Calculating Beam (and RF) Currents: Review of Variational Principles, Adjoints and Higher-Order Moments

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References (Beams)

- J. G. Cordey, et. al. Nucl. Fusion 19, 249 (1979): direct numerical solution for $j_{||}$ including e-e collisions, arbitrary v_b/v_e
- S. P. Hirshman, Phys. Fluids 23, 1238 (1980): variational/adjoint “analytical” solution for classical beam driven current

References (cont'd)

- S. I. Braginskii, Reviews of Plasma Physics vol 1 (1965): Generating functions for Coulomb collisions

- Helander and Sigmar, “Collisional Transport in Magnetized Plasmas”, 2002: Moment methods

Steady-state beam current

- Ignoring time-derivatives, drifts (neoclassical effects), trapping, etc, one has the “classical” linearized ($n_b/n_e << 1$) equation to solve for $f_{e1} = f_e - f_{Me}$ (Ohkawa, shielded current) :

$$
\begin{aligned}
C_{ee}(f_{e1}, f_{Me}) + C_{ee}(f_{Me}, f_{e1}) + C_{ei}(f_{e1}, f_{Mi}) &\equiv S_b(\vec{v}) \\
&= -C_{eb}(f_{Me}, f_b)
\end{aligned}
$$

Only l=1 (Momentum) Harmonic Needed for Current

$$J_{\|} \equiv e_b \int v_{\|} f_b d\vec{v} - |e| \int v_{\|} f_{e1} d\vec{v}$$

- Harmonics uncouple classically (although **not** neo-classically)
- Solve using a “Green’s Function” delta-fcn (in $|v_b| = v_b$), integrate at end of slowing down distribution:

$$f_b^{l=1} = \tfrac{3}{2} a_b(v) P_1(v_{\|}/v)$$

$$a_b(v) = n_b u_{\|b} \delta(v - v_b)/(2\pi v_b^3)$$

Beam Equation has "jump" Source

- Singular scattering kernel (|v-v'|-1) in Coulomb operator leads to jump in "source" *s** at $v_b = v_e$:

$$C_e(f_e^*) = S_b(\vec{v}), f_e^* = f_{e1} - (2v_{\|}u_{\|b} / v_{Te}^2)(n_b e_b / n_e |e|) f_{Me}$$

$$C_e(f_e^*) \equiv C_{ee} + C_{ei}$$

$$S_b(\vec{v}) \sim v_{\|} s^*(x = v / v_{Te})$$

$$s^*(x) = x^{-3}\left[(1 - Z/Z_b) + \tfrac{6}{5}\bar{v}_b^2\right], \bar{v}_b = v_b / v_{Te} < x$$

$$= -x^{-3}\left[Z/Z_b - \bar{v}_b^{-3}(2 - \tfrac{6}{5}x^2\right], \bar{v}_b > x$$

$$j_{\|} = -|e| \int v \| f_e^* d\vec{v}$$

Discontinuity Occurs in Electron Bulk (for usual case $v_b << v_{Te}$)

- Smoothed by integration over beam velocity distribution and 2nd order energy scattering (e-e) of electrons
- However, need for greater accuracy (because of jump) than the usual Spitzer problem (no jump)
  - Variational methods
  - Higher-order (>2) moment methods: extendable to time-dependent calculations (CEL-Callen, et.al.)

Classical Variational Principle (CVP) Does NOT Work

- The CVP (Robinson and Bernstein, 1962) fails for this problem because of the complex |v|-dependence of the beam "source" kernel $s_b(x)$ [i.e., !~ const]

- Put another way, the stationary state of CVP is NOT the beam current, which is what we want!

