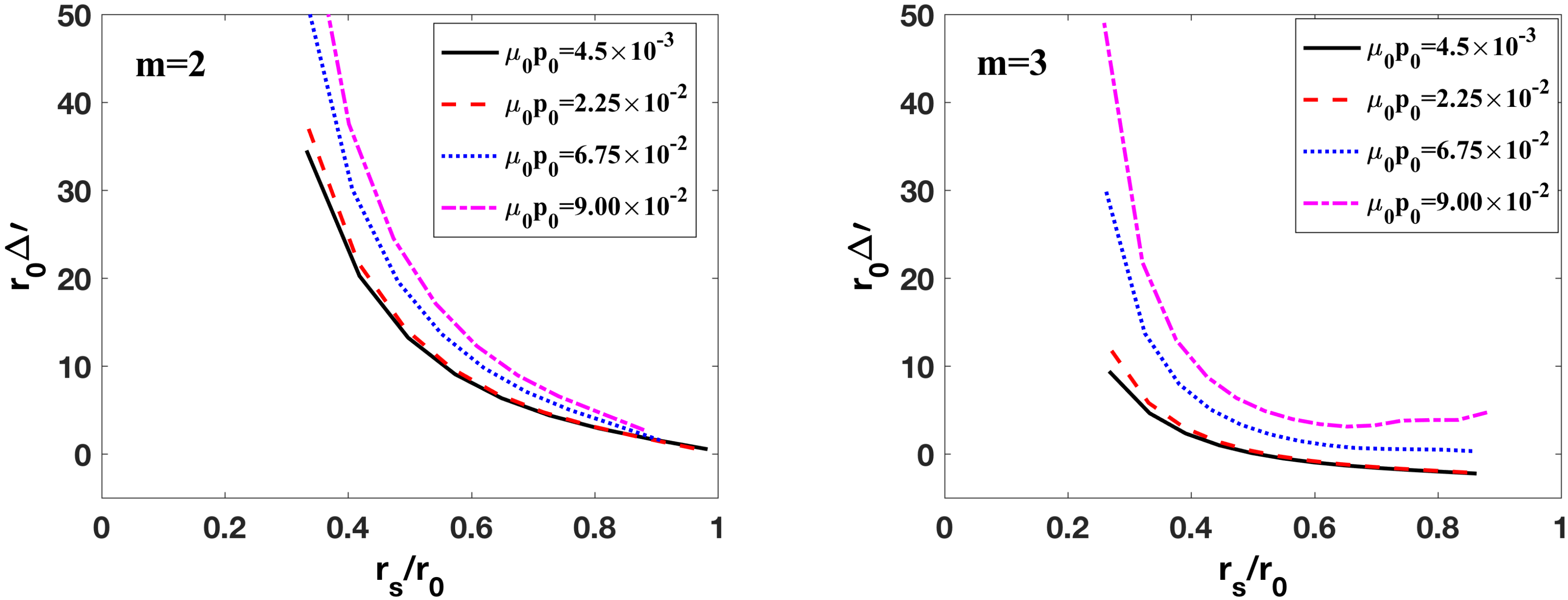
Finite β Effects on Δ′



Instability for $\Delta' > \Delta_C$.

Δ' increases with β, which is destabilizing,

but Δ_C increases faster with β, which is stabilizing.

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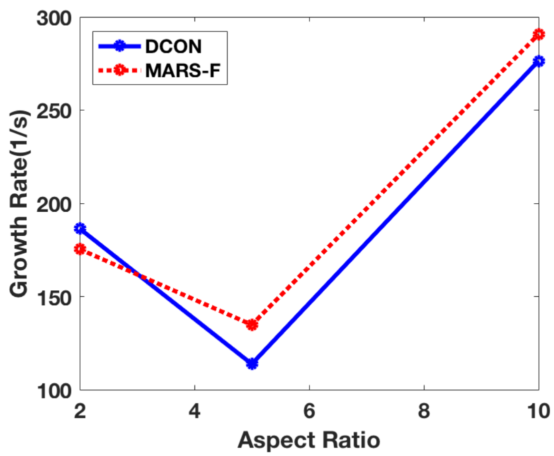
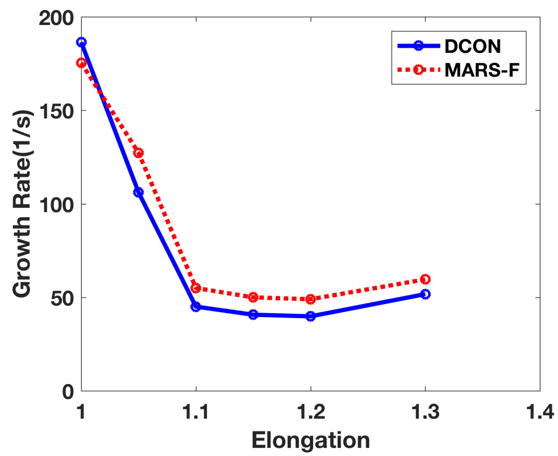
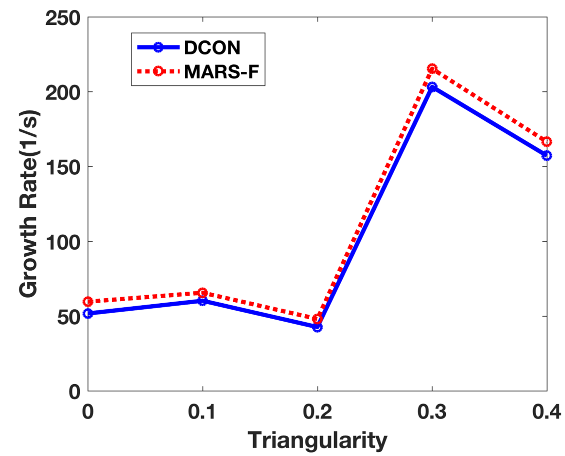
MARS Benchmarks, One Singular Surface