Intro to the Adjoint Equation

- Recall that the Spitzer-Härm equation – which is similar to the beam equation but with $s_b(x) \sim 1$ - leads to a CVP for the electric-field driven current:

$$C_e(f_{eS}) = v_{\|} f_{Me}$$

$$\dot{S} = \int (f_{eS} / f_{Me}) C_e(f_{eS}) d\vec{v} - 2\int v_{\|} f_{eS} d\vec{v}$$

$$\delta\dot{S} = 0 \quad \text{(CVP)}$$

The Adjoint (cont'd)

- Multiply SH equation by f_e/f_{Me} and using the *self-adjointness* of the linearized collision operator yields the adjoint expression for the beam current (also works for RF current drive, replacing S_b with QL operatior-Fisch):

$$j_{\|b} = \int f_{eS} S_b(\vec{v}) d\vec{v}$$

Advantage of Adjoint Method

- Compared with direct numerical solution, f_{eS} can be obtained (variationally) much more accurately than f_e^* using a few-term trial function approximation

Disadvantage of Adjoint Method

- Really anchored to “steady-state” calculations (beam slowing-down changing much slower than electron distribution function “adiabatic” response) and linearized (small beam density) limit.
- Both of these might be excellent assumptions (for beam problem, but what about RF???). If not, then an extended moment approach might work better.

Why Higher-Order Moments?

- Moments methods have been used to accurately compute classical and neoclassical currents (SH, bootstrap, conductivity reduction) to high accuracy with few moments (2 or 3) – see Helander-Sigmar.
- Reason: neoclassical “sources” have benign energy dependence
  - $S_{neo} \sim 1, v^2$

NB Injection/RF Tails Violate "Benign" Velocity Dependence

- NB Injection source "jump" in energy space not "smooth"

- Lower hybrid RF tails at v ~ nv_{TE}, for n ~ 2-3 or more, produce localized structures in velocity space that are not well represented by a few low-order Laguerre polynomials (which form the basis set for low order moment methods)

Hi-Order Expansions

- Consider Grad's tensor Hermite expansion projected along B and (for simplicity) the l=1 spherical harmonic only: (-> Laguerre polynomials of order 3/2):

$$f^{(l=1)} = \frac{2v_{\parallel}}{v_{Te}^2} f_M(x^2) \sum_{n=0} f_n L_n^{3/2}(x^2)$$

$$L_0 = 1; L_1 = (5/2 - x^2)$$

Lo-Order vs Hi-Order Moments

- Lo-Order moments (L_0,L_1) correspond to flow and heat flow along field lines (or friction/heat friction collisional forces)

- Hi-Order moments represent distortions of distribution function in |v| space

Require Hi-Order Coulomb Matrix Elements

- Projection of the kinetic equation onto the L_n basis requires evaluation of "matrix elements" of *C* ("test" particle *M* and "field particle" *N* contributions):

$$M(i,j) = \int v_{\parallel} L_i C(v_{\parallel} L_j, f_M) d\vec{v}$$

$$N(i,j) = \int v_{\parallel} L_i C(f_M, v_{\parallel} L_j) d\vec{v}$$

Nasty (*really*) 6-D v-space integrals

- To evaluate a few of these matrix elements is something grad-students are useful for…(if they survive, give them their degree)

- However, to evaluate hi-orders, it is very desirable to limit the calculations to as few as possible, and use recursion thereafter.

Two Schemes: Generating functions and Recursion

- Generating function for L's allows a single evaluation of matrix elements for a two-parameter set (Braginskii) and subsequent Taylor expansion yields the desired elements:
  - Must use MACSYMA or MATLAB for very high orders (~20 is practical but takes a while…)

Recursive Method

- After *a lot* of algebra (Hirshman and Houlberg, Savannah Sherwood) one finds

$$M_{ij} = \frac{-n}{\tau} m_{ij} H_{ij}$$

$$N_{ij} = \frac{n}{\tau} n_{ij} H_{ij}$$

$$H_{ij} = \frac{4\Gamma[\frac{1}{2}(L_{ij}+4)]}{\sqrt{\pi}}$$

$$L_{ij} = i + j + 1$$

Recursive method (cont'd)

- The matrix elements m_{ij}, n_{ij} can be expressed in terms of Gauss' hypergeometric function (1D integral) and satisfy recurrence relations which makes them easy to evaluate (in a computationally efficient manner).