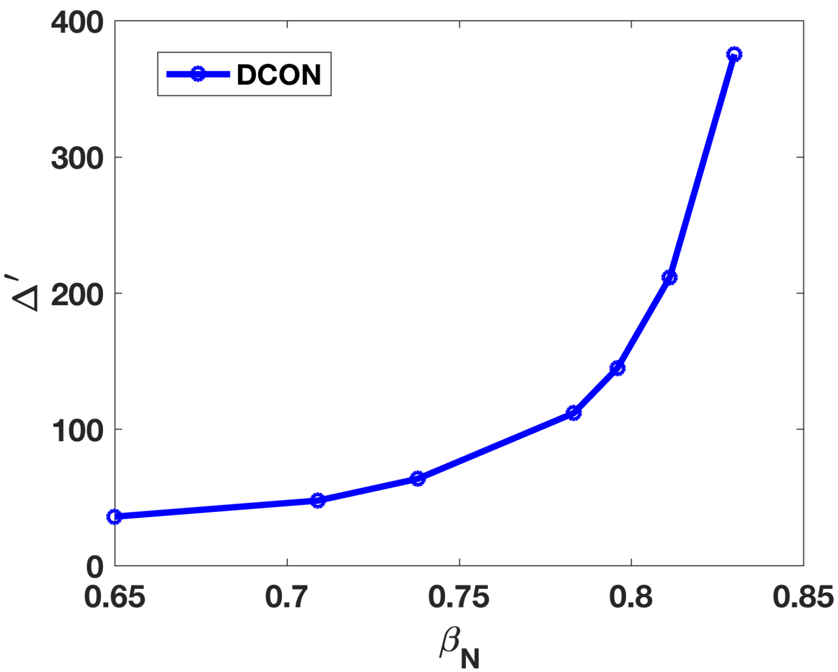


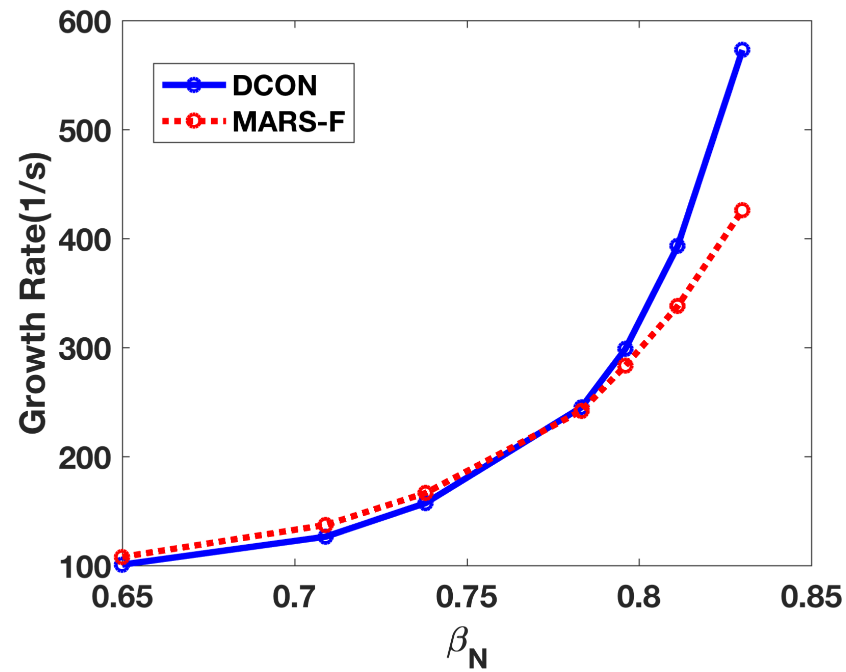
κ = elongation = 1
τ = triangularity = 0
$q_0 = 1.1$, $q_a < 3$

R/a = aspect ratio = 2
τ = triangularity = 0
$q_0 = 1.1$, $q_a < 3$

R/a = aspect ratio = 2
κ = elongation = 1.3
$q_0 = 1.1$, $q_a < 3$

MARS Benchmarks: Growth Rate Increases with β





$R/a = 2, \kappa = 1.3, \tau = 0.4, q_0 = 1.1, q_a < 3$

Growth rate increases on approach to ideal stability boundary.

MARS Benchmarks, Two Singular Surfaces

$q_0 = 1.1$, $q_a \sim 3.1$, $S = 2\text{x}10^7$

#	R/a	κ	τ	DCON	MARS
1	10	1.0	0.0	476.5	406.32
2	5	1.0	0.0	101.6	130
3	4	1.0	0.0	47.5	50.97
4	3	1.0	0.0	161.3	73.7
5	2	1.0	0.0	(-62.4,122.8)	105
6	2	1.3	0.4	(-68.1,132.1)	55

Discrepancies, not yet understood

Conclusions

1. The mathematical foundation of the ideal MHD DCON fixed-boundary stability criterion is now proven.

2. Benchmarks of resistive DCON against previous results and other codes, partially complete, some remaining discrepancies